An efficient two-time-level semi-Lagrangian semi-implicit integration scheme

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SUMMARY

The semi-implicit semi-Lagrangian integration technique enables numerical weather prediction models to be run with much longer timesteps than permitted by a semi-implicit Eulerian scheme. The choice of timestep can then be made on the basis of accuracy rather than stability requirements. To realize the full potential of the technique, it is important to maintain second-order accuracy in time; this has previously been achieved by applying it in the context of a three-time-level integration scheme. In this paper we present a two-time-level version of the technique which yields the same level of accuracy for half the computational effort. Unlike other efficient two-time-level schemes, ours does not rely on operator splitting.

We apply this scheme to a variable-resolution barotropic finite-element regional model with a minimum gridlength of 100 km, using timesteps of up to three hours. The results are verified against a control run with uniformly high resolution, and are shown to be of similar accuracy to those of a semi-implicit Eulerian integration with a timestep of 10 minutes.

1. INTRODUCTION

The semi-Lagrangian technique has recently been used to permit stable integrations of the meteorological equations with timesteps longer than those permitted by models having an Eulerian treatment of advection. It has been successfully coupled with the semi-implicit method (Robert 1981, 1982; Robert et al. 1985; Ritchie 1986; McDonald 1986; Staniforth and Temperton 1986), the split-explicit method (Bates and McDonald 1982) and the alternating-direction-implicit method (Bates 1984).

Present semi-Lagrangian schemes are based on discretization over either two time levels (Bates and McDonald 1982; Bates 1984; McDonald 1986) or three (Robert 1981, 1982; Robert et al. 1985; Ritchie 1986; Staniforth and Temperton 1986). Two-time-level schemes have, in principle, several virtues; they are simpler to code, make fewer demands on computer memory and have no time computational modes. Furthermore they are potentially twice as efficient as three-time-level leapfrog-based schemes. This is because three-time-level schemes require timesteps half the size of two-time-level ones in order to achieve the same level of time truncation error (which is governed by the interval of time over which time differences are taken; i.e. over an interval $\Delta t$ in the case of a two-time-level scheme, but $2\Delta t$ in the case of a three-time-level one).

In practice the advantages of two-time-level schemes over three-time-level ones have not yet been fully realized. For example, Bates and McDonald (1982) were limited in their integrations with a two-time-level model to a timestep of 30 min because of accuracy considerations, whereas Robert et al. (1985) with a three-time-level model were able to integrate at comparable resolution with a larger timestep of 90 min with no loss of accuracy over the central part of their limited area. It was argued in Staniforth and Pudykiewicz (1985) that this behaviour could be attributed to the use by Bates and McDonald of an advection method having first-order time truncation error, rather than one having second-order accuracy, as in Robert et al. The two-time-level schemes of Bates (1984) and McDonald (1986) also share this limitation, although they all permit significantly longer timesteps than comparable schemes having an Eulerian treatment of advection. We further expand on this point regarding the time truncation errors in section 5.
To conclude this brief review of two-time-level schemes, we note that Cohn et al. (1985) and Augenbaum et al. (1985) have formulated efficient two-time-level schemes based on operator splitting. (We use this term to mean the approximate factorization of a two-dimensional horizontal operator as the product of two one-dimensional operators.) However, their accuracy with large timesteps remains to be demonstrated in practice. In fact Yakimiw and Robert (1986) and Tanguay and Robert (1986) have recently demonstrated that operator splitting, in two different contexts, leads to unacceptable time truncation errors when used with large timesteps.

In a recent paper (Staniforth and Temperton 1986) we applied the semi-implicit semi-Lagrangian integration technique to a three-time-level finite-element barotropic model, using the shallow-water equations on a variable-resolution mesh. We were able to integrate the model stably with timesteps at least six times longer than the limiting timestep of the corresponding Eulerian scheme. After 48 hours of integration the r.m.s. height and wind differences over the high-resolution central area were shown to be of the order of 5 m and 1 m s$^{-1}$ respectively, compared with an Eulerian control integration run with a small timestep and uniform high resolution over the whole domain. We concluded that the semi-implicit semi-Lagrangian scheme is a promising technique for the efficient and accurate integration of finite-element models as well as for finite-difference models.

The aim of the present study is to present an (almost) unconditionally stable, accurate two-time-level semi-Lagrangian semi-implicit scheme which is twice as efficient (for comparable accuracy) as our previous three-time-level one. To achieve this we pay particular attention to maintaining the second-order time truncation accuracy of our previous scheme. We believe that the key idea of the present study, an improved method for computing the trajectories in the context of two-time-level schemes, may be easily applied to existing two-time-level semi-Lagrangian models, such as those of Bates and McDonald. (After this paper was submitted, we learned that Bates and McDonald (1987a) had independently proposed method 2 of section 3.) This should result in improved accuracy or, equivalently, should permit larger timesteps with no loss of accuracy.

