Modification of turbulence characteristics in flow over hills

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SUMMARY

A new model has been developed to investigate turbulence in flows over two-dimensional hills: the von Mises transformation was applied in mean momentum equations, and turbulence equations (second-order-closure type) were solved in a coordinate system aligned with streamlines; the hill perturbation pressure was calculated by means of potential flow theory. The model predictions were compared with some experimental results from the Askervinen Hill project with good agreement. It was found that apart from rapid distortion the curvature of streamlines has an important dynamical effect on turbulence. The curvature effect is shown to be particularly pronounced in the vicinity of the inner layer height on the hill top, where it attenuates turbulence. Overall, the salient features of the hill top turbulence profile were successfully explained by analysing the turbulence conservation equations in streamline coordinates.

1. INTRODUCTION

The presence of hills induces pressure perturbations in the atmospheric boundary layer (ABL) and both the mean and turbulence fields are modified. Knowledge of the effect of hills on the flow field in the ABL is important for many aspects of meso- and micro-scale meteorology and wind energy applications. In the past ten years the mean wind field over hills has been extensively studied both theoretically and experimentally (Jackson and Hunt 1975; Taylor 1977, 1980; Mason and Sykes 1979; Sykes 1980; Bradley 1980; Britter et al. 1981; Walmsley et al. 1982; Teunissen et al. 1982; and others). Today the physics of mean flow over gentle topography in neutrally stratified surroundings is fairly well understood, and numerical or analytical models are capable of predicting the mean wind field with reasonable accuracy.

The effect of the presence of hills on turbulence has not so far been studied in detail and is not fully understood. Until recently turbulence observations on real hills have not been available. Based on theory, Hunt (1980) has suggested the existence of a two-layer turbulence structure: in the so-called inner layer (near the hill surface) turbulence is expected to be in equilibrium with the current boundary conditions, hence the turbulence structure can be predicted on the basis of the surface layer laws; in the overlying outer layer, turbulence is expected to be modified by the so-called rapid distortion effect as described by Batchelor and Proudman (1954) and Townsend (1972). Accordingly, the longitudinal turbulence component is expected to decrease and the vertical one increase with respect to the upstream level. In the case of the vertical component this description was corroborated by the observations of Bradley (1980) on Black Mountain in Australia although this hill was wooded and very steep. Britter et al. (1981) suggested parametric expressions for the rapid distortion effect which agreed with their wind tunnel experiment on the turbulent air flow over a rough 2-D hill.

Taylor (1977) presented a numerical study of a neutral ABL flow over a 2-D hill using a model which incorporated a turbulence energy conservation equation; turbulent stresses were related to mean velocity gradients by eddy viscosity. His calculated turbulent energy profiles correctly show the perturbation of the turbulence energy on the hill top in the surface (inner) layer. However, in principle this eddy viscosity model is incapable of representing the rapid distortion or relaxation effects. Taylor (1980) was aware of this

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drawback in his model and the need for representing the stress relaxation effects. The calculated Reynolds shear stresses ($\overline{uw}$) were compared with observations on a real hill (Bradley 1980). The observations indicated that $\overline{uw}$ does not follow the change in sign of the mean wind gradient as predicted by the model (above the wind maximum on the hill top) but remembers more or less its upstream value.

Sykes (1980) formulated an asymptotic theory for flow over two-dimensional humps which incorporates a second-order turbulence closure model of Launder et al. (1975). He computed Reynolds stress tensor perturbations for flow over a periodic (sinusoidal) topography. His results showed that the effects of rapid distortion and relaxation are important in regions away from the surface. His conclusion was that the local eddy viscosity closure for Reynolds stresses as in Taylor (1977) gives spuriously large stress perturbations which in turn affect, for example, the perturbation force on the hill. An interesting aspect of Sykes's theory is that it relaxes the limitations of the Jackson–Hunt theory, namely, that the hill height must be smaller than the inner layer depth, which translates into the restriction that the maximum (absolute) speed-up, $\Delta U$, has to be smaller than the friction velocity, $u_\ast$. Sykes's theory allows $\Delta U > u_\ast$, a condition which is true for natural flows of practical interest.

Recent observational studies of turbulence, notably those on the Askervein Hill in the Scottish Outer Hebrides, suggest that the turbulence modification as observed on the hill top cannot be satisfactorily explained by present theories or methods, and we felt it was desirable to investigate turbulence dynamics with models which are consistent with the observed background (upwind) ABL structure and possibly without limitations (as far as the turbulence modelling is concerned) of asymptotic theories. For this purpose we developed a relatively simple model which is intended for investigation of turbulence in neutral ABL flows over isolated (or periodic) two-dimensional hills. The model formulation is described in section 2. Section 3 is a preliminary analysis of turbulence dynamics in flows over hills, and in section 5 computed Reynolds stress profiles are compared with the measurements on Askervein Hill (section 4 gives some specific details regarding hill parameters etc.). Other presently available turbulence data are discussed in section 6. Conclusions are given in section 7.

2. THE HILL FLOW MODEL

The purpose of the model described in the following is to study the response of turbulence in the ABL to the presence of hills and to assess the effect of various physical processes that contribute to alteration of the turbulence structure as represented by the Reynolds stress tensor $\overline{u_iu_j}$, and by the turbulence length scale and dissipation rate, $l$ and $\overline{\varepsilon}$.

The optimal level of turbulence modelling for this type of flow is a second-order closure technique. We chose the model developed by Zeman (1975) and described in part also in Zeman and Tennekes (1975) and Zeman and Lumley (1979). The model was specifically designed for applications in planetary boundary layers and has been successfully tested against rapid distortion experiments as shown in Zeman (1975). The model is sufficiently flexible to simulate different upwind distributions of the stress tensor $\overline{u_iu_j}$ (in the horizontally homogeneous ABL) and represents well the physical processes that are important in the hill flow.

In order to simplify the model and reduce the system of equations to a manageable level computationally, we devised a special method which allows us to solve turbulence equations along streamlines. The major aspects of the method are as follows:

(a) the von Mises transformation is applied in the mean momentum equation, which
allows us to specify mean flow streamlines;
(b) the turbulence equations are formulated in the curvilinear coordinate system following the streamlines;
(c) the mean (driving) pressure gradient is assumed to be given by the linearized potential solution based on: (1) an unperturbed wind speed \( U_0 = U(z \ll L) \), upstream at a height proportional to the length scale of the hill, \( L \); (2) a hill contour \( \eta(x) = h(x/L) \), where \( h \) is the hill height and \( f \) represents the hill shape.

The length scale \( L \) is chosen in agreement with Jackson and Hunt (1975) as the distance from the hill top to the point where \( \eta(L) = \pm h \). In general, the perturbation pressure can be calculated from the known formula (see e.g. Nørstrud 1982)

\[
P(x, z) = -U_0^2 \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta'(s)(x - s)}{(x - s)^2 + z^2} \, ds.
\]

For the inverse polynomial \( (1 + (x/L)^2)^{-1} \), \( P \) is an analytical function. However, for a general function \( \eta \), \( P \) must be obtained by numerical integration.

The method outlined has the advantage of greatly simplifying the system of equations and thus of numerical computation. The equations may be integrated numerically by marching forward in streamwise direction. The turbulence stress equations are cast in the natural coordinate system aligned with streamlines, thus allowing for a clear interpretation of different terms in model equations whose meanings are not obscured by cross-terms resulting from generalized transformation of turbulence stress equations.

The most critical assumption, (c), demands some justification. Jackson and Hunt showed that within the validity of their theory the linear (potential) pressure solution is correct to the leading order in the hill slope \( h/L \). Sykes (1980) arrived at the same conclusion with somewhat different and more realistic assumptions about the flow geometry. Apart from the theoretical justifications, we found that the potential pressure solution yielded, with some correction, hill top velocity profiles in good agreement with measurements for a fairly large slope of \( h/L = 0.5 \).

It should be remarked that our method is not, in principle, limited to hill geometries with small slopes. The nonlinear pressure field can be calculated by standard numerical techniques. Preliminary tests showed, however, that the mean and turbulence fields are a little sensitive to higher-order pressure corrections.

\[(a)\] Mean flow equations

Using the standard Reynolds decomposition for velocity components \( \bar{u}_i = U_i + u_i \) in a fixed Cartesian coordinate system \((x, y, z)\) oriented as shown in Fig. 1, we write the mean momentum equations in a neutral incompressible barotropic ABL as follows:

\[
\begin{align*}
\frac{DU}{Dt} + (V - V_g)f &= - \partial P/\partial x - \partial \bar{u}w/\partial z - \partial w^2/\partial x \\
\frac{DV}{Dt} - (U - U_g)f &= - \partial P/\partial y - \partial \bar{v}w/\partial z - \partial w^2/\partial y \\
\frac{DW}{Dt} &= - \partial P/\partial z - \partial \bar{u}w/\partial x - \partial \bar{v}w/\partial z \\
\partial U/\partial x + \partial W/\partial z &= 0
\end{align*}
\]

where \( U, W \) are horizontal and vertical mean velocities in the plane normal to the 2-D hill, the surface wind is aligned with the \( x \) axis so that the cross-stream wind component \( V \) and the Reynolds stress \( \bar{uw} \) are zero near the ground. \( U_g, V_g \) are the geostrophic wind components and \( P(x) \) is the mean perturbation pressure due to the presence of the hill. The turbulence (one-point) moments \( \bar{u} \) are elements of the Reynolds stress tensor (subscripts 1, 2, 3 refer to the directions along \( x, y, z \), respectively; averaged quantities
are in capital letters or are designated by overbar).

