On the two-dimensional steady upshear-sloping convection

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SUMMARY

A 2-D inviscid steering-level model due to M. W. Moncrieff is reexamined both analytically and numerically. It is found that the thermal effects on the distribution of energy and vorticity play opposite roles in determining the interface slope, with the former being dominant. By separating the two effects, a useful insight is obtained into how the interface is controlled by the internal flow regime. Consequently, a mixed jump and steering-level model is proposed, in which the returning updraught and downdraught are separated by adjacent jump updraught and downdraught. By solving for the internal flow, upshear-sloping convection is obtained in the mixed model. Numerical results indicate that the interface slope increases towards the vertical, or even becomes downshear-sloping, as (i) the jump outflow becomes shallow, (ii) the lifting condensation level (LCL) becomes low, and/or (iii) the Richardson number (proportional to the ratio between convective available potential energy and kinetic energy in the inflow) increases (except for the cases where the jump outflow is very deep and LCL is very high). These features are discussed in connection with the recent results of A. J. Thorpe et al., and the observed differences between tropical propagating squall lines and mid-latitude steering-level-type squall lines.

1. INTRODUCTION

It has been known for many years that squall line thunderstorms tend to occur in environments of strong vertical shear and their internal circulations are slantwise (e.g. Ludlam 1963). This slantwise feature has been recognized to have important thermo-dynamic consequences for maintaining the storms in such a way that precipitation formed in the updraught falls into the downdraught. The dynamics which may account for the slantwise convection can be very different depending on a variety of storm–environment interactions. One type of slantwise convection, in which the line of convection is approximately perpendicular to the basic shear, has been extensively studied by theoretical methods (Moncrieff and Green 1972; Moncrieff and Miller 1976; Moncrieff 1978, 1981). The main theme of this paper is related to the steering-level model in the classification of Moncrieff (1981).

The original concept of the steering-level model proposed by Moncrieff and Green (1972) was two-dimensional, in which the moist updraught slanted upshear over the downdraught. However, after having solved for the internal flow for such a 2-D steady state model, Moncrieff found that his results were in sharp disagreement with the original conceptual model. In every case for a convectively unstable atmosphere the updraught was found to slope downshear (see Fig. 2). This has cast doubt on the existence, or at least the steadiness, of upshear-sloping convection in a 2-D steering-level model and, in consequence, the concept of the steering-level model has been extended to three dimensions (Moncrieff 1978, 1981).

In the current paper, we choose to deal with the 2-D steering-level model mainly for the following considerations: (i) quasi-2-D upshear-sloping convection is observed in squall lines; (ii) 2-D flow has many advantages for both analytical and numerical studies due to its mathematical simplicity; and (iii) the challenge on the existence and steadiness of upshear-sloping convection in a 2-D steering-level model is still open: more recent studies are given by Thorpe et al. (1982) and Seitter and Kuo (1983). In those two studies, squall lines were simulated numerically as initial value problems, which were very different from the inviscid steady state model of Moncrieff (1978). By including shallow
downdraught cooling and non-constant shear, Thorpe et al. (1982) were able to obtain a quasi-steady upshear-sloping updraught in their 2-D model. They concluded that both non-constant shear and shallow downdraught cooling were indispensable for maintaining an upshear-sloping updraught in two dimensions. By including the negative buoyancy due to liquid water loading, Seitter and Kuo (1983) were able to produce an upshear-sloping updraught in a more general environmental shear in their 2-D model. They emphasized that “the vorticity production due to the liquid water distribution leads to an upshear slope of the updraught/downdraught interface”. However, the storms in their model could not maintain a quasi-steady stage for long (less than half an hour). On the other hand, although Thorpe et al. (1982) simulated a relatively long-lasting 2-D upshear updraught, there still exist questions as to (i) whether the quasi-steady convection produced in their model can be qualitatively reproduced in an inviscid 2-D steady state model; and (ii) why both non-constant shear and shallow downdraught were important in producing quasi-steady convection in their model? Since the inviscid 2-D steady flow can be studied semi-analytically and since models of this type (Moncrieff 1981) have merits in providing a theoretical framework for analysis and design of numerical simulations and observations, we will examine the above questions by furthering the study of Moncrieff (1978). In the following section we solve the internal flow and examine the interface properties for the steering-level model both analytically and numerically and explain why upshear-sloping convection cannot exist in such a model. In section 3 we develop a mixed jump and steering-level model and show that upshear-sloping convection exists in this mixed model. Discussion and comparison of our results with those of Thorpe et al. (1982) follow in section 4.

2. REEXAMINATION OF THE STEERING-LEVEL MODEL

In the steering-level model of Moncrieff (1978) (referred to as M78 hereafter), the interface slopes were closely related to the thermal stratification. When the inflow stratification was neutral, stable or unstable, the interface slope was vertical, upshear or downshear respectively. In the model’s formulation, the thermal effects appear in both the governing vorticity equation and the interface dynamic condition (see the last terms in Eqs. (8) and (17) of M78). The thermal term in the former exerts an influence on the vorticity distribution, while the thermal term in the latter contributes to the energy distribution. We will examine these two different thermal effects in detail.