2. DISCRETIZING THE GOVERNING EQUATIONS

(a) Governing equations

The governing equations are the shallow-water equations over a rotating sphere, using Cartesian coordinates on a polar stereographic projection, true at 60°N. They are:

$$\frac{dU}{dt} + \Phi_x - fV = -\frac{1}{2}S_x(U^2 + V^2)$$

(1)

$$\frac{dV}{dt} + \Phi_y + fU = -\frac{1}{2}S_y(U^2 + V^2)$$

(2)

$$d(\ln \Phi)/dt + D = 0$$

(3)

where

$$D = S(U_x + V_y)$$

(4)

$$\frac{d}{dt} = \partial/\partial t + S(U \partial/\partial x + V \partial/\partial y)$$

(5)

$$U = u/m, \ V = v/m \text{ and } S = m^2.$$
the free surface, \( m \) is the map-scale factor, \( f \) is the Coriolis parameter and \( U \) and \( V \) are termed the wind images.

We define a domain \( \Omega \), encompassed by a solid rectangular wall \( \Gamma \). This wall is placed in the vicinity of the projection of the equator as in Staniforth and Mitchell (1977). Decomposing the wind images in terms of a velocity potential \( \chi \) and a streamfunction \( \psi \), we have:

\[
U = \chi_y - \psi_x, \quad V = \chi_x + \psi_y. \tag{6}
\]

With these definitions we have the diagnostic relations

\[
\chi_{xx} + \chi_{yy} = U_x + V_y = D/S, \tag{8}
\]

\[
\psi_{xx} + \psi_{yy} = V_x - U_y = \xi/S, \tag{9}
\]

where \( \xi \) is the relative vorticity. The appropriate boundary conditions on \( \chi \) and \( \psi \) to ensure no cross-boundary flow are thus

\[
\chi_n = 0 \quad \text{on } \Gamma, \tag{10}
\]

and

\[
\psi = 0 \quad \text{on } \Gamma, \tag{11}
\]

where subscript \( n \) denotes differentiation in the direction of the outward pointing normal vector \( \mathbf{n} \) at the boundary.

(b) Derivation of the elliptic equation for \( \Phi \)

A semi-Lagrangian semi-implicit discretization of Eqs. (1)–(3) is

\[
\frac{U(x,t+\Delta t) - U(x-\alpha,t)}{\Delta t} + \frac{1}{2}[(\Phi_x)|_{(x,t+\Delta t)} + (\Phi_x)|_{(x-\alpha,t)}] - \frac{1}{2}[(fV)|_{(x,t+\Delta t)} + (fV)|_{(x-\alpha,t)}] = -\left[\frac{1}{2}S_x(U^2 + V^2)\right]|_{(x-\alpha,t+\Delta t)} \tag{12}
\]

\[
\frac{V(x,t+\Delta t) - V(x-\alpha,t)}{\Delta t} + \frac{1}{2}[(\Phi_y)|_{(x,t+\Delta t)} + (\Phi_y)|_{(x-\alpha,t)}] + \frac{1}{2}[(fU)|_{(x,t+\Delta t)} + (fU)|_{(x-\alpha,t)}] = -\left[\frac{1}{2}S_y(U^2 + V^2)\right]|_{(x-\alpha,t+\Delta t)} \tag{13}
\]

\[
\frac{(\ln \Phi)|_{(x,t+\Delta t)} - (\ln \Phi)|_{(x-\alpha,t)}}{\Delta t} + \frac{1}{2}[D(x,t+\Delta t) + D(x-\alpha,t)] = 0 \tag{14}
\]

where

\[
\alpha = x - x_f. \tag{15}
\]

Here \( x_f \) is the initial position vector of a particle that arrives at \( x \) after a time interval \( \Delta t \);
\( x = (x, y) \) is a mesh-point of a Cartesian (but not necessarily uniform) mesh (see Fig. 1 of Staniforth and Temperton (1986) for an example).

We have used centred time differences and averages along the straight line trajectory from \( x_f \) to \( x \). All terms, except the metric terms on the right-hand side of Eqs. (12) and (13), are treated implicitly in time. In particular, the Coriolis terms are treated implicitly since an explicit treatment of these terms in the context of a two-time-level scheme (but
not a three-time-level one) would be computationally unstable. The winds in the metric terms are computed by extrapolating those at the two or three preceding time levels. Product terms and derivatives are computed by an appropriate numerical method; we use bilinear finite elements as in Staniforth and Temperton (1986).