Because of the large Rossby number $U_0/fL$ the cross-stream field will remain frozen during the time $T_s = L/U_o$ of flow passage over the hill and thus only solutions of equations for $U$ and $W$ are needed if $V$ and $V_g$ are specified upstream.

The von Mises transformation $(x, z) \rightarrow (x, \psi)$ (where $\psi$ is the streamfunction such that $U = \partial \psi/\partial z$ and $W = -\partial \psi/\partial x$) can now be applied to steady state equations (2) and (3). By the transformation operation rules (see e.g. Milne-Thompson 1949)

$$\frac{\partial}{\partial x} \bigg|_z = \frac{\partial}{\partial x} \bigg|_\psi - W \frac{\partial}{\partial \psi} \quad \text{and} \quad \frac{\partial}{\partial z} \bigg|_x = U \frac{\partial}{\partial \psi} \bigg|_x$$

Eq. (2) transforms into

$$\frac{\partial U}{\partial x} \bigg|_\psi = -\frac{1}{U} \left( \frac{\partial P}{\partial x} + (V - V_g)f \right) - \frac{\partial}{\partial \psi} \left( u \bar{w} \right) - \frac{1}{U} \left( \frac{\partial}{\partial x} - W \frac{\partial}{\partial \psi} \right) u^2. \quad (4)$$

We note that in Eq. (4) the gradients $\partial/\partial x$ are along the streamlines rather than along constant $z$; however, the components of mean or fluctuating velocities are still in the orthogonal Cartesian system $(x, y, z)$. Because the pressure field is known a priori as a function of $(x, z)$ there is no need to transform $\partial P/\partial x$. If the Reynolds stresses $u \bar{w}, u^2$ are known, Eq. (4) can be solved by marching along streamlines. At every point along $x$ the position of a streamline $\psi_1(x, z/z_1) = \text{constant}$ originating at a height $z_1$ upstream $(x = -\infty)$ can be determined by the relationships

$$\psi_1(z_1) = \int_0^{z_1} U_o(z) \, dz \quad x = -\infty$$

and

$$\xi(x, \psi_1(z_1)) = \int_0^{\psi_1} \frac{d \psi}{U(x, \psi)} + \eta(x) \quad (5)$$

where the streamline $\psi_0(x, 0) = 0$ following the hill surface is assigned zero value; $\xi$ is the vertical coordinate of $\psi_1$. 

Figure 1. Sketch defining some of the key parameters and variables used in the text.
Our next step is to determine the Reynolds stress components necessary to close Eq. (4). Since the streamlines are known, it is convenient to formulate the Reynolds stress equations in an orthogonal coordinate system locally aligned with the streamlines as shown in Fig. 1. The mean velocity and stress tensor components in the coordinate systems \((x, y, z)\) and \((\hat{x}_1, \hat{x}_2, \hat{x}_3)\) are related through the transformation

\[
\bar{U}_i = A_{ij} U_j, \quad \bar{a}_i \bar{a}_j = A_{ik} A_{jl} \bar{a}_k \bar{a}_l.
\]  

(6)

Here \(A_{ij}\) is the transformation tensor

\[
A_{ij} = \begin{bmatrix}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-\sin \alpha & 0 & \cos \alpha
\end{bmatrix}
\]  

(6a)

with \(\alpha = \tan^{-1}(\partial \hat{z}/\partial x)\), the angle between a given streamline and the \(x\) axis. The hat designates the streamline coordinate system \((\hat{x}_1, \hat{x}_2, \hat{x}_3)\).

(b) Turbulence closure

In Cartesian coordinates and with tensor notation the general expression for the conservation equation for \(\bar{u}_i \bar{u}_j\) is (see e.g. Zeman 1981)

\[
\frac{D\bar{u}_i \bar{u}_j}{Dt} = -\bar{u}_i \bar{u}_k \frac{\partial U_i}{\partial x_k} - \bar{u}_j \bar{u}_k \frac{\partial U_i}{\partial x_k} - \bar{P}_{ij} \frac{\partial \bar{u}_j}{\partial x_i} - \frac{\partial}{\partial x_i} \left( \bar{u}_j \bar{u}_j \bar{u}_k \right) - 2\nu \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_j}{\partial x_i} \]  

(7)

where the terms \(P_{ij}, \Pi_{ij}, T_{ijk}\) and \(D_{ij}\) are respectively called the shear production, pressure, transport and viscous (dissipation) terms, and \(\nu\) is the (constant) kinematic viscosity of air.

We shall use the standard approximation for the viscous term \(D_{ij} = \frac{3}{2} \bar{e} \delta_{ij}\) where \(\bar{e}\) is the average dissipation rate of turbulence kinetic energy \((q^2/2 = \bar{u}_i \bar{u}_j)/2\); \(\bar{e}\) is determined from the model equation

\[
\frac{D\bar{e}}{Dt} = -3.8 \frac{\bar{e}}{q^3} (\bar{e} - \beta \bar{P}) + \frac{\partial}{\partial x_j} \left( K_e \frac{\partial \bar{e}}{\partial x_j} \right)
\]  

(8)

where \(\bar{P} = P_{ij}\) is the total rate of turbulence energy production, the model constant \(\beta = 3/4\), and \(K_e\) is an eddy viscosity allowing for turbulent transport of \(\bar{e}\) in an inhomogeneous turbulence field.

Equation (8) is a simplified version of the model equation used by Zeman and Lumley (1979). The present form is a standard approximation used by many researchers (see e.g. Launder et al. 1975). With respect to the present application, the major effect to be captured in Eq. (8) is a correct rate of relaxation of the turbulence time scale \(T = q^2/\bar{e}\) (or length scale \(l \propto q^3/\bar{e}\)) as the turbulent fluid elements pass through the rapidly changing mean field, over the hill. This mechanism is contained in Eq. (8). Other plausible forms of the \(\bar{e}\) equation have been suggested and tested against experiments, for example by Zeman and Lumley (1976), Launder et al. (1975), Newman et al. (1981) and others. We are fortunate that in this particular flow the results are insensitive to the precise form of the \(\bar{e}\) equation, as long as the upstream profile of \(\bar{e}(z)\) is consistent with the surface layer laws.
The pressure term $\Pi_\eta$ is known formally to have two contributions: due to turbulence self-interactions and due to the mean field–turbulence interactions. The first contribution is approximated by the so-called Rotta model:

$$
\Pi_\eta^R = -\frac{C}{T}(\overline{u_i u_j} - kq^2 \delta_{ij}) = -C\overline{b_{ij}}
$$

whose effect is to relax the turbulence to isotropy ($b_{ij} = 0$) on the time scale $T/C$ where the value of the model constant $C$ is typically in the vicinity of 3; $b_{ij} = \overline{u_i u_j}/q^2 - k\delta_{ij}$ is called a departure-from-isotropy tensor.

The second, and in our flow more important, contribution to $\Pi_\eta$ is associated with the so-called rapid distortion effect (Batchelor and Proudman 1954; Townsend 1972). In homogeneous turbulence this contribution ($\Pi_\eta^D$) has a general form (Lumley 1975)

$$
\Pi_\eta^D = \frac{\partial U_e}{\partial x_q} F_{ipq}(u_i u_k, q^2)
$$

where $F_{ipq}$ (a 4th-rank tensor) is a function of $\overline{u_i u_j}$. Gallagher et al. (1981) derived a full expansion of $F_{ipq}$ which contains, in general, 81 terms. We employ a relatively simple ‘rapid’ model of Zeman (1975) which is particularly suited for ABL simulation because it is capable of reproducing the observed distribution of Reynolds stresses (in a homogeneous ABL). This is a desirable quality of the model where the rapid distortion mechanism is important. The form of the Zeman model, described in detail in Zeman and Tennekes (1975), is reproduced in an appendix (Eqs. (A1), (A2)). We note that the contributions due to the mean strain-rate (symmetric) tensor $S_{ij}$ and the rotation (antisymmetric) tensor $R_{ij}$ are associated with two different constants $\alpha_1$ and $\alpha_2$. The so-called realizability condition (Schumann 1977; Lumley 1979), $\lim(u_3 \to 0)\{\partial/\partial t \overline{u_i u_j}\} = 0$, requires that $\alpha_1 = \alpha_2 = 0.3$. As shown later in section 5 the match of computed and observed values $\overline{u_i u_j}/u^2_* \phi^2$ is achieved by small departures of $\alpha_1$ and $\alpha_2$ from the realizable values. We note that in the isotropic limit ($b_{ij} = 0$), Eq. (A1) reduces to

$$
\Pi_\eta^D = -(q^2/5)(\partial U_i/\partial x_j + \partial U_j/\partial x_i)
$$

as inferred by Crow (1968). Additional contributions to $\Pi_\eta$ due to the curvature of streamlines are discussed later and presented in the appendix.