(a) Some new considerations for the model formulation

The thermal term in the governing equation (9) of M78 is nonlinear and not amenable to analytical treatment, so we will use the method suggested by D. Raymond (1984 pp. 202–203) to develop a vorticity equation with a similar thermal effect but in a simple form. In the stream coordinates $(\psi, z)$, the nondimensional vorticity equation for a twodimensional steady Boussinesq flow can be integrated along the streamline (see Eq. (2) of M78):

\[
\nabla^2 \psi - \int \frac{\partial \theta}{\sqrt{\psi}} \bigg|_{z} \, dz = G_1(\psi)
\]

(1)

where the operator $\nabla^2$ is in $(x, z)$ space but the calculus of the second term is in $(\psi, z)$ space, $\theta$ is the nondimensional perturbation of the (equivalent) potential temperature. Here the scaling is $(x, z) \rightarrow (x, z)/H$, $\psi \rightarrow \psi/2AH^2$, $\theta \rightarrow \theta \bar{\theta}/(4A^2H\bar{\theta}_o)$, $A$ and $\bar{\theta}_o$ are
the reference shear and potential temperature scales respectively. The next step of development will be different from M78. In M78 a remote inflow with a linear shear and a constant thermal stratification was prescribed. In this case the inflow level $z_0$ was related linearly to $\nabla \psi$ (rather than to $\psi$), $\theta$ was related linearly to $z$, and $z$, and thus the thermal term (see the second term in Eq. (1)) was a nonlinear function of $\psi$. Apparently, this complicated thermal term resulted from the simple form prescribed for the remote inflow. Now, we prefer to reverse the derivation of M78 and show that a simple form can be prescribed for the thermal term but the (mathematically rather than physically) resulting remote inflow has a slightly complicated form.

To find a simple form for the thermal term in (1), we note that

$$\int \frac{\partial \theta}{\partial \psi} dz = F(\psi, z) - F(\psi, z_0)$$  \hspace{1cm} (2)

where $\partial F(\psi, z)/\partial z = \partial \theta(\psi, z)/\partial \psi$. Since the inflow level $z_0 = z_0(\psi)$ only depends on $\psi$ in $(\psi, z)$ coordinates, the last term in (2) can be incorporated into $G_1(\psi)$ in (1). We choose

$$G_1(\psi) - F(\psi, z_0) = 1$$  \hspace{1cm} (3)

because the unit linear shear is expected to be recovered for the remote flow when the thermal term in (1) vanishes. Obviously, $F(\psi, z)$ will be a linear function of $\psi$ if $\theta(\psi, z)$ is a quadratic function of $\psi$. Here, even more simply, a linear form can be prescribed for $\theta$:

$$\theta = R z \psi / \psi_m$$  \hspace{1cm} (4)

where $\psi_m$ is the value of $\psi$ along the interface and boundaries, and thus $\psi / \psi_m$ is the normalized streamfunction. To see the physical meaning of $R$, we integrate (4) along the interface $\psi = \psi_m$ from $z = 0$ to $z = 1$, and obtain

$$R = 2 \int_0^1 \theta(\psi_m, z) dz = \text{CAPE} / (4U^2),$$

where CAPE is the convective available potential energy and $U = AH$ is the velocity scale. Thus, $R$ is a Richardson number similar to Eqs. (9)–(11) of M78. Substituting (2)–(4) into (1) gives

$$\nabla^2 \psi = 1 + (R/2\psi_m)z^2.$$  \hspace{1cm} (5)

Solving (5) for the remote flow, we obtain

$$\psi \rightarrow \psi_\infty(z) = \frac{\alpha_1}{2} (z - z_u)^2 + \frac{\alpha_2}{3} (z - z_u)^3 + \frac{\alpha_3}{4} (z - z_u)^4 \quad \text{as } x \rightarrow \infty,$$  \hspace{1cm} (6)

where $\alpha_1 = 1 + b z_u$, $\alpha_2 = b z_u$, $\alpha_3 = b / 3$, $b = R/(2\psi_m)$ and $z_u$ is the steering level. The boundary conditions are

$$\psi = \psi_m \quad \text{on } z = 0, 1 \text{ and along the interface } x = \Gamma(z)$$  \hspace{1cm} (7)

which give two constraints on the remote streamfunction, Eq. (6):

$$\psi_m = \psi_\infty(0) = \psi_\infty(1).$$  \hspace{1cm} (8)

Thus, there is only one free external parameter $R$. For a given $R$, the other three parameters $b$, $z_u$ and $\psi_m$ are determined by (8) and $b = R/(2\psi_m)$. 
Now the nonlinearity appears only in the interface dynamic condition:

\[ \Delta(\frac{1}{2}v^2) = \Delta PE \quad \text{along } x = \Gamma(x) \]  