Quantities at the points \((x - a)\) and \((x - b)\) are evaluated by a suitable interpolation formula (Robert 1981, 1982; Bates and McDonald 1982; McDonald 1984). In the present study we have used bicubic spline interpolation (Purnell 1976; Pudykiewicz and Staniforth 1984; Staniforth and Temperton 1986). The displacements are determined by approximate integration of

\[
\frac{dx}{dt} = V = (SU, SV)\]  

(16)

over the period \([t, t + \Delta t]\); details are given in section 3.

Algebraic manipulation of Eqs. (12) and (13) yields

\[
[U + (a\Delta t/2)\Phi_x + (b\Delta t/2)\Phi_y]_{(x, t + \Delta t)} = ar_1 + br_2
\]

(17)

\[
[V + (a\Delta t/2)\Phi_y - (b\Delta t/2)\Phi_x]_{(x, t + \Delta t)} = ar_2 - br_1
\]

(18)

where

\[
r_1(x) = [U + (f\Delta t/2) - (\Delta t/2)S_x]_{(x - a, t + \Delta t)} - (\Delta t/2)[S_x(U^2 + V^2)]_{(x - a, t + \Delta t)}
\]

(19)

\[
r_2(x) = [V - (f\Delta t/2) - (\Delta t/2)S_y]_{(x - a, t + \Delta t)} - (\Delta t/2)[S_y(U^2 + V^2)]_{(x - a, t + \Delta t)}
\]

(20)

\[
a(x) = 1/[1 + (f\Delta t/2)^2]
\]

(21)

\[
b(x) = (f\Delta t/2)a.
\]

(22)

Adding the \(x\) derivative of Eq. (17) to the \(y\) derivative of (18) gives the following relation between \(D/S\) and \(\Phi\) at time level \((t + \Delta t)\):

\[
(D/S)_{(x, t + \Delta t)} = (\chi_{xx} + \chi_{yy})_{(x, t + \Delta t)}
\]

\[
= \frac{1}{2}\Delta t[(a\Phi_x)_x + (a\Phi_y)_y + (b\Phi_y)_x - (b\Phi_x)_y]_{(x, t + \Delta t)} + (r_3)_x + (r_4)_y
\]

(23)

where

\[
r_3 = ar_1 + br_2
\]

(24)

\[
r_4 = ar_2 - br_1.
\]

(25)

Dividing Eq. (14) by \(S\) and eliminating \((D/S)_{(x, t + \Delta t)}\) from (23) yields the elliptic equation

\[
[(a\Phi_x)_x + (a\Phi_y)_y + (b\Phi_y)_x - (b\Phi_x)_y - 4(\ln(\Phi))/(S\Delta t^2)]_{(x, t + \Delta t)} = (2/\Delta t)[(r_3)_x + (r_4)_y] - 4r_5/(S\Delta t^2)
\]

(26)

where

\[
r_5 = [\ln(\Phi) - (\Delta t/2)D]_{(x - a, 0)}.
\]

(27)

The boundary conditions for Eq. (26) are obtained from (17) and (18) by setting the normal component of the wind images at time \((t + \Delta t)\) to zero at the boundary. Thus along boundaries parallel to the \(y\) and \(x\) axes respectively, we have

\[
[a\Phi_x + b\Phi_y]_{(x, t + \Delta t)} = 2r_3/\Delta t
\]

(28)

and

\[
[a\Phi_y - b\Phi_x]_{(x, t + \Delta t)} = 2r_4/\Delta t.
\]

(29)
Note that Eqs. (28) and (29) correspond to the cancellation of the boundary flux terms associated with the differential operators in (26).

Assuming that all quantities (including the displacements \( \alpha \)) are known at times \( t \) and \( (t + \Delta t)/2 \), we can explicitly compute the right-hand sides of Eqs. (26), (28) and (29) and thus solve (26) for \( \Phi(x, t + \Delta t) \). The discretized form of (26) is solved by an appropriate method (e.g. successive over-relaxation, alternating direction implicit). For the bilinear finite elements of the present study, Eq. (26) gives a nine-point stencil. This is solved using four iterations of a semi-direct solver similar to that described in Staniforth and Mitchell (1977, 1978), with \( \Phi(x, t) \) as the first guess.