Concerning the transport terms $T_{ij3}$ in Eq. (7), only vertical transport is considered significant. We make a simple (scalar-gradient) approximation

$$
T_{ij3} = -K_e \partial \overline{u_i u_j}/\partial x_3
$$

where the eddy diffusivity $K_e = 0.075 Tu^3_3$ was adopted from previous modelling studies (e.g. Zeman and Lumley 1979). A similar approximation is made in the dissipation equation (8) with $K_e = K_i$ and $\partial/\partial x_i = \partial/\partial x_3$.

Now the turbulence closure model is complete and it may be adapted to the streamline coordinate system $\{\tilde{x}_i\}$ as sketched in Fig. 1. The $\{\tilde{x}_i\}$ system is locally orthogonal, curvilinear so that the inviscid momentum and conservation equations become (see e.g. Townsend 1976)

$$
\begin{align*}
D\tilde{u}_i/ Dt + \varepsilon_{i\ell j} \tilde{u}_\ell \tilde{u}_j/R + \varepsilon_{i\ell j} \tilde{u}_j f &= -\partial P/\partial \tilde{x}_i \left\{ 
\begin{array}{l}
\partial \tilde{u}_1/\partial \tilde{x}_1 + (1/R) \partial (\tilde{u}_3 R)/\partial \tilde{x}_3 = 0 
\end{array}
\right. 
\end{align*}
$$
where $R(x, z)$ is the radius of streamline curvature defined by

$$R^{-1} = -\frac{\partial^2 \hat{z}}{\partial x^2} \{1 + (\partial \hat{z} / \partial x)^2\}^{-3/2}, \quad \psi = \text{constant}$$

and $\hat{z}$ is defined in Eq. (5).

Equations (11) are valid for weakly distorting flow and discard terms of order $(h/L)^2$ or higher. This is consistent with the assumed pressure field in Eq. (1). In the forward numerical integration, the curvature at $x$ has to be calculated one step $\Delta x$ backwards (at $x - \Delta x$); however, this introduces a small error of order $\Delta x / R \ll O(\Delta x / L)$. Because the mean velocity $\hat{U}_3$ (across the streamline) is zero, the Reynolds decomposition applied to Eq. (11) is

$$\hat{U}_1 = \hat{U}_1 + \bar{u}_1, \quad \hat{U}_2 = V + \bar{u}_2, \quad \hat{U}_3 = \bar{u}_3.$$  \hspace{1cm} (12)

That also implies that since $\hat{x}_1$ is a streamline $\partial \hat{U}_3 / \partial \hat{x}_1 = 0$; however, $\partial \hat{U}_3 / \partial \hat{x}_3$ has to be retained to satisfy the mass conservation law in Eq. (11).

Because in the upstream ABL the cross-flow vertical gradient $\partial V / \partial z$ is $O(f)$ while the streamwise gradient $\partial U / \partial z$ is $O(u'_w/z')$ (in the logarithmic region) we neglect the terms containing $V$ when forming the conservation equations for $\bar{u}_i \hat{u}_i$. This substantially simplifies the model equations although for very long hills where $L \approx u'_w/f$ the cross-flow terms must be retained. Substituting Eq. (12) into Eq. (11), neglecting the curvature effect in transport terms, and dropping for convenience the hat (designating the streamline coordinate quantities) we obtain five equations needed to determine the turbulent stresses (see also Townsend 1976) and the dissipation rate $\bar{\varepsilon}$ (in the rest of the paper, unless otherwise noted, $\{x, U\}$ denotes the streamline system and $\{x, y, z; U, V, W\}$ the fixed system):

$$\begin{align}
\bar{u}_1 & = -2 \left( u^0_1 / \partial x_1 + u_1 u_3 / \partial x_3 + u_1 u_3 / R \right) / \Pi_{11} - \frac{\partial}{\partial x_3} T_{113} - \tilde{3} \bar{\varepsilon} \hspace{1cm} (13) \\
\bar{u}_3 & = 2 \left( u^0_3 / \partial x_1 + 2 u_1 u_3 / R \right) / \Pi_{33} - \frac{\partial}{\partial x_3} T_{333} - \tilde{3} \bar{\varepsilon} \hspace{1cm} (14) \\
\bar{u}_1 / \partial x_1 & = -u^0_3 / \partial x_3 + 2 \left( u^0_1 / \partial x_1 - u^0_3 / 2 \right) / \Pi_{11} - \frac{\partial}{\partial x_3} T_{113} \hspace{1cm} (15) \\
q^2 & = 2 \left( u_1 u_3 / \partial x_3 + (u^0_1 - u^0_3) / \partial x_1 / \partial x_3 - u_1 u_3 / R \right) / \Pi_{13} - \frac{\partial}{\partial x_3} T_{113} - 2 \varepsilon \hspace{1cm} (16) \\
\bar{\varepsilon} & = -3 \cdot 8(\bar{\varepsilon} - \beta P) / q^2 + \frac{\partial}{\partial x_3} \left( K_e \frac{\partial \bar{\varepsilon}}{\partial x_3} \right) \hspace{1cm} (17)
\end{align}$$

The terms retained in (13)–(16) are up to the order $u^2_w h / L^2$. Higher-order terms in $(h/L)^2$ and $(h^2/RL)$ were discarded. For a similar approach see, e.g., Gibson and Rodi (1981).

With curvature terms included in (13)–(17) it is necessary to construct a curvature-dependent contribution $\Pi^l_{ij}$ to the total pressure term $\Pi_{ij}$ which is consistent with the rapid model in Eq. (A1). This is done by analogy between the curvature and Coriolis forces. In other words, locally the curvature effect on turbulence is identical to the effect of fluctuating rotation $\Omega_2 = i(U_1 + u_1)/R$ of the coordinates around the $x_2$ axis. The expressions for $\Pi^l_{ij}$ derived in the appendix in Eq. (A6) are an extension of the general model in Eq. (A1) with the idea that the anti-symmetric tensor $R_{ij}$ in (A2) contains the
rotation terms (containing $\Omega_2$ but not the earth’s rotation terms). With the typical value $\alpha_2 = 0.3$, the model in Eq. (A6) yields individual terms which are quantitatively similar to the model of Gibson and Rodi.

The mean momentum equation (4) can now be closed by expressing the turbulent stresses on the r.h.s. of Eq. (4) in terms of the stresses in the streamline system represented in Eqs. (13)–(17). This is done by rotating the stress tensor $\bar{\tau}_{ij}$ by an angle $\alpha = \tan^{-1}(\partial \zeta/\partial \sigma)$ into the fixed system $(x, y, z)$ by transformation in Eqs. (6) and (6a). By including on the right-hand side of Eq. (4) only terms up to order $\tan \alpha = \frac{1}{4} h/L$ on the basis of small $\alpha$ ($\sin \alpha = \tan \alpha$, $\cos \alpha = 1$) and upon substituting $\bar{u} \bar{w} = \bar{u}_1 \bar{u}_3 + (\bar{u}_1^2 - \bar{u}_3^2) \tan \alpha$, $\bar{u}_z^2 = \bar{u}_1^2 - 2(\bar{u}_1 \bar{u}_3) \tan \alpha$, (4) can be reduced to

$$
\ddot{U} \left|_\psi \right. = - \frac{1}{U} \left[ \frac{\partial P}{\partial x} + (V - V_g) \right] - \frac{\partial}{\partial \psi} \left( \bar{u}_1 \bar{u}_3 - \bar{u}_3^2 \tan \alpha \right).
$$

Equation (18) is correct to the leading order in $h/L$. All other (neglected) terms can be shown to be of order $(h/L)^2$ or higher.