(9)

where \( \Delta( ) \) represents the jump of ( ) across the interface from the updraught to the downdraught and \( PE \) is the released thermal potential energy. Here Eq. (9) is derived from the continuity of pressure and the Bernoulli theorem:

\[ p + \frac{1}{2}v^2 - PE = \text{constant} \quad \text{along a streamline}, \]

where \( p \) is nondimensionalized by \( p \rightarrow p/(4A^2H^2\rho_o) \). The difference in \( PE \) between the two adjacent streamlines on the two sides of the interface gives the thermal term \( \Delta PE \) in (9). If the lapse rates in the updraught and downdraught are equal, then the system is antisymmetric with respect to \( (x, z) = (\Gamma(\frac{1}{2}), \frac{1}{2}) \) and

\[ \Delta PE = R(z - \frac{1}{2}) \quad \text{along } x = \Gamma(z). \]  

(10)

Here (10) is the nondimensional form of (17) in M78.

The system (5)–(9) is similar to the original problems of M78, where the moist updraught and downdraught were assumed to have the same basic potential temperature \( \theta_o = \bar{\theta}_o = \text{constant} \). In this case, the spatial variation of the (moist) potential temperature remains only in the (nonlinear) perturbation part, i.e. \( \theta \) given by Eq. (4). When the updraught and downdraught have different basic potential temperatures, for example, \( \theta_o^+ \) and \( \theta_o^- \) respectively, the reference potential temperature scale can be chosen as \( \theta_o = (\theta_o^+ + \theta_o^-)/2 \). However, the basic temperature difference \( \Delta \theta_o = \theta_o^+ - \theta_o^- \) results in a basic hydrostatic pressure difference between the two sides of the interface (nondimensional form with the scaling \( \Delta P_o \rightarrow \Delta P_o/(4A^2H^2\rho_o) \) and \( \Delta \theta_o \rightarrow \Delta \theta_o g/(4A^2H\bar{\theta}_o) \)):

\[ \Delta P_o = (z - z_0)\Delta \theta_o \]  

(11)

where \( z_0 \) is the level of \( \Delta P_o = 0 \). In this case, the dynamic condition at the interface should be obtained from the continuity of the total pressure \( P_o + p \) rather than only the perturbation pressure \( p \). Thus, instead of (9), we have

\[ \Delta(\frac{1}{2}v^2) = \Delta PE + \Delta P_o \quad \text{along } x = \Gamma(z). \]  

(12)

Here \( \Delta P_o \) in Eq. (12) can play a similar role to \( \Delta PE \). For example, if both the updraught and downdraught have the same unstable stratification \( N_o^2 < 0 \), then \( R > 0 \) in (10) and \( \Delta PE \) increases with \( z \). The same is true for \( \Delta P_o \) if the updraught is warmer than the downdraught \( (\Delta \theta_o > 0) \) and \( z_0 = 0.5 \) in Eq. (11). However, physically they are different, \( \Delta P_o \) is due to the thermal effect on the basic pressure distribution while \( \Delta PE \) is due to the thermal effect on the energy distribution. In real storms, both effects may come into play. In the next subsection we will examine the simplest case—antisymmetric flow where \( z_0 = 0.5 \). In this case the interface dynamic condition is

\[ \Delta[\frac{1}{2}(1 + \Gamma'^2)w^2] = (R + \Delta \theta_o)(z - \frac{1}{2}) \]  

(13)

where \( \Gamma' = d\Gamma/dz = u/w \) along the interface \( x = \Gamma(z) \).

(b) Asymptotic solutions

Now we seek solutions for (5)–(7) and (13). Since the coupled updraught–downdraught flow is antisymmetrical, we only need to examine the updraught branch. When \( R = \Delta \theta_o = 0 \), the interface is vertical, i.e. \( x = \Gamma(z) = 0 \), and the updraught solution is

\[ \psi = \psi^{(0)}(x, z) = \psi_{o}^{(0)}(z) + \sum_{n=1}^{\text{odd}} a_n e^{-n\pi z} \sin n\pi z \]  

(14)
where \( \psi_{\infty}^{(0)} = \frac{1}{2}(z - \frac{1}{2})^2 \) is the remote \((x \to \infty)\) streamfunction given by Eq. (6) with 
\( R = 0, \; a_n = 4(n\pi)^{-3}, \) and \( n = 1, 3, 5, \ldots \).