\((c)\) Updating the winds

Since \( \Phi(x, t + \Delta t) \) is known from the computations of section 2(b), \( U \) and \( V \) can in principle be computed explicitly from Eqs. (17) and (18). This may be appropriate for a finite-difference model that uses a suitable staggering of the winds with respect to \( \Phi \). It is, however, inappropriate for the unstaggered variable-resolution finite elements of this study, since it leads to an undesirable damping of the slow-mode solution (see the stability analysis of Staniforth and Mitchell (1977)). The remainder of this sub-section is devoted to the formulation of an acceptable procedure for unstaggered finite elements. With a suitable interpretation it should also be applicable to finite-difference formulations on unstaggered meshes.

On an unstaggered mesh, Eq. (23) is solved for \( \chi(x, t + \Delta t) \) subject to boundary condition (10). The first terms on the right-hand sides of (6) and (7) then give the divergent component of the wind. The implicit constraint on the data (the right-hand side of (23)) of a Poisson problem with a Neumann boundary condition, is automatically satisfied by the bilinear finite-element formulation of the present study (Staniforth and Temperton 1986).

Equations (1)–(3) lead to the potential vorticity equation

\[
\frac{d}{dt} \left( \frac{Q}{\Phi} \right) = 0 \tag{30}
\]

where

\[
Q = \xi + f \tag{31}
\]

Discretizing Eq. (30) using the semi-Lagrangian method gives

\[
Q(x, t + \Delta t) = (Q/\Phi)_{(x-\alpha, t)} \times \Phi(x, t + \Delta t) \tag{32}
\]

Equations (9), (31) and (32) yield

\[
(\psi_{xx} + \psi_{yy})_{(x, t + \Delta t)} = [(Q - f)/S]_{(x, t + \Delta t)} \tag{33}
\]

which is solved for \( \psi(x, t + \Delta t) \) subject to boundary condition (11). The second terms on the right-hand sides of (6) and (7) then give the rotational component of the wind, thus completing the updating of the winds.

\((d)\) Alternative forms

Several alternative forms for the discretization of Eqs. (1)–(3) are possible, and we describe some that we tried.

The metric terms on the right-hand sides of Eqs. (12) and (13) are relatively small, since the gradient of the map-scale factor is small. At the price of decentering the scheme (and potentially increasing the time truncation error) we may evaluate these terms at
\((x - \alpha, t)\) instead of \((x - \frac{1}{2}\alpha, t + \frac{1}{2}\Delta t)\). It is computationally advantageous to do so, since it halves the number of interpolations required in (19) and (20). It turns out that treating these terms in this manner has a negligible impact on the solution, confirming that these terms are indeed small.

An alternative discretization of the continuity equation is

\[
\frac{\Phi(x, t + \Delta t) - \Phi(x - \alpha, t)}{\Delta t} + \frac{1}{2} \Phi(x, t + \Delta t) = - [\Phi \Phi_{x} D(x, t + \Delta t) + \Phi(x - \alpha, t)] (x - \alpha, t)
\]

(34)

where \(\Phi_{x}\) is the mean value of \(\Phi\). The divergence \(D(x, t + \Delta t)\) can be eliminated from (34) using (23), again yielding an elliptic equation for \(\Phi(x, t + \Delta t)\) which may be solved in a similar manner to (26). This alternative form also decouples the scheme and potentially increases the time truncation error. Although the impact was found to be negligible in the present study, this might not be the case for a \(\Phi\) having an appreciably smaller mean value.

An alternative discretization of the continuity equation which maintains second-order accuracy in time is

\[
\frac{\Phi(x, t + \Delta t) - \Phi(x - \alpha, t)}{\Delta t} + \frac{1}{2} \Phi(x, t + \Delta t) + \Phi(x - \alpha, t)] = 0.
\]

(35)

Again the divergence \(D(x, t + \Delta t)\) may be eliminated from this equation via (23) to obtain an elliptic equation for \(\Phi(x, t + \Delta t)\). However, we chose to use the logarithmic form (14) in this study since the solution procedure is then a little more direct, and the results were virtually identical.

Finally it is possible to discretize

\[
dQ/dt + QD = 0
\]

(36)

instead of (30). Thus we iteratively solve

\[
\left(Q + \frac{1}{2}\Delta tQD\right)(x, t + \Delta t) = \left(Q - \frac{1}{2}\Delta tQD\right)(x - \alpha, t)
\]

(37)

for \(Q(x, t + \Delta t)\), instead of using (32), where \(D(x, t + \Delta t)\) is given by (23). Again the results were very similar, and we chose the more economical approach using Eq. (32).