Because of the identity $\bar{U}_1 \partial/\partial \bar{x}_1 = U \partial/\partial x$ the coupled system of equations (13)–(18) can be solved by integrating forward in $x$ from some upstream location $-x_u$ sufficiently far from the hill. The integration proceeds along streamlines whose coordinates $(x, \zeta)$ are simultaneously calculated from the relationship (5). Thus the normal distance between two neighboring streamlines changes as the flow accelerates toward the hill top. The boundary conditions for (13)–(17) are applied at the first streamline $\psi_b$ which is at the height $\Delta z = \zeta(x, \psi_b) - \eta(x)$ above ground. Below this streamline we assume the constant stress layer relationships thus,

$$
\partial(u_0 u_0)/\partial x_3 = 0 \quad \alpha = 1, 2, 3
$$

$$
- \bar{u}_1 \bar{u}_3 = u_*^2 = U_1(x, \Delta z) \kappa [\ln(\Delta Z/z_0)]^{-1}
$$

$$
\bar{e}/\partial x_3 = -e/\Delta z, \quad \partial U_1/\partial x_3 = u_* / \kappa \Delta z
$$

where $u_*$ is the friction velocity, $\kappa = 0.4$ is the von Kármán constant and $z_0$ is the surface roughness length. The upper boundary at $\zeta(x, \psi_u) = H = \text{constant}$ is the streamline which originates upstream at height $z = H$ and coincides with the top of the ABL. The upper boundary conditions are

$$
\bar{u}_1 \bar{u}_j = \bar{e} = 0, \quad \partial U_1/\partial x_3 = \partial U/\partial z = 0, \quad V = V_g.
$$

The initial (upstream) conditions are identical with the steady (horizontally homogeneous) ABL which is initially calculated by prescribing the geostrophic departure $-(V - V_g) = u_*^2 g(z/H) H^{-1}$ where $g(z/H)$ was inferred from computations of Zeman and Tennekes (1975); the best fit approximation was found to be

$$
g(s) = 3(1 - s)^2, \quad H = 0.4 u_* f^{-1}.
$$

With this approximation we obtained a calculated velocity profile which agreed with the reference ABL data upstream of the hill.

3. PRELIMINARY ANALYSIS

It is evident that many of the hill flow effects on turbulence can be inferred from the turbulence model equations. We begin by defining the depth scale $\delta$ of the so-called ‘inner layer’ within which the turbulence (turnover) time scale $T_\delta$ is smaller than the advective time scale $T_a = LU^{-1}$. It is appropriate to define $T_\delta$ as a return-to-isotropy time
scale, i.e., \( T_i = q^2/\epsilon C \), and by equating \( T_i = T_a \) at \( z' = z - \eta = \delta \) we obtain

\[
(\delta/L) \ln(\delta/z_o) = C u_*^2/q^2 = 1/3.
\]  

(19)

This relationship for \( \delta \) is, as expected, equivalent to that of Jackson and Hunt (1975) for their inner layer depth scale \( l \). In cases of practical interest \( L/z_o \approx 10^3 \) to \( 10^5 \) so that \( \delta/L = 1/20 \). Within this inner layer the terms involving horizontal gradients \( \partial/\partial x \) and curvature \( 1/R \) are \( O(z'/\delta) \) smaller than those involving the vertical shear \( \partial U/\partial z \) and the dissipation \( \epsilon \). Hence, for sufficiently small \( z'/\delta \ll 1 \), turbulence will be in equilibrium with its current boundary conditions because \( T_i \ll T_a \) and the primary balance in a turbulence equation will involve (as in a surface layer on a flat terrain ABL) shear, dissipation and pressure terms. Hence, in this equilibrium layer, the wind profile will be logarithmic with \( u_*^2 = -\bar{u}'\bar{u}_3 \) approximately constant with height and with the normalized stress tensor components \( u_iu_j*/u_*^2 \) remaining invariant.

The existence of the equilibrium layer similarity is limited by the magnitude of the pressure gradient \( \partial P/\partial x \), and the depth of the constant stress layer is therefore quite shallow: the depth \( l \) over which a significant stress gradient exists may be estimated from \( \partial P/\partial x = \partial \tau/\partial z \), \( (z/L \ll 0) \). Evaluation of the magnitude of \( \partial \tau/\partial z \) gives a result of the order \( (u_*^2h/L)/l \), which with the above equation gives

\[
l \sim u_*^2 \frac{h}{L} \left| \frac{\partial P}{\partial x} \right|^{-1} = \frac{h}{L} L_c.
\]

The natural vertical length scale \( L_c = u_*^2 \left| \frac{\partial P}{\partial x} \right|^{-1} \) is determined by a pressure gradient signifying the strength of the perturbation, and \( u_*^2 \), the original surface stress. Thus the heights \( z' \) to which the constant stress boundary condition should be applied is limited by \( z' < (h/L)L_c \). For the Askervein Hill \( L_c \) is of the order of 1 to 2 m.

This represents a serious problem in determining the surface stress \( u_*^2 = \bar{u}_1\bar{u}_3(0) \) from measurements which are typically at elevations above 2 m, since much lower elevations are prone to serious distortion from small-scale surface irregularities.

In the flow outside the inner layer \( z' > \delta \) the turbulence time scale \( T_i \) is larger than \( T_a \) and hence the turbulence dynamics may be dominated by the rapid distortion mechanism. The necessary condition for this is that the rapid terms (of order \( u_*^2 \partial U/\partial x = u_*^2 \bar{U}_h/L^2 \)) are larger than the shear terms (of order \( u_*^2/\kappa z' \)), or that \( z'/\delta \ll L/h \). For a typical hill \( \delta = L/20 \) and \( h/L = 0.1 \) to 0.5, and the above inequality reduces to \( z'/L \geq 0.1 \) to 0.5. Since the pressure perturbation effects are known to extend to about \( z' = 2L \) we shall define an 'outer layer' \( \delta < z' < 2L \) which has a property \( T_i > T_a \) and within which rapid distortion may dominate turbulence dynamics. The mean flow in the outer layer is dominated by the Euler (non-turbulent) equations and since the vertical variation \( \partial U \) of the upstream velocity within the outer layer is small (\( \delta U/U_o \approx \{\ln(L/z_o)\}^{-1} \), the flow solution is very nearly potential as argued in theories of Jackson and Hunt (1975) and Sykes (1980).

An interesting aspect of the flow over a hill is that the curvature terms in the outer layer are of the same order as the rapid terms since \( U/R \approx -\bar{u}'' \approx \bar{U}_h/L^2 \) which is of the same order as \( \partial U/\partial x \). As shown later, the curvature is also important in the vicinity of the inner layer.

To analyse the combined effect of rapid distortion and curvature in the outer layer we write a simplified version of Eqs. (13)–(16) where inertial, shear and transport terms are neglected by the above argument; we obtain
The pressure terms $\Pi_i^P$ and $\Pi_i^C$ (defined in Eqs. (A4) and (A5) of the appendix) are a linear combination of the terms containing $\partial U / \partial x$ and $U / R$, and their major effect is to reduce the anisotropy induced by rapid distortion and curvature.

For certain initial conditions Eqs. (20) yield analytical solutions which can serve as guiding formulae for estimates of variances $\sigma^2_{\alpha} = \mathbf{u}^2_{\alpha}$ in the outer layer. With the curvature effect neglected, and with the rapid pressure term $\Pi_i^P$ in Eq. (A4), (20) can be solved analytically for small enough fractional speedup:

$$\Delta S(z') = \left\{U(x = 0, z') - U_w(z')/U_w(z') \right\} \approx h/L \ll 1. \quad (21)$$

Defining the ratio $A = (\mathbf{u}^1_{\alpha})_w$ as an initial (upstream) value we obtain the following rapid distortion estimates

$$\frac{\Delta\sigma_{\alpha}^2}{\sigma_{\alpha w}^2} = -\frac{6}{5} \left(1 - \frac{1}{3A}\right) \Delta S, \quad \frac{\Delta\sigma_{\alpha}^4}{\sigma_{\alpha w}^4} = \frac{6}{5} \left(1 - \frac{A}{3}\right) \Delta S \quad (22)$$

where $\Delta\sigma_{\alpha}^2/\sigma_{\alpha w}^2$ refers to relative change of variances of turbulent velocity ($u_\alpha$) on the hill top with respect to the upstream (reference) values at the same height above the ground, and $\Delta S$ is the fractional speedup at that height. For initial isotropic turbulence ($A = 1$), these estimates are the same as those given by Britter et al. (1981) inferred from the rapid distortion theory of Townsend (1972). Hence, the primary effect of the (irrotational) rapid distortion in the flow up to the hill top is to transfer turbulence energy from the longitudinal ($u^1_{\alpha}$) to the vertical ($u^3_{\alpha}$) component. In reality departures from isotropy in an ABL are significant ($\sigma^2_{\alpha}/\sigma^2_{\alpha} = 2.5$ to 3.5) and it is seen from Eqs. (22) that the energy transfer to $\sigma^2_{3 \alpha}$ is reduced or even reversed if $A \gg 3$. This result is not intuitively obvious but it is predicted by rapid distortion theory as well (Hunt and Newley—personal communications). The outcome of this result is that the upstream ABL anisotropy induces more energy extraction from the longitudinal component and less effective energy transfer to the vertical one. The overall effect is a larger total energy loss, as seen in Eq. (20d) and possibly a reversal of energy transfer to the vertical energy component.