When \( R \) and \( \Delta \theta_0 \sim O(\epsilon) < 1 \), we may assume the following asymptotic expansions:

\[
\psi = \sum_{k=0}^{\infty} \epsilon^k \psi^{(k)}(\xi, \zeta) \quad \text{in coordinates} \quad \begin{cases} \xi = x - \Gamma(z) \\ \zeta = z \end{cases} \tag{15}
\]

\[
\Gamma(z) = \Gamma(\zeta) = \sum_{k=1}^{\infty} \epsilon^k \Gamma^{(k)}(\zeta). \tag{16}
\]

Substituting (15)--(16) into (5)--(8) and (13), we find that the leading order solution has the same form as (14) except that \((x, z)\) is replaced by \((\xi, \zeta)\). The interface dynamic condition is trivial in the leading order, so to see the dependence of the interface geometry on the thermal effect we need to solve the next-higher-order problem, i.e.

\[
\left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2} \right) \psi^{(1)} = B \xi^2 + 2 \frac{d\Gamma^{(1)}}{d\zeta} \frac{\partial^2 \psi^{(0)}}{\partial \xi \partial \zeta} + \frac{d^2 \Gamma^{(1)}}{d \zeta^2} \frac{\partial \psi^{(0)}}{\partial \xi} \tag{17}
\]

\[
\psi^{(1)}(\xi) = 0 \quad \text{on} \quad \zeta = 0, 1 \quad \text{and} \quad \zeta = 0 \tag{18}
\]

\[
\psi^{(1)}(\xi) \to \psi^{(1)}(\xi) = -\frac{B}{12} \xi(1 - \xi^3) \quad \text{as} \quad \xi \to \infty \tag{19}
\]

\[
\Delta(\psi^{(0)})^{(1)} = E(z - \frac{1}{4}) \tag{20}
\]

where \( \epsilon B = b \) and \( \epsilon E = R + \Delta \theta_0 \). Note that \( \Gamma^{(1)} \) is unknown and needs to be solved together with \( \psi^{(1)} \) in the Fourier space composed of the base functions

\[
\{e^{-\pi \xi}\}_{n=0}^{\infty} \{\sin m\pi \zeta\}_{m=1}^{\infty} \quad \text{for} \quad \psi^{(1)} \quad \text{and} \quad \{\cos m\pi \zeta\}_{m=1}^{\infty} \quad \text{for} \quad \Gamma^{(1)}.
\]

Here the odd- and even-periodic extensions with respect to \( \zeta = 0 \) have been made for \( \psi^{(1)} \) and \( \Gamma^{(1)} \) respectively, which is consistent with (14) and (17) and uniquely satisfies the continuity of the interface geometry at \( z = 0, 1 \). Furthermore, we note that the updraught and downdraught are antisymmetric, so \( \Gamma^{(1)} \) is an odd function with respect to \( z = \frac{1}{2} \) and thus belongs to the subspace \( \{\cos \pi \zeta\}_{m=1}^{\text{odd}} \) and has the following complete expression

\[
\Gamma^{(1)}(\zeta) = \sum_{m=1}^{\text{odd}} \beta_m \cos m\pi \zeta. \tag{21}
\]

Substituting (21) into (17) and solving for \( \psi^{(1)} \) from (17)--(19), we obtain (see appendix A)

\[
\psi^{(1)}(\xi, \zeta) = \sum_{m=1}^{\text{odd}} \sum_{n=1}^{\text{odd}} n a_n \alpha_m [\sin(n + m)\pi \zeta + \sin(n - m)\pi \zeta] e^{-\pi \xi} + \sum_{m=2}^{\text{odd}} \sum_{n=1}^{\text{even}} \alpha_n [(m + n)a_{m+n} + |m - n|a_{n-m}] e^{-\pi \xi} \sin m\pi \zeta + B \sum_{m=1}^{\infty} f_m (1 - e^{-m\pi \xi}) \sin m\pi \zeta \tag{22}
\]

where

\[
\alpha_m = -\frac{\pi}{2} \beta_m. \tag{23}
\]
and
\[ f_m = 2(-1)^m (m\pi)^{-3} + 4[1 - (-1)^m](m\pi)^{-5} \]  
(24)
is the \(m\)th component of \(\psi_2^{(1)}(\xi)\) in the Fourier space. As \(\xi \to \infty\), the last term in (22) gives (19) and the other two terms on the r.h.s. of (22) vanish. (Note that, in the last term of Eq. (22), 1 is the first component of \(\{e^{-n\pi\xi}\}_{n=0}^\infty\).)

To determine \(\{\alpha_m\}\) or \(\{\beta_m\}\), we substitute (14) with \((\xi, \xi)\) coordinates into \(w^{(0)} = -\partial \psi^{(0)}/\partial \xi + O(\epsilon)\) and (22) into \(w^{(1)} = -\partial \psi^{(1)}/\partial \xi + O(\epsilon)\) for the updraught along the interface \(\xi = 0\). The downdraught along \(\xi = 0\) is antisymmetric to the updraught. Substituting these results into (20) and moving \(O(\epsilon)\) terms to the next higher order (i.e. \(O(\epsilon^2)\)), we obtain (see appendix A) in the Fourier space
\[ M_{kn} \alpha_n = Ed_k' - Bd_k'', \quad k \text{ and } n = 1, 3, 5, \ldots \]  
(25)
where
\[ M_{kn} = \frac{4}{\pi^2} \sum_{m=2}^{\text{even}} C_{mk} \left[ \frac{n - 2m \delta}{(m - n)^2} - \frac{n}{(m + n)^2} \right], \quad \delta = \begin{cases} 0 & \text{when } m > n \\ 1 & \text{when } m < n \end{cases} \]  
(26)
\[ d_k' = k^{-2}, \quad d_k'' = \frac{2}{\pi^2} \sum_{m=2}^{\text{even}} C_{mk} m^{-2} \]  
(27)
\[ C_{mk} = (m + k)^{-2} + (m - k)^{-1}|m - k|^{-1}. \]  
(28)