3. COMPUTING THE DISPLACEMENTS

The displacements in the semi-Lagrangian method are computed approximately from Eq. (16). Perhaps the simplest such approximation (method 1) is that of Bates and McDonald (1982), where \(\alpha\) is computed explicitly from the relation

\[
\alpha = \Delta t V(x, t).
\]

(38)

However, as we show in sections 5 and 6, method 1 leads to serious truncation errors when used with large timesteps.

Since enhanced stability is of little benefit if achieved at the expense of accuracy, we examined various alternatives for improving the accuracy of the trajectory computations. A straightforward method (method 2) is to extrapolate the winds at mesh points to time \((t + \frac{1}{2}\Delta t)\) using

\[
V(x, t + \frac{1}{2}\Delta t) = \frac{1}{2}[3V(x, t) - V(x, t - \Delta t)]
\]

(39)

and then to solve iteratively

\[
\alpha = \Delta t V(x - \frac{1}{2}\alpha, t + \frac{1}{2}\Delta t)
\]

(40)
for $\alpha$, using interpolation to evaluate quantities at $(x - \frac{1}{2}\Delta x)$. We found only small differences between linear and cubic interpolation in our trajectory computations. This iterative procedure is similar to that introduced by Robert (1981), and adopted in our recent work (Staniforth and Temperton 1986). A sufficient condition for convergence is given in Pudykiewicz et al. (1985).

Extending method 2, we replace the two-time-level extrapolation (39) by the three-time-level extrapolation

$$V(x, t + \frac{1}{2}\Delta t) = (1/8)[15V(x, t) - 10V(x, t - \Delta t) + 3V(x, t - 2\Delta t)]$$

(41)

in the iterative solution of (40). This method (method 3) turns out to yield the most accurate trajectories of all the various methods we tried. All of these extrapolation schemes yield winds at $(t + \frac{1}{2}\Delta t)$, which can also be used to compute the time-centred metric terms in (12) and (13).

An analysis of each method when applied to the problem of solid-body rotation provides a useful guide as to its expected accuracy in more general circumstances. It was found that those methods which keep a particle on its exact trajectory (a circle) for this simple problem, give better results for the more general problem than those that do not. For uniform rotation, several methods keep a particle on its exact trajectory. One such method (method 4) is to solve

$$\alpha = \frac{1}{2}\Delta t[3V(x - \alpha, t) - V(x - 2\alpha, t - \Delta t)]$$

(42)

iteratively for $\alpha$. This corresponds to extrapolation along a trajectory, rather than at a point as in methods 2 and 3. However, methods 2 and 3 additionally retain a particle on its exact trajectory for solid-body rotation when the rotation rate is time dependent, and also for circular motion in a wind field with radial shear. The superior performance of pointwise extrapolation for these simple problems is found to hold true also for the more general problem.

4. Discussion of the Scheme

The solution procedure may now be summarized by the following four-step algorithm:
(a) Find the displacements $x$ using method 1, 2, 3 or 4 of section 3.
(b) Evaluate $r_1, r_2, r_3, r_4$ and $r_5$ using Eqs. (19), (20), (24), (25) and (27) respectively.
(c) Solve Eq. (26) for $\Phi(x, t + \Delta t)$ subject to boundary conditions (28) and (29).
(d) Update the winds.
On a staggered mesh, Eqs. (12) and (13) would be solved for $U(x, t + \Delta t)$ and $V(x, t + \Delta t)$.
On an unstaggered mesh (as in this study), (23) and (33) are solved for $\chi$ and $\psi$ respectively, and $U(x, t + \Delta t)$ and $V(x, t + \Delta t)$ are then obtained diagnostically from (6) and (7).

We now compare the computational complexity of this scheme with that of other semi-Lagrangian schemes. Such a comparison can, however, only be very approximate, since much depends on how well different parts of the code are optimized, and on the architecture of the computer on which the code is run. With this reservation in mind, a comparison with our earlier three-time-level scheme (Staniforth and Temperton 1986) shows that the present two-time-level scheme requires two fewer applications of horizontal interpolation per timestep. On the other hand, the elliptic equation to be solved at each timestep for the new value of $\Phi$ is slightly more complicated. For our model code run on a Cray-1, the combined effect was that one timestep of the new scheme required
10% less CPU time than one timestep of the previous scheme. Since our new scheme only needs half the number of timesteps, it runs 2.2 times faster than our previous one.