The curvature effect in the outer layer is less pronounced than the rapid distortion. The first-order correction to Eqs. (22) for the curvature effect, $1/R = -\eta'(x)$, is obtained by integrating Eqs. (20a, b) with $\mathbf{u}^1_{\alpha} = -u^3_{\alpha w}$; then

$$\mathbf{u}^1_{\alpha} U^1_{\alpha} \bigg|_{x = 0} \propto u^3_{\alpha w} U^0_{\alpha 1} (h/L)^2 \quad (23)$$

and

$$\mathbf{u}^3_{\alpha} U^3_{\alpha} \bigg|_{x = -\infty} \propto 2u^3_{\alpha w} U^0_{\alpha 3} (h/L)^2$$

where $n_1 = (6/5)(1 - (1/3)A)$ and $n_3 = (6/5)(1 - A/3)$. It is seen that above the inner layer the cumulative effect of curvature is always positive and of order $(h/L)^2$ and hence smaller than the rapid distortion effect. However, the curvature effect is very important on the hill top in the vicinity of the inner layer height at $z' = \delta$ where, as seen in Fig. 3, there is negligible mean shear. There the inertial, rapid, and curvature terms are of the
same order of magnitude. The convex curvature on the hill top affects most significantly momentum transfer, i.e. the shear stress, and to a lesser degree $u'_z$. As seen in Eqs. (15) and (20c) the factor multiplying $U/R$ could be of order $10u'_w^2$. Model calculations indicate that the observed reduction of the shear stress at the hill top is mainly attributable to the convex curvature effect (discussed also in section 4).

Some quantitative estimates of the influence of finite $R$ can be made near the surface, where $\partial U/\partial z \gg U/R$. Bradshaw (1969) and Townsend (1976) invoked an analogy between the curvature and buoyancy effects (first suggested by Prandtl). Based on energy considerations Townsend inferred a curvature Richardson number such that the gradient of the square of angular momentum $UR$ is equivalent to a density gradient. The turbulence equations (13)-(16) indicate that an equally important parameter in curved flows is $T_tUR^{-1} \approx R_e \approx I_u/u'_w R$, which compares the curvature acceleration force $(Uu'_w/R)$ to the turbulence (inertial) force $(u'_w^2/l)$. It shows that the curvature effect is primarily dependent on $l/R \sim z'/R$. For small $R_e^2 \ll 1$, the eddy viscosity relationship for $\bar{u}w$ can be written as

$$\bar{u}w = -\{K_0/(1 + \alpha R_e)\} \partial U/\partial z$$

(24)

where $K_0$ is a reference eddy viscosity for $R = \infty$. This expression is analogous to those suggested, for example, by So and Mellor (1973) if, in $R_e$, $u'_w/l \sim u'_w/z'$ is replaced by $\partial U/\partial z$. The coefficient is primarily dependent on the ratio $u'_w^2/u'_z^2$ and hence on anisotropy. We should stress that Eq. (24) is realistic only for $z' \ll \delta$ and definitely not in the vicinity of $\delta$.

The concave curvature ($R < 0$) at the foot of the hill intensifies turbulence mixing as in an unstable ABL. This effect is clearly seen in the numerical simulation and is discussed in the end of section 5.

To summarize the preceding analysis: (1) We established that the thickness of an equilibrium (constant stress) layer is limited by the horizontal pressure gradient, or by the associated length scale $L_c$; and that this layer may be inaccessible to measurements. (2) Near the inner–outer layer interface (at $z' = \delta$) the Reynolds stresses are expected to be significantly affected by the curvature of streamlines, particularly near the hill top where the curvature dampens turbulence. (3) The rapid distortion is likely to dominate in the outer layer. (4) The initial (upstream) anisotropy is expected to play an important role in energy component redistribution along the hill. Concerning the numerical model it is thus important to simulate correctly the upstream-measured Reynolds stress tensor. In the following section we compare the Askervein Hill flow simulations with observations. Further analysis of data is presented in section 5.

4. THE ASKERVEIN HILL: MODELLING THE MEAN FLOW

The Askervein 1982, 1983 experiments were conducted as a combined effort by research laboratories from Canada, Denmark, Germany, New Zealand and the United Kingdom. The experimental data used in this paper are primarily from the final Askervein 1983 experiment and are published with kind permission of participating scientists from AES Canada, Risø Denmark and the Meteorological Institute of the University of Hanover, Germany. Due acknowledgments are presented at the end of this paper.

The Askervein experimental set-up and the surrounding terrain have been described in a report by Taylor and Teunissen (1983). Askervein is an oblong hill located in the Outer Hebrides. The ridge is sufficiently long to warrant the assumption of approximately two-dimensional mean wind field in the vicinity of the experimental towers when the
Figure 2. (a) The Askervein contour along the minor axis (SW–NE direction) through the hill top observation tower (solid line); the dashed line is a Gaussian curve fit in Eq. (25); for a more detailed hill description see Taylor and Teunissen (1983). (b) Measured and modelled relative speed-up \( \Delta S \) along the minor axis of Askervein at 10 m level; the data points (with error bars) are selected data for SW–NE wind direction from Walmsley and Salmon (1984).

wind blows mainly in a direction, SW–NE, normal to the ridge. The hill contour along the principal direction through the hill top observation tower is shown in Fig. 2(a) along with the best-fit Gaussian curve. The inverse polynomial function (the Witch of Agnesi) fits the hill less well because the tails of the function curve do not decay sufficiently rapidly.

Noting that Askervein is a fairly steep hill yielding \( h/L \approx 0.5 \), we have to consider carefully the application of our method to the flow in question. Previous numerical models (Taylor 1977) indicate that for a Gaussian hill, separation should occur in the lee of the hill at about \( h/L \approx 0.52 \). Hence Askervein is a marginal case; an intermittent flow reversal has been observed in the lee at Askervein but not a large-scale persistent separation.

Should large-scale separation occur, it could significantly alter the pressure distribution on the windward side. The weak intermittent separation at Askervein is not expected to affect the windward side pressure, and it is therefore justifiable to use our method to calculate the mean flow, at least up to the hill top. The ultimate test of the correct pressure distribution is the agreement of calculated and observed mean wind in the outer layer. In the inner layer the wind profile is primarily dependent on turbulence, i.e. on the turbulence closure. The best agreement between the model and data was achieved with the pressure distribution calculated from Eq. (1) with the contour \( \eta \) in (1)
prescribed as

$$\eta(x) = h \exp\{- (x/L)^2 \ln 2\}$$

(25)

where $L = 225$ m and $h = 115$ m; the effective slope, $h/L$, is 0.51.

The upstream ABL (in the absence of the hill) was computed with the roughness length $z_0 = 0.03$ m, and $u_{\text{ref}} = 0.5$ m s$^{-1}$ was chosen as a reference state (corresponding to $U_0 = U_\infty(4L) = 10$ m s$^{-1}$). When comparing with data, this is simply rescaled to the appropriate situation. The roughness length was estimated at the reference station (RS) some miles upstream from the hill (Taylor and Tenneissen 1983). As shown later the best agreement with data was achieved when $z_0$ was reduced by a factor of three within a hundred metres before the hill top. This reduction may be justified on the basis of visual inspection of the peak of Askervein.

As discussed earlier, it is expected that the modification of the turbulence field on the hill will be dependent on the upstream anisotropy. It was therefore important to match as closely as possible the observed distribution of the Reynolds stresses, $u_i u_j$, in the upstream surface layer. At the RS tower the following ratios have been observed at 10 m height (Taylor and Tenneissen 1983):

$$\frac{u^2}{u_{\text{*}}^2} = 5.9, \quad \frac{w^2}{u_{\text{*}}^2} = 1.5, \quad \frac{q^2}{u_{\text{*}}^2} = 10.6.$$  

(26)

In order to achieve an approximation to this distribution, we adjusted the model constants $\alpha_1$, $\alpha_2$ (in the rapid term model $\Pi^D_j$ (Eq. (A3)) and $\Pi^T_j$ (Eq. (A4))) away from their realizable values, $\alpha_1 = \alpha_2 = 0.3$, so that $\alpha_1 = 0.3 + \Delta \alpha$ and $\alpha_2 = 0.3 - \Delta \alpha$ with $\Delta \alpha = 0.075$. In this way we obtained the model values

$$\frac{u^2}{u_{\text{*}}^2} = 4.68, \quad \frac{w^2}{u_{\text{*}}^2} = 1.56, \quad \frac{q^2}{u_{\text{*}}^2} = 8.84$$

which have almost the same degree of anisotropy as the measured values. Another possible way to reproduce (26) is to consider $\alpha_1$, $\alpha_2$, and the return-to-isotropy constant $C$ as the unknown model constants and calculate these by the method described in Zeman and Tennekes (1975).