Applying the inverse of \(\{M_{kn}\}\), i.e. \(\{M_{kn}\}^{-1}\), to Eq. (25) gives
\[ \alpha_n = E\alpha_n' - B\alpha_n'', \quad n = 1, 3, 5, \ldots \]  
(29)
where \(\{\alpha_n'\} = \{M_{kn}\}^{-1}\{d_k'\}\) and \(\{\alpha_n''\} = \{M_{kn}\}^{-1}\{d_k''\}\). In practice, (29) is obtained numerically in the truncated subspace \(n \leq N\). The numerical values of \(\alpha_n'\) and \(\alpha_n''\) are shown in Fig. 1, where \(\{\alpha_n''\}\) converges rapidly, but \(\{\alpha_n'\}\) converges slowly. A truncated \((N = 19)\) solution \(\psi = \psi^{(0)} + \epsilon\psi^{(1)}\) with \(R = 0.1\) and \(\Delta\theta_o = 0\) is given in Fig. 2(a), which is similar to Fig. 4 of M78. This solution is also comparable to the direct numerical solution of the finite element method in Fig. 2(b) (see appendix B).

Although the above truncated solutions can be very close to the numerical solutions as \(\epsilon\) is small, they do not converge uniformly as \(N \to \infty\). First we note that \(w = w^{(0)} + \epsilon w^{(1)} \to 0\) as \(z \to 0\) and 1. Thus Eq. (13) cannot be satisfied at the two corners.
Figure 2. Analytical (a) and numerical (b) solutions \( \psi \) for the steering-level model with \( R = 0.1 \) and \( \Delta \theta_c = 0 \). The truncation for the analytical solution is \( N = 19 \). Since the flow is antisymmetrical, the downdraught streamlines are not shown in (b). Instead, the element meshes for the numerical solution are shown in the downdraught region. A, B, C and D in (a) indicate the corner regions. The contour intervals for \( \psi \) are 0.01.

unless \( |\Gamma'| \to \infty \) and \( \Delta w/w \to 0 \) as \( z \to 0 \) and 1. In the asymptotic solutions \( \Gamma \) and \( \Gamma' \) are assumed to be of \( O(\epsilon) \), but for the above reason the solutions of \( \Gamma \) in Fig. 3 show that \( \Gamma \) increases rapidly as \( z \to 0 \) and 1. The higher the truncation \( N \), the larger the values of \( \Gamma \) and \( |\Gamma'| \) near the boundaries. Note that \( \Gamma' = 0 \) at \( z = 0 \) and 1, but \( |\Gamma'| \to \infty \) at \( z = \varepsilon \) and \( 1 - \varepsilon \) as \( N \to \infty \) for any \( 0 < \varepsilon \ll 1 \), so the even-periodic extension of \( \Gamma \) (for \( z < 0 \) and \( z > 1 \)) in Eq. (21) gives two cusps, at \( z = 0 \) and 1 as \( N \to \infty \). Thus, the asymptotic expansion (16) will become invalid near the corners if \( N \) is large enough. This indicates that (20) can be satisfied only to a certain extent on the interior segment of the interface. In fact, as the above truncated solution is substituted into (20), the error can be as large as the term on the l.h.s. of (20) for either low or high truncation. For small \( N \) the error is due to the lack of enough modes, while for large \( N \) the error is due to the divergence of the derivatives of the high modes. When \( 15 < N < 25 \) the error is relatively small \( (m_1 \sim 0.1 \) and \( m_2 \sim 0.6 \), see appendix B).

So far we have seen that the asymptotic expansion (15)–(16) is not uniformly valid and a different expansion of \( \psi \) should be matched in the two corner regions where \( |\Gamma'|^{-1} \) is small and decreases to zero as the rigid boundaries are approached. These two matched corner regions will become infinitely small as \( \varepsilon \to 0 \). For \( \varepsilon = 0 \) the solution degenerates