If our new scheme were formulated on a staggered grid, we could eliminate the Poisson equations (23) and (33), and be left with the single elliptic equation (26). In any case the solution of elliptic equations does not represent a major overhead in a realistic model. For example, the Canadian operational baroclinic finite-element regional model is based on a grid similar to that used here, and has been highly optimized. It requires a similar pair of Poisson equations to be solved at each level every timestep, yet the resulting overhead is less than 5% of the overall computation time. The three-dimensional elliptic equation resulting from the use of a semi-implicit scheme is solved using a semi-direct method like that used here, and again represents less than 5% of the cost per timestep. Thus the time spent on all the elliptic boundary value problems is less than 10% of the total model time, and the added Poisson problems are not a major consideration in practice.

Comparing semi-implicit semi-Lagrangian schemes, it follows that the computational work per timestep for our two-level scheme is comparable to that for our three-level scheme (Staniforth and Temperton 1986), the three-level scheme of Robert et al. (1985), and the two-level scheme of McDonald (1986).

The two-level semi-Lagrangian alternating direction implicit (SLADI) scheme of Bates (1984) requires less work per timestep than the above semi-implicit schemes; in particular there are no two-dimensional elliptic equations to be solved. However, in the latest version of this scheme (Bates and McDonald 1987b), some additional interpolations are required; according to the CPU times quoted in their paper, the cost per timestep is now greater than that for the semi-implicit semi-Lagrangian (SISL) scheme of McDonald (1986). The disadvantage of the SLADI scheme, as noted by Bates and McDonald (1987b), is that the timestep is more limited by considerations of accuracy than for the SISL scheme. Our hypothesis is that this is because the ADI scheme implicitly uses the approximate factorization of a two-dimensional operator into a product of one-dimensional operators. As shown by Tanguay and Robert (1986), this can result in rapidly increasing time truncation errors when used with long timesteps.

5. Analysis

For the purposes of our truncation error analysis, it is convenient to define the operators

\[ \frac{dF}{dt} = \frac{F(x, t + \Delta t) - F(x - \alpha, t)}{\Delta t} \]

\[ \bar{F}^* = \frac{F(x, t + \Delta t) + F(x - \alpha, t)}{2} \]

Expanding \( F \) as a Taylor series in three variables we obtain

\[ \frac{dF}{dt} = F_i^* + (\alpha/\Delta t)F_i^* + (\beta/\Delta t)F_y^* + O((\Delta t)^2, \alpha^2/\Delta t, \beta^3/\Delta t, \alpha^2, \alpha \beta, \beta^2, \alpha \Delta t, \beta \Delta t) \]

\[ \bar{F}^* = F^* + O((\Delta t)^2, \alpha^2, \alpha \beta, \beta^2, \alpha \Delta t, \beta \Delta t) \]

where an asterisk denotes evaluation at \((x^*, t^*) = (x - \alpha/2, t + \Delta t/2)\), and \(\alpha\) and \(\beta\) are the \(x\) and \(y\) components respectively of \(\alpha\).

The displacements \(\alpha\) are determined from the approximate solutions of (16) (see section 3), and so

\[ \alpha/\Delta t = V^* + O(\Delta t^n) \]
where the value of $n$ depends upon the details of the chosen approximation. For methods 1–4 of section 3, it is easily shown that $n$ takes the values 1, 2, 3 and 2 respectively. It follows that

$$dF/d\tau = (dF/dt)^* + O(\Delta t^2)$$

for methods 2–4 (i.e. a second-order accurate approximation to $dF/dt$ at $(x^*, t^*)$), whereas for method 1

$$dF/d\tau = (dF/dt)^* + O(\Delta t)$$

which is only first-order accurate. For all four methods, Eq. (46) yields a second-order accurate approximation to $F$ at $(x^*, t^*)$.

In the formulation of section 2, the governing equations have been approximated along a trajectory using the operators $dF/d\tau$ and $F^\tau$, and an $O(\Delta t^2)$-accurate extrapolation of the metric terms to $(x^*, t^*)$. The importance of a sufficiently accurate evaluation of the displacements $\alpha$ is now evident. Evaluation using methods 2, 3 or 4 leads to a second-order accurate formulation whereas method 1 (the method of Bates and McDonald (1982)) yields only first-order accuracy because of Eq. (49).

Regarding stability, it is easily shown (in a manner similar to that employed in Staniforth and Temperton (1986)) that the formulation is unconditionally stable for the linearized equations. This is only to be expected, since all the linear terms (including the Coriolis ones) are now treated implicitly in time.