The geostrophic pressure gradient $(V - V_\eta)f$ in Eq. (18) was as in Eq. (18a) with the ABL top taken at $H = 2000$ m. The upstream steady (homogeneous) ABL was computed only once and used as the upstream conditions for the hill flow computations, which start at $x = -5L$, where the hill perturbation is negligible. The spacing of streamlines along which the flow is computed is distributed logarithmically upstream with the lowest computed level at $z = \Delta z = 1$ m. The forward integration of the model equations (13) to (18) proceeds along streamlines to just beyond the top of the hill. A typically used forward step length is 1 m.

The comparison of calculated and observed relative speed-up at the height $z' = z - \eta = 10$ m is presented in Fig. 2(b), and of the profiles of $U_\infty$ at the RS point and $U(0, z')$ at the hill top in Fig. 3. The agreement of calculated and observed values above the inner layer suggests that the assumed perturbation pressure gives correct acceleration along the hill.

There are some interesting features of the mean wind field. We observe in Fig. 2(b) that the reduction of the wind at the foot of the hill is fairly large, reaching a maximum of 25% at about $x = -2L$ in agreement with data. The inverse polynomial shape $\eta(x) = h(1 + (x/L)^2)^{-1}$ does not fit with the Askervein contour and the computed uphill maximum reduction is only 5%.

Visual evidence at Askervein suggests that the hill top is smoother than the rest of the hill surface, and we therefore assumed that the roughness length, $z_0$, decreased within a hundred metres of the hill top. We obtained the best agreement with data if the
reduction in $z_o$ was three-fold with respect to the upstream value, i.e.

$$
\begin{align*}
    z_o &= 0.03 \text{ m} \\
    z_{ao} &= \frac{z_o}{3} \left[ 1 + 2 \exp \left\{ - \left( \frac{x + L}{\frac{1}{3}L} \right)^2 \right\} \right] \quad \text{ for } 0 \geq x \geq -L
\end{align*}
$$

(27)

Figure 3 includes also computations with uniform roughness $z_o = 0.03$ m, and it shows that the roughness change does not affect the wind profile above $z' = 4$ m. The hill top
smoothing and the associated acceleration of the surface flow are likely to occur on a majority of grass covered hills and should be considered (Walmsley et al. 1986).

To defend our choice of roughness modulation it should be noted that if the surface roughness was in fact uniform, i.e. \( z_o = 0.03 \) m, then the estimated hill top stress \( \bar{u} \bar{w} \) based on the measured wind at 1 m, would have to be larger than \( 5u_{+w} \). Evidently, according to Fig. 4, such a value is unrealistic.

An important parameter of the flow is the depth \( \delta_m \) of the inner layer defined here as the elevation of the maximum speed-up (\( \Delta \bar{U}_{max} \)); it roughly coincides with the height of the minimum \( \bar{u} \bar{w} \) (see Fig. 4). As shown in Fig. 3, \( \delta_m \approx 5 \) m which is \( 1/3 \) of the depth (l) predicted by the Jackson–Hunt formula as in Eq. (19). Numerical experiments with different slopes \( h/L \) (but constant \( L \) and \( z_o \)) suggest an interesting variation of \( \delta_m \) with \( h/L \). For \( h/L > 0.25 \), \( \delta_m \) is constant, however \( \delta_m \) triples as \( h/L \) changes from 0.2 to 0.05. This suggests that the compression of streamlines (proportional to \( h/L \)) on the hill top is instrumental in reducing \( \delta_m \) with respect to the value in (19) which is valid for asymptotically small slopes only.

The good agreement between computed and measured mean flow fields assures us that the mean deformation tensor \( \partial \bar{U}/\partial x_i \) (the input to the turbulent stress equations) is correctly predicted. An area of uncertainty is in the correct specification of boundary conditions near the surface at the first streamline where \( z^\prime = \Delta z = 10z_o \). At these heights the logarithmic law for \( U/u_\infty \) is in principle no longer valid, although the assumption of a constant stress region is likely to be satisfied up to \( L_c = -(\partial P/\partial x)^{-1} u_\infty \), the depth of an equilibrium layer (discussed in the preceding section). The Askervein parameters yield \( L_c = 1 \) m. This is in agreement with computed stress profiles in Figs. 4–6, which indicate the existence of the constant stress layer below \( z^\prime = L_c \).

5. THE ASKERVEIN TURBULENCE COMPARISON

The Askervein turbulence data shown in this paper were obtained from the following instruments: two sonic anemometers placed at the upstream reference site, one at 50 m (Kajio-Denki DAT-300, owned by AES) and one at 10 m; and four placed along another 50 m tower on the hill top (all five units of the older PAT-311 type, owned by Risø); and in addition three propeller anemometer arrays (Gill) placed along a 16 m mast situated on the hill crest but at some distance from the hill top tower. These latter anemometers were run by the University of Hanover.

The reference site instruments were recorded by AES on a computerized system. The hill top instruments were recorded on two 8-channel HP-3968A FM tape recorders. These and the instruments were powered by a portable generator. The analogue tapes from this system were later digitized to a Nyquist frequency of 50 Hz and the time series of the raw sonic signals transformed into series of \( u, v, w \). This transformation is done in two steps. First the measured components were transferred into Cartesian components (using accurate measurements of the actual geometry of the individual probe heads). In this step corrections for the dependence on air temperature and humidity were also made. Then the average values of the resulting Cartesian velocity components were computed providing the input for the last transformation where the coordinate system is tilted and turned to eliminate \( W \) and \( V \). Time series prepared in this manner are then available for computation of turbulence statistics. No detrending was applied.

Regarding other procedures and details about the various hardware systems, we refer to the data reports (Taylor and Teunissen 1983, 1984). We should mention that the results we use here are only a small fraction of what will become available from these experiments. The hill top sonic anemometer data presented here were recorded on 3
Figure 4. This and figures 5 and 6 present a comparison of measured and modelled turbulence statistics for the mean wind in Fig. 3, although the averaging time was 2.5 hours instead of 1 hour (from 1430 to 1700). This figure shows vertical profiles of shear stress. The dash–dotted line is the modelled upstream distribution and the heavy solid line is the modelled distribution above the hill top. The dashed line represents a model run (for the hill top) with no curvature terms in the turbulence equations, everything else being unchanged.

October 1983 between 1430 and 1700 h (British summer time). The model prediction of turbulent stresses and data points are presented in Figs. 4, 5 and 6. We use the fixed system notation \( (x, y, z) \) which on the hill top is virtually identical with the streamline system \( (x_i) \).

Considering the complexity of the turbulence processes in flow over a hill the model predicts the turbulent stresses on the hill top quite well. This is true, in particular, for the shear stress \( \overline{uw} \) shown in Fig. 4. There is evidently no observed region of constant stress above the height \( L_e \approx 1 \text{ m} \) and it is not possible to determine \( u_* \) from the data. The large stress gradient above 1 m is due to rapid decrease of the mean shear with height, but apart from that it is the curvature effect which attenuates both \( \overline{uw} \) and \( w^2 \) on
the hill top. Deletion of curvature terms in the equations gives the stress profile shown by a dashed line and obviously leads to poor prediction. An interesting aspect of the data is a slight and perhaps not quite significant overshoot of \( \overline{uw} \) at \( z' = 50 \text{ m} \). It follows the similar overshoot in \( w^2 \) in Fig. 5 which is attributed to the rapid distortion and cumulative curvature effects. The dip in the stress at a height \( \delta_m \) (\( \delta_m \) identifies the maximum speed-up) is a rather remarkable aspect of the dynamics. It could not be predicted by either eddy viscosity (numerical or theoretical) model. Sykes's results (1980) for flow over sinusoidal topography show similar effects.

In Figs. 5 and 6 the vertical and horizontal r.m.s. fluctuating velocities \( \sigma_1 = \sigma_u \), \( \sigma_3 = \sigma_w \) are shown normalized by the upstream values at 10 m. Although the model predictions follow a trend similar to the observations, the model is not capable of predicting well the magnitudes of the reduction in \( \sigma_u \) at 4 m. This drawback, we believe, is primarily connected with the rather imperfect predictive equation for the dissipation \( \overline{\epsilon} \) and hence for the turbulence length scale. Both the rapid distortion and curvature reduce the length scale of large eddies (in the vertical), and this effect is not properly
represented in the $\bar{e}$ equation. The behaviour of the observed $\sigma_w$ can be explained as follows: the minimum in $\sigma_w$ at $z' = \delta_m$ is associated with the attenuating effect of the convex curvature and, to a lesser degree, with the disappearance of the mean shear which maintains $\sigma_w^2$ upstream. The overshoot in $\sigma_w$ at $z' \approx 50$ m is apparently not a result of the rapid distortion. Referring to the estimates in Eqs. (22) and (23) it is obvious that rapid distortion in the $w^2$ component is ineffective due to large anisotropy in the ABL. The computed profiles with and without curvature effects support this notion. It is primarily the $u^2$ component which is affected by rapid distortion, as seen in Fig. 6.