Figure 3. Interface geometries solved with different truncations: \( N = 9, 19, 49 \). The dashed curve is for \( N = 49 \) with the modification of Eq. (30).
into (14), which is uniformly valid. The real atmospheric flow near the rigid boundaries and interface, especially in the corner regions, is often observed to be highly viscous or turbulent, so (15)–(16) should really be matched with a viscous or turbulent boundary solution in each of these regions. However, we will not go further in this direction. Instead, we will show that the gross pattern of the solution $\psi(x, z)$ and interface geometry $\Gamma(z)$ is not sensitive to the corner interface condition. Note that the asymptotic expansion (15)–(16) can be uniformly valid if the function $z - \frac{1}{2}$ on the r.h.s. of (13) and (20) is modified as

$$G(z, \delta) = \begin{cases} (1 - \delta - \frac{1}{2}) \sin^2 \left( \frac{\pi}{2 \delta} \frac{1 - z}{\delta} \right), & 1 - \delta < z \leq 1 \\ (z - \frac{1}{2}), & \delta \leq z \leq 1 - \delta \\ (\delta - \frac{1}{2}) \sin^2 \left( \frac{\pi}{2 \delta} \frac{z}{\delta} \right), & 0 \leq z < \delta \end{cases} \quad (30)$$

where $0 < \delta \ll 1$. As $\delta \to 0$, (30) gives $(z - \frac{1}{2})$ except at $z = 0$ and 1. If $\delta$ is set equal to the resolution of a numerical solution, e.g. $\delta = \Delta z = 0.05$ for the finite element model in appendix B, then (30) may represent the actual modification of the interface condition (13) in the numerical model where the l.h.s. of (13) is zero at the two corner grid points (or nodes) and thus cannot equal the r.h.s. of (13) in the unresolved corner regions.

With the above modification, $\Gamma^{(1)}$ and $\psi^{(1)}$ are solved as in (21)–(29) except that $d'_k$ in (25) and (27) is replaced by

$$\tilde{d}_k = \frac{-\cos k\pi \delta}{k^2} + \frac{\pi}{1 - (k\delta)^2} \frac{0.5 - \delta}{2k} \sin k\pi \delta, \quad k = 1, 3, 5, \ldots \quad (31)$$

and thus $a'_n$ in (29) is replaced by $\tilde{a}_n = \{M_{kn}\}^{-1}\{\tilde{d}_k\}$. As shown in Fig. 1, $\{\tilde{a}_n\}$ converges much more rapidly than $\{a'_n\}$. (Note that the whole spectrum of $\{a'_n\}$ increases slowly with $N$, but the spectrum of $\{\tilde{a}_n\}$ does not.) It is found for this modified case that $\Gamma^{(1)}$, $\psi^{(1)}$ and their derivatives all converge rapidly and the modified interface condition can be satisfied to any accuracy if $N$ is large enough. Figure 3 shows that the interface (dashed curve) is uniformly smoothed (in comparison with the solid curves on which the small-scale waviness is visible) and becomes less extended near the rigid boundaries. Away from the corner region, the slope of the interface and the flow patterns are almost unaffected by the above modification. This indicates that the interior solutions of $\psi^{(1)}$ and $\Gamma^{(1)}$ are insensitive to the corner interface condition. The modified solutions are found to be more consistent, than the non-modified solutions, with the numerical solutions of the finite element method. Examples are shown in Figs. 4(a)–(d).

(c) **Thermal effects on the interface slope**

The thermal effect on the vorticity distribution is measured by $B$ in (17)–(19), which leads to the second term on the r.h.s. of (29). The thermal effect on the energy and basic pressure distributions along the interface can be measured by $E$ in (20), which leads to the first term on the r.h.s. of (29). Since $\varepsilon B = R/(2\psi_m) = 4R$, $\varepsilon E = R + \Delta \theta_0$, and $a'_n$ (or $\tilde{a}_n$) $> \alpha'_n > 0$ (see Fig. 1), our results show that the thermal effects on the distribution of energy and vorticity play opposite roles in determining the interface slope, with the former being dominant. As shown by Fig. 3, the interface slope is nearly constant in the interior region. At the middle level the interface slope (with respect to the vertical coordinate $z$) is

$$k_0 = \Gamma'(\frac{1}{2}) = \gamma'_0 (R + \Delta \theta_0) - \gamma''_0 b \quad (32)$$
where \((\gamma', \gamma'') = 2\Sigma_{n=1}^{\infty} n(\alpha', \alpha'')(n-1)/(n+1/2)\). For the modified case \(\gamma'\) and \(\alpha'\) are replaced by \(\bar{\gamma}\) and \(\bar{\alpha}_m\), respectively. As the truncation \(N\) increases from 9 to 99, \(\gamma''\) and \(\bar{\gamma}\) converge to \(-0.26\) and 3.1 respectively, but \(\gamma'\) vacillates between 2.5 and 4.4. However, the middle-level mean slope (over \(\frac{1}{4} \leq z \leq \frac{3}{4}\))

\[
k_m^{-1} = \int_{1/4}^{3/4} \Gamma' dz = \gamma_m^*(R + \Delta\theta_o) - \gamma_m'' b(R)
\]

is not sensitive to \(N\), where \(\gamma_m' = 0.25\) and \(\gamma_m'' = 3.8\) (or \(\bar{\gamma}_m = 3.5\)) for the non-modified (or modified) case. Here the function \(b = b(R)\) is defined implicitly by (7)–(8).