6. Experiments

(a) Procedure

In order to test the various methods proposed in section 3 for calculating the displacements, we ran a series of numerical experiments using our finite-element barotropic model. The experimental procedure followed closely that of our previous work (Staniforth and Temperton 1986). Thus, the model was run over a square domain of side 20 000 km centred at the north pole, using a stereographic projection true at $60^\circ$N and bounded by a solid wall in the vicinity of the equator. This domain was covered by a non-uniform mesh of $101 \times 101$ points, with a $61 \times 61$ uniform high-resolution (100 km) mesh over the North American 'area of interest' (see Fig. 1 of Staniforth and Temperton (1986)). We also used the same carefully balanced initial conditions as before, based on an operational 500 mb analysis for 12 GMT on 28 February 1984. The model was run to 48 hours with various choices of the scheme for calculating displacements, and with various timestep lengths.

In principle, a two-time-level integration scheme does not require the special 'smooth start' procedure needed for a three-time-level scheme. However, for the first timestep there is a similar problem when using methods 2, 3 and 4 of section 3 in that they require winds extrapolated to time $\Delta t/2$ in order to calculate the displacements $\alpha$. For these methods we thus computed the first timestep iteratively, using

$$V^{(k)}(x, \Delta t/2) = \frac{1}{2}[V(x, 0) + V^{(k-1)}(x, \Delta t)]$$

for three iterations, where $V^{(0)}(x, \Delta t) = V(x, 0)$.

For the second timestep onwards, two-time-level extrapolation (methods 2 or 4) can be applied. In the case of integrations using method 3 (three-time-level extrapolation), method 2 was used for the second timestep and the full scheme was used for the third and subsequent timesteps. The displacement equations (40) and (42) were solved with
Figure 1. Forecast height fields at 48 hours (contour interval 10 dam). (a) Control run on uniform grid (all other forecasts are on non-uniform grid); (b) scheme B with $\Delta t = 90$ minutes; (c) new scheme with method 1 for displacements and $\Delta t = 10$ minutes; (d) new scheme with method 2 and $\Delta t = 3$ hours; (e) new scheme with method 3 and $\Delta t = 3$ hours; (f) new scheme with method 3 and $\Delta t = 6$ hours.
four iterations using linear interpolation. No explicit horizontal diffusion or time-filtering was used in any of the experiments.

To evaluate the results obtained using variants of the proposed scheme on the non-uniform grid, the forecasts were compared against those of a control run on a uniform 201 × 201 100 km mesh covering the same domain. This control run used the three-time-level semi-implicit Eulerian formulation of Staniforth and Mitchell (1977, 1978), with a timestep small enough (Δt = 10 min) to make the time truncation errors negligible.

(b) Results

For a preliminary assessment of the accuracy of our schemes, we present in Fig. 1 a series of maps of the forecast height field after 48 hours. Figure 1(a) shows the ‘control’ forecast using the Eulerian scheme on a uniform high-resolution grid with a small timestep. The remaining forecasts all used the non-uniform grid. Figure 1(b) shows the forecast using scheme B of Staniforth and Temperton (1986) with a timestep of 90 minutes. Over the North American high-resolution ‘window’ the forecast appears identical to the control run, with only small differences over the outer low-resolution areas. Since scheme B is a centred three-time-level scheme, time-derivatives are evaluated over an interval of 2Δt = 3 hours, and our goal was to obtain similar accuracy using a two-time-level scheme with a 3-hour timestep.

Figure 1(c) shows the forecast using our two-time-level scheme together with method 1 for the displacement calculations, and a timestep of 10 minutes. Even with such a small timestep the errors are considerable (e.g. 4 dam in the central value for the low over Quebec). Experiments with larger timesteps showed a clear O(Δt) growth of the errors, and this method is evidently inadequate for our purposes.

Figure 1(d) shows the corresponding forecast with a timestep of three hours, using method 2 (two-time-level extrapolation) for the displacement calculations. This forecast, though reasonable in view of the large timestep, is by no means as accurate as that of scheme B with half the timestep shown in Fig. 1(b). We also ran forecasts with method 4 (‘two-time-level extrapolation along trajectories’) for the displacement calculations, but the results were consistently worse than those for the simpler method 2 with the same timestep and we will not discuss this method further.

For the forecast shown in Fig. 1(e), we again used a timestep of three hours but with method 3 (three-time-level extrapolation) for the displacement calculations. With this scheme our goal was achieved, in that the control forecast was reproduced with essentially the same accuracy as that of scheme B with half the timestep.

Finally, Fig. 1(f) shows the corresponding forecast using method 3 with a timestep of six hours. The error level is now unacceptable, but we show the figure to make the point that this scheme produces a forecast which is stable and not completely unreasonable, using a timestep 30 times longer than the stability limit for an Eulerian semi-implicit scheme at the same horizontal resolution. With such a scheme we are clearly in a better position to choose the timestep on the basis of balancing the temporal and spatial truncation errors, rather than being constrained by considerations of stability.