The data indicate that $\sigma_u$ is attenuated at higher levels than $\sigma_w$; the minimum of $\sigma_u$ occurs at 10 to 15 m compared with 4 m in the case of $\sigma_w$. The longitudinal fluctuations are more affected by the absence of shear than the vertical ones. In fact, above 10 m the mean shear changes sign, thus locally a counter-gradient momentum transfer occurs which in turn necessitates extraction of turbulence energy from the longitudinal component. This leads to the reduction of $\sigma_u$ above 6 m. The effect of curvature on $\sigma_u$ is small although
some energy, in convexly curved flow, is transferred from $\sigma_v^2$ to $\sigma_u^2$ (see Eqs. (13) and (14)). The overall decrease of total energy, $q^2$, more than compensates for this transfer.

The rapid distortion adds further to the reduction of $\sigma_u$ above 20 m (the region where the turbulence time scale $T_i > L/U_0$), with the aid of Eq. (22) we estimate $\sigma_u/\sigma_{uo}$ at 50 m as 0.8 which agrees with the model prediction for $R = \infty$. However, the cumulative contribution of curvature as suggested by Eq. (23) and valid for $T_i > L/U_0$ (or $z/\delta > L/h$) moves this estimate towards the observation point (Fig. 6). In conclusion, the overall agreement between the data and model predictions suggests that the fundamental physical mechanisms that alter the turbulence field in the flow over hills are captured by the model equations. Our technique allows for the separation of the curvature, (irrotational) rapid distortion, and shear effects in the streamline coordinate system and as such it has a great advantage over the techniques which employ the generalized coordinate transformation for solving turbulence. In generalized coordinates, the turbulence physics are obscured by the plethora of terms which do not usually have a distinct physical meaning.

The treatment of turbulence and mean field by the present method has of course its limitations, i.e. it is suited only to gentle two-dimensional hills without separation. Finnigan (1983) has proposed a method to form a general coordinate system using streamlines. His approach is not limited to small slopes; however, the turbulence equations for quantities such as $\bar{u}_1\bar{u}_3$ have to be formulated in the generalized coordinates and the number of terms that such systems of equations contain could be forbidding.

The last point to be mentioned concerns the possible horizontal inhomogeneities upstream of the Askervein Hill due to the changes in roughness from sea to land, changes in stability, lateral flow divergence, etc. These could not be estimated with any confidence and were not considered in the numerical simulations. At the RS tower, the observations of turbulence variations exhibited anomalously large differences between the two observation levels at 10 and 50 m. The low values of $\sigma_v^2$ and $\sigma_u^2$ at 50 m (by 30 to 40%) have been attributed to the remnants of the upstream marine boundary layer advected from the nearby coast which has not been fully replaced at the RS tower by the (internal) boundary layer characteristics of greater roughness (the coastline is located at about 3.5 km and the RS tower at 2 km upstream of the hill when the wind is in the principal wind sector 220°). We assumed that immediately upstream of the hill, the internal layer has grown deep enough so that the anomalously low 50 m data at the RS tower could be safely ignored.

The response of turbulence to the flow distortion along the windward side of the hill is displayed in Figs. 7-9. In this case the calculations were performed with the ‘Witch of Agnesi’ shape, $\eta(x) = h(1 + (x/L)^2)^{-1}$; the best fit to the Askervein contour gives $L = 250$ m, $h/L = 0.45$. A uniform roughness, $z_0 = 0.03$ m, was used in these calculations. Figures 7-9 depict vertical profiles of $\sigma_3$, $\sigma_1$ and $\bar{u}_1\bar{u}_3$ normalized by their respective 10 m-level upstream values, at five locations along the hill. The depicted quantities are in the frame of reference aligned with the streamlines, i.e. with the local wind vector. The hill top profiles are almost identical to those shown in Figs. 4-6 suggesting that turbulence response is fairly independent of the details of the hill shape and pressure distribution. The significant alteration of the profiles, particularly of $\sigma_3$ and $\bar{u}_1\bar{u}_3$, as the flow advances along the hill indicates interesting dynamical effects. The marked variation with $x$ of $\sigma_3$ and $\bar{u}_1\bar{u}_3$ at heights between $\delta_m$ and $2\delta_m$ is attributed to the change of the streamline curvature from concave (unstable) to convex (stable). The inflection point of $\eta(x)$ (where curvature changes sign) is at $x/L = -0.8$ and this is also the point where $\sigma_3$ and $\bar{u}_1\bar{u}_3$ reach their maxima (with values slightly larger than those shown in Figs. 7 and 9 at $x/L = -1$).
Figure 7. Model predictions of $\sigma_3 = (\bar{u}^2)^{1/2}$, the standard deviation of velocity fluctuations normal to streamlines normalized by the upstream surface value $\sigma_u$ at 10 m; computed at a few locations along a hill $\eta(x) = h[1 + (x/L)^2]^{-1}$ with $h/L = 0.45$, $L = 250$ m and $z_0 = 0.03$ m. The thin line represents the upstream profile ($x = -\infty$), dashed line at $x = -3L$, dotted line at $x = -2L$, dash-dotted line at $x = -L$, and solid line at the hill top $x = 0$.

6. OTHER EXPERIMENTAL RESULTS

It is appropriate to comment on other observational studies in comparison with the presented experimental and numerical results. One of the first experiments in natural hill flow was conducted by Bradley (1980). As mentioned earlier, data were taken on the summit of a wooded hill (Black Mountain) with an extreme estimated roughness length $z_0 = 1$ m, $L = 275$ m and $h/L = 0.62$. The Black Mountain data exhibit different behaviour from those in Figs. 4 to 6. No significant reduction below the upstream value was observed in $\sigma_u$ or $\sigma_v$. The composite profiles of $\sigma_u/\sigma_{uo}$, $\sigma_v/\sigma_{vo}$ showed a monotonic decrease from the surface values (2.0, 1.6, respectively) to unity. The rapid distortion effect was only weakly evident, perhaps because the observation was limited to heights below 60 m, which is about 3 times in that case. A small reduction in $\bar{u}w$ of about 20% below the upstream value was observed.

It is difficult to speculate why the Black Mountain data are different from those at Askervein and from the model predictions. There is, of course, the problem of extreme roughness associated with the wooded hill. Flow displacement and anomalous generation of turbulence due to the trees induce unknown perturbations both in the mean and turbulence fields; the large slope ($h/L = 0.62$) probably causes flow separation fairly near the summit. This probably reduces streamline curvature and, in turn, intensifies
Figure 8. As Fig. 7, but for velocity fluctuations ($\sigma_v$) in the streamline direction.

turbulence mixing compared with a non-separating flow. There is indirect evidence for the above argument in Bradley's data on the mean flow: the mean flow angle at the top was measured to be $10^\circ$ (at $z' \approx 50$ m), hence the flow streamlines could not follow the hill contour.

There are a few wind tunnel experiments which also furnish turbulence measurements. The model hill used by Britter et al. (1981) had $h = 0.1$ m and $L = 0.25$ m, i.e. $h/L = 0.4$, and the shape was the inverse polynomial. Thus we can directly compare their results with the calculations discussed in the end of section 5; however, only longitudinal velocity fluctuations were measured (single hot-wire). The normalization in their Fig. 4(b) is different from the one used here as they plot $\Delta \sigma_u^2 / \sigma_{u0}^2$, but the agreement with our Fig. 8 can be deduced to be quite good. Their data points are somewhat scattered, but close to the surface they observe this quantity to be $\sim -0.3$. This gives $\sigma_u / \sigma_{u0} \sim 0.84$, compared with 0.82 in our calculations (the minimum in the hill top profile in Fig. 8). At $z/L = 1$, our calculations show a difference in $\sigma_u / \sigma_{u0}$ from upstream to hill top of about $-0.04$, which renormalized to the upstream value of $\sigma_u$ at that level ($\sigma_{u0}$) corresponds to about $-0.05$, or $\Delta \sigma_u^2 / \sigma_{u0}^2 = -0.1$, which again is quite close to an interpolated value of the Britter et al. results. At higher levels the Britter et al. data seem to decay towards the undisturbed value at a slower rate than in our case, but that is again caused by the difference in normalization. Observations in the inner layer could not be obtained as it was too shallow compared with the height that simulation requirements forced their roughness elements to be.
In another wind tunnel study by Tuenissen et al. (1982) of flow over a model of the Kettles Hill (Walmsley et al. 1982) all three turbulence components were measured. In the inner layer (an aerodynamically smooth model was used) turbulence levels were observed to increase as expected. In the outer layer, however, all components were observed to decrease relative to upstream undisturbed values. For the $u$ component this is in agreement with rapid distortion; for the $v$ and $w$ components this trend could be explained by our analysis in Eq. (22) assuming that the wind tunnel turbulence was sufficiently anisotropic.