The results of (33) are shown in Fig. 5, which indicates that the effect of an unstable stratification \((R > 0)\) and/or warm updraught \((\Delta\theta_o > 0)\) in the interface condition tends to produce a downshear-sloping interface \((k_m > 0)\) and vice versa. The thermal effect of an unstable stratification in the vorticity equation tends to turn the interface towards the upshear direction. The latter effect is weaker than the former, so the steering-level model cannot produce upshear-sloping convection for \(R \approx 0\) and/or \(\Delta\theta_o \approx 0\). This is consistent with the results of M78.
Figure 5. The inverse of the interface slope \( (k_m > 0 \text{ is downshear sloping}) \) versus \( R \) or \( \Delta \theta_0 \) for steering-level model (see Eq. (33)). The solid curves are the analytical results for (a) \( R = 0 \), (b) \( \Delta \theta_0 = 0 \), and (c) \( R + \Delta \theta_0 = 0 \). The dashed curves are for the cases modified by Eq. (30). Numerical results are shown by the symbols: \( \circ \) for \( R = 0 \), \( \bigcirc \) for \( \Delta \theta_0 = 0 \), and \( \triangle \) for \( R + \Delta \theta_0 = 0 \).

Based on the interface dynamic condition the valid parameter range for the existence of solutions was obtained in M78 as \(-\frac{1}{4} \leq R \leq 1\). In our case, we find that the existence of solutions requires at least (see appendix A and Fig. A1)

\[
-\frac{1}{4} \leq \Delta \theta_0 \leq \frac{1}{4} \quad \text{for } R = 0 \tag{34a}
\]

\[
-0.176 \leq R \leq 2.45 \quad \text{for } \Delta \theta_0 = 0. \tag{34b}
\]

As we test our analytical results in the numerical model of finite elements, we find for the latter case \( (\Delta \theta_0 = 0) \) that the interface dynamic condition (12) cannot be satisfied for \( R \geq 0.75 \) no matter how the interface slope is adjusted (see appendix B). This may suggest that the upper bound of \( R \) is over-estimated in (34b) and a closer estimate seems to be

\[
-0.15 \leq R \leq 0.75 \quad \text{for } \Delta \theta_0 = 0. \tag{34c}
\]

Within the range of (34a) and (34c), both \( R + \Delta \theta_0 \) and \( k_m^{-1} \) (see Fig. 5) are smaller than 1, so the asymptotic expansion (15)–(16) and the analytical curves in Fig. 5 should be valid. This is verified by the numerical solutions of the finite element model and the results which are shown by the symbols in Fig. 5.

To summarize this section we propose the following intuitive explanation for the thermal effects on the interface slope. As mentioned at the beginning of this subsection, the thermal effects can be separated into two parts: the thermal energetic effect measured by \( \varepsilon E = R + \Delta \theta_0 \) in Eqs. (13) or (20) and the thermal vorticity effect measured by \( b = R/(2 \psi_m) = \varepsilon B \) in (5) or (17). The latter effect has been proven to be significantly smaller than the former. To analyse the former effect we may assume \( B = 0 \) but \( E \neq 0 \) (the exact case is \( R = 0 \) and \( \Delta \theta_0 \neq 0 \)). Note that the interface dynamic condition (12) is obtained from the continuity of the total pressure and Bernoulli theorem along the two adjacent streamlines on the two sides of the interface. The difference in \( P_E + P_o \) between these two streamlines gives the thermal term in (12). When, for example, \( \Delta \theta_0 = 0 \) and
the stratification is unstable \((R > 0)\), \(\text{PE}\) increases with height in the updraught due to
the release of buoyant energy from condensation heating. In the downdraught, \(\text{PE}\)
decreases with height due to the release of negative buoyant energy from evaporation
cooling. Thus at the lower levels, the downdraught contains high released \(\text{PE}\) and high
dynamic pressure \(p + \frac{1}{2}v^2\), whereas the dynamic pressure is low in the updraught because
\(\text{PE}\) has largely not been released. To obtain the pressure balance on the low-level interface,
the downdraught speed should be higher than that of the updraught. On the other hand,
since the thermal term is neglected \((b = 0)\) in the governing equation, both the updraught
and downdraught branches conserve vorticity. As they turn clockwise in the low-level
corner regions (see regions A and D in Fig. 2), their vorticities are largely manifested as
rotation rather than shear. In this case, to conserve vorticity, the high speed downdraught
should make a smooth turn and the low speed updraught turns sharply in the low-level
corner regions, which means a downshear interface at low levels. The upper corner flow
(regions B and C in Fig. 2) can be similarly analysed, so the interface is downshear
everywhere. The above explanation can also be easily extended to the case of \(\Delta \theta_o \neq 0\).