In order to investigate the growth of the time-truncation error as a function of timestep length of our two-time-level scheme using methods 2 and 3, we ran series of forecasts on the non-uniform grid with different values of Δt. The accuracy of each forecast was measured by computing the r.m.s. differences (in the height and wind fields) from the control forecast run on the uniform high-resolution grid with a small timestep. The r.m.s. differences were computed over the ‘area of interest’ where the non-uniform and uniform grids coincide.
The results at 48 hours are plotted in Figs. 2(a) and (b) for the height and wind fields respectively. As a comparison is made with the corresponding results of a three-time-level scheme, we have chosen to plot the r.m.s. differences as a function of $\Delta \tau$, the interval over which time-derivatives are calculated. Thus $\Delta \tau = 2\Delta \tau$ for the two-time-level schemes, but $\Delta \tau = 2\Delta \tau$ for a three-time-level scheme. For a given value of $\Delta \tau$, the two-time-level scheme requires approximately half the work of the three-time-level scheme, but the time-truncation errors are in principle formally equivalent.

In Figs. 2(a) and (b), the solid circles plotted at $\Delta \tau = 20$ min show the r.m.s. differences from the control run for a forecast using the same Eulerian scheme but on the non-uniform grid. Since the timestep is small, this provides an approximate measure of the spatial truncation error incurred by running with lower resolution outside the area of interest.

The crosses show the results for the three-time-level scheme B of Staniforth and Temperton (1986). As noted in our earlier paper, the forecast over the area of interest using this scheme, with timesteps of at least one hour ($\Delta \tau = 120$ min), was more accurate than that of the small-timestep Eulerian scheme, and was still of comparable accuracy with a timestep of 90 minutes ($\Delta \tau = 180$ min).

The goal of this study was to reproduce this level of accuracy with half the computational work, using a two-time-level scheme. The results using method 2 (two-time-level extrapolation) for the displacements are shown in Figs. 2(a) and (b) by open circles. For timesteps ($\Delta \tau$) of up to 90 minutes, our goal is clearly achieved, while for longer timesteps the growth of time-truncation errors becomes evident. Nevertheless, this extrapolation scheme might well be adequate if, for example, a smaller timestep is chosen in a more complete model to resolve the time-evolution of physical processes.

The results using method 3 (three-time-level extrapolation) for the displacements are shown by the 'plus' symbols. This scheme yields acceptable accuracy with timesteps as long as three hours ($\Delta \tau = 180$ min); the r.m.s. differences are similar to those of scheme B throughout the range, while remaining lower than or similar to those of the corresponding semi-implicit Eulerian scheme with a much smaller timestep. It seems from our results that the optimal timestep at this resolution is approximately three hours, since the spatial and time truncation errors contribute about equally to the total error. There seems to be little virtue in using timesteps greater than three hours since the time truncation errors will become dominant.

7. CONCLUSIONS

In a previous study (Staniforth and Temperton 1986) we had shown that the three-time-level semi-implicit semi-Lagrangian technique of Robert (1981, 1982) could be applied to a model in which the finite-element method was used for the horizontal discretization. We concluded that this technique is very promising both for finite-difference and finite-element models, since it permits much longer timesteps than an Eulerian scheme, with no loss of accuracy.

In the present study we have shown that the same level of accuracy can be obtained for half the computational effort by changing from a three-time-level to a two-time-level scheme. To achieve this, it is important to maintain the second-order accuracy in time of the semi-Lagrangian technique. Our scheme does not require operator splitting in the discretization of the equations, and it has essentially no restriction on the timestep length to maintain stability. The advantage of this enhanced stability is that the timestep can be chosen so that the time and space truncation errors are of approximately equal importance.
Figure 2. (a) Root mean square differences from the control run, in the height field over the region of interest at 48 hours, as a function of $\Delta t$. ● Eulerian; × scheme B; ○ new scheme with method 2 for displacements; + new scheme with method 3. (b) As (a) but for wind field.
In a finite-element regional barotropic model with a 100 km mesh over the area of interest, we were able to obtain similar accuracy, using our scheme with a 3-hour timestep, to that obtained using an Eulerian scheme with a 10-minute timestep.

Use of the new two-time-level scheme does not seem to be restricted to regional models with wall boundary conditions. It is presently being tested in a global barotropic spectral model, and we see no inherent difficulty in applying it to regional models with open boundary conditions. A more difficult challenge, however, will be to extend the scheme to a multi-level model; this is the subject of current research.

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