Much better agreement with the present results is found in the recent full-scale observations reported by Mason and King (1985). Their hill, Blashaval, is much like Askervein in its size and surface roughness (as well as general geographical location) although its plan-form is more circular. Thus their stress profile observations on the hill top show a minimum at a height of $\sim 5m$ with a value of about 1/3 of the upstream (compare with Fig. 4). The reduction in $\sigma_u$ is slight but of order 10% at the 5 and 15 m levels. This is quite in agreement with Fig. 6. It would have been interesting had they had an observation level between these two levels where we would expect with present knowledge a minimum to occur. Another agreement with Askervein (as well as with the Black Mountain) observations is the quite large overshoot in $\sigma_w$ at the higher levels (qualitatively predicted by our model). To point to a difference, the reduction in $\sigma_w$ at the $\delta_m$ level observed and modelled at Askervein, is not present in the observations by
Mason and King although their $\sigma_w$ profile shows a distinct relative minimum at that height. An explanation of this difference could be in the plan-form of the two hills: Blashavel is clearly a round three-dimensional hill where the streamline curvature at heights near $\delta_m$ is smaller than at the two-dimensional Askervein.

7. Conclusions and future work

We have developed a relatively simple model for turbulent flow over two-dimensional hills, where turbulence (second-order) equations are solved in a streamline coordinate system. The numerical solution of the equations requires little computer time (for a typical run we used about 10 seconds of CPU on Burroughs 7800, or 3 seconds on Cyber 175). Good agreement has been obtained between the model predictions and a set of measurements from the Askervein Hill project. Our major findings are:

(a) The height $\delta_m$ where the absolute speed-up is maximum on the hill top is about 1/3 of the inner layer depth as defined by the Jackson–Hunt formula.

(b) An analysis of data and model results shows that there are three distinguishable layers on the hilltop:

1. close to the ground (below $z' = \frac{1}{2}\delta_m$) magnitudes of all non-zero stress components are increased relative to upstream values in accordance with the increased wind speed and thus shear. This enhancement is strongly height dependent and disappears at a height of around $\delta_m$.

2. In the vicinity of $\delta_m$ the values of all stress components reach a minimum although the observed cross-stream component $\overline{u'z'}$ does not. The model calculations show that this is a consequence of streamline curvature.

3. At even higher levels, between about 4$\delta_m$ and $L$, rapid distortion causes yet another change in $w'^2 = u'_3$ and $\overline{u'^2} = \overline{u'^2_i}$. In the case of the $w$ components the combined (contrary) effects of rapid distortion and curvature result in a peculiar vertical profile as shown in Figs. 5 and 7.

(c) The result of rapid distortion of, especially, the vertical component is highly dependent on upstream anisotropy.

(d) Turbulence calculations indicate that the streamline curvature is also instrumental in modifying turbulence upstream of the hill top. As shown in Figs. 5 and 9, both the shear stress $\overline{u'u'_2}$ and the vertical variance $u'_3$ increase significantly in the vicinity of $x = -L$ due to the cumulative effect of concave streamline curvature. It remains to be seen whether this effect is real. Preliminary data from Askervein suggest that this is so.

Preliminary results with different parameter sets $h$, $L$ and $z_0$, indicate that the depth $\delta_m$, the maximum relative speed-up $\Delta S_m$, and extrema in the turbulence stress profiles (on the hill top) can be parametrized in terms of the basic parameters $h/L$ and $L/z_0$, at least for a family of similar hill shapes. Our future work will be directed towards the generalization of the present results to hills of different shape and size and to parametrization of turbulence in hill flow.

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The general form of the rapid pressure term model (Zeman and Tennekes 1975) is

\[
\Pi^D_{ij} = \left\{ \frac{\partial p}{\partial x_i} u_j + \frac{\partial p}{\partial x_j} u_i - \frac{2}{3} \frac{\partial p}{\partial x_1} u_1 \delta_{ij} \right\}_D = -2q^2(\delta_{ij} S_{ij} + \alpha_1 (S_{ik} b_{kj} + S_{jk} b_{ki} - \frac{5}{9} S_{ik} b_{1k} \delta_{ij}) + \alpha_2 (R_{ik} b_{kj} + R_{jk} b_{ki})).
\] (A1)

Here \( b_{ij} = \overline{u_i u_j / q^2 - \frac{1}{3} \delta_{ij} S_{ij}} \) and \( R_{ij} \) are symmetric and anti-symmetric deformation tensors defined as

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad R_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right) - 2\epsilon_{ijk} \Omega_k
\] (A2)

where \( \Omega_k \) is the angular velocity of the frame of reference.

In two-dimensional flow with \( S = \partial U_1 / \partial x_3, D = \partial U_1 / \partial x_1 - \partial U_3 / \partial x_3, U_2 = 0 \) and \( \partial U_3 / \partial x_1 \) neglected, (A1) becomes

\[
\begin{align*}
\Pi^D_{11} &= -q^2 \left\{ \frac{5}{6} \alpha_1 + 2\alpha_2 \right\} S_{13} + \left\{ \frac{5}{6} \alpha_1 (b_{11} + \frac{1}{9} b_{33}) + \frac{2}{9} \right\} D \\
\Pi^D_{12} &= -q^2 \left\{ -\frac{5}{6} \alpha_1 S_{13} - \frac{5}{6} \alpha_1 (b_{11} - b_{33}) D \right\} \\
\Pi^D_{33} &= -\Pi^D_{11} - \Pi^D_{22} \\
\Pi^D_{31} &= -q^2 \left\{ \frac{5}{6} + b_{11} (\alpha_1 - \alpha_2) + b_{33} (\alpha_1 + \alpha_2) \right\} S
\end{align*}
\] (A3)

To satisfy realizability for \( \overline{u_1 u_3} \), \( \alpha_1 = \alpha_2 = 0.3 \), and (A3) becomes

\[
\begin{align*}
\Pi^D_{11} &= -\frac{5}{6} \left\{ \overline{2u_1 u_3 S} + (2u_1^2 + u_3^2) D \right\} \\
\Pi^D_{33} &= +\frac{5}{6} \left\{ \overline{u_1 u_3 S} + (2u_3^2 + u_1^2) D \right\} \\
\Pi^D_{31} &= -\Pi^D_{11} - \Pi^D_{33} \\
\Pi^D_{31} &= -\frac{5}{6} \overline{u_3^2 S}
\end{align*}
\] (A4)

The contribution \( \Pi^c_{ij} \) due to the streamline curvature to \( \Pi_{ij} \) rests on the analogy between the curvature and Coriolis forcing. By equating \( \frac{3}{4} (U_1 + u_1') R \) to fluctuating rotation \( \Omega_2 + \omega_2 \) in the \( x_2 \) direction, the virtual Coriolis terms in Eqs. (13)-(16) can be written as

\[
D\overline{u_1 u_j} / Dt \propto \varepsilon_{pq} (\Omega_p \overline{u_q u_j} + U_q \overline{u_p u_j}) + \varepsilon_{jq} (\Omega_q \overline{u_p u_j} + U_q \overline{u_p u_j}) - \frac{5}{9} \varepsilon_{pq} \overline{U_q u_p u_j} \delta_{ij}.
\]

Then a tensorially invariant model \( \Pi^c_{ij} \) is formed by associating the rotation tensor \( R_{ij} \) with \( \varepsilon_{jk} \Omega_k \) and with additional terms due to correlations \( \overline{u_i u_j} = \frac{1}{3} \overline{u_1 u_3} \delta_{ij} R^{-1} \). The general form of \( \Pi^c_{ij} \) is then

\[
\Pi^c_{ij} = -4 \alpha_2 (\varepsilon_{pq} (\Omega_p \overline{u_q u_j} + U_q \overline{u_p u_j}) + \varepsilon_{jq} (\Omega_q \overline{u_p u_j} + U_q \overline{u_p u_j}) - \frac{5}{9} \varepsilon_{pq} \overline{U_q u_p u_j} \delta_{ij}).
\]

In two-dimensional flows with \( \Omega_p = \frac{3}{4} U_1 R^{-1} \delta_{ij} \) we obtain

\[
\begin{align*}
\Pi^c_{11} &= -(16/3) \alpha_2 \left\{ (U_1 / R) \overline{u_1 u_3} \right\} \\
\Pi^c_{32} &= -\Pi^c_{11} - \Pi^c_{33} \\
\Pi^c_{33} &= + (20/3) \alpha_2 \left\{ (U_1 / R) \overline{u_1 u_3} \right\} \\
\Pi^c_{31} &= 2 \alpha_2 (u_1^2 - \frac{1}{9} u_3^2) U_1 / R.
\end{align*}
\] (A5)
The return-to-isotropy pressure term $\Pi_R^b$ is modelled as

$$\Pi_R^b = \frac{(C/T)b_{ij}q^2}{Ceb_{ij}}$$

(A6)

with $C = 3.25$.

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