When \(R + \Delta \theta_o \to 0\) but \(R \neq 0\), the thermal term in the interface dynamic condition
vanishes. In this case, as indicated by Eq. \((12)\), across the interface the updraught and
downdraught have the same magnitude of velocity. If the stratification is unstable \((R > 0)\)
then the thermal term in the vorticity equation produces clockwise vorticity following
either the updraught or downdraught. In the low-level corner regions, the vorticity on
the downdraught side is higher than that on the updraught side. Since the vorticity in the
corner regions is largely manifested as rotation, in order to obtain the same magnitude
of velocity on both sides of the interface, the low-level high vorticity downdraught should
make a sharp turn, while the low vorticity updraught turns smoothly. This means an
upshear interface. Since the interface condition is directly related to the energy distri-
bution along the two sides of the interface, the thermal effect on the vorticity distribution
influences the interface slope indirectly and less effectively. In other types of interface
convective problems, the thermal energetic and vorticity effects may interact differently
in controlling the interface slope. This will be further discussed in the next section for
the mixed jump and steering-level model.

3. A mixed jump and steering-level model.

\(a\) Some qualitative analyses

It is typical in an observed squall line that the stratification is unstable \((R > 0)\) and
the moist updraught is warm \((\Delta \theta_o > 0)\) and tilts upshear while the downdraught has a
sharp turn at the low-level corner region. In this case, the downdraught speed in the low-
level corner region should be reduced due to the vorticity constraint. This means a high
pressure on the downdraught side of the interface, so the low-level interface cannot be
steady. It tends to move fast and go ahead of the convection. This leads us to consider
a new model, i.e. a mixed jump and steering-level model. In this mixed model, as the
downdraught hits the ground, it splits into two branches: a backward-turning flow and a
forward-jump flow (see Fig. 6). Similarly as the updraught hits the upper boundary it
also splits into two branches. An interface is formed between the two adjacent jump
branches (IIu and IId in Fig. 6), by which the two turning branches (Iu and Id in Fig. 6)
are further separated.

The flow properties near the interface of the mixed model can be qualitatively
analysed as follows. First we note that as the updraught turns clockwise along the low-
level concave side of the interface, its speed is reduced due to the vorticity constraint.
However, as the downdraught turns anticlockwise along the low-level interface, to satisfy the vorticity constraint its speed has to increase. The upper-level flow contains similar features. Thus as the interface becomes steeper (more vertical), its curvature increases so $\Delta(v^2)$ in Eq. (12) becomes increasingly negative (positive) in the lower (upper) levels. In this case, the interface condition (12) may be satisfied if the stratification is unstable or $\Delta \theta_o > 0$.

As the convective instability or $\Delta \theta_o$ increases, the maintenance of a steady upshear-sloping interface requires the inflow speed (in regions IIu and IIId) and shear (in regions Iu and Id) to increase. Otherwise the (upshear) interface slope will increase. Thus, upshear-sloping convection may exist in a mixed jump and steering-level model, whose interface slope depends on the competition between these thermal and kinetic effects.

**(b) Model formulation**

In the steering-level model of Moncrieff (1978), the condensational heating (evaporative cooling) in the moist updraught (downdraught) was taken into account by an unstable stratification and the inflow region is uniformly convectively unstable. The real atmosphere is conditionally unstable, finite rather than infinitesimal lifting being required to release potential energy. An observational parameter measuring this property is the lifting condensation level (LCL). For the mid-latitude storms the LCLs are usually higher than the top of the PBL and sometimes reach the middle troposphere. However, the LCLs in tropical storms are often very low and contiguous to the top of PBL or mixing layer. To include these features, we consider the following nondimensional form of $\theta(\psi, z)$ in the updraught branch:

$$\theta = \begin{cases} 
R(z-z_c)\psi/\psi_m & \text{for } z > z_c \\
0 & \text{for } z \leq z_c 
\end{cases} \quad (35)$$

where $R$ is the Richardson number similar to that in Eq. (4), $z_c$ represents the LCL and $\psi_m$ is the value of $\psi$ along the interface between region IIu and region IIId (see Fig. 6). The scaling is the same as in the previous section. In Eq. (35) the low-level inflow below the LCL is assumed to be neutrally stratified, while above the LCL up to the top level the moist stratification is unstable, i.e. $N^2 z_c = -4A^2 R < 0$.

As with Eqs. (4)–(5), the following nondimensional vorticity equation is obtained, corresponding to (35):

$$\nabla^2 \psi = \begin{cases} 
1 + R(z-z_c)^2/(2\psi_m) & \text{for } z > z_c \\
1 & \text{for } z \leq z_c 
\end{cases} \quad (36)$$
Solving (36) for the remote flow at \( x \rightarrow \infty \) and \( 1 \geq z \geq z_a \) (see Fig. 6), we obtain

\[
\begin{align*}
\mathbf{u} &\rightarrow u_\infty(z) = \begin{cases} 
\alpha_1(z-z_u) + \alpha_2(z-z_u)^2 + \alpha_3(z-z_u)^3 & \text{for } z \leq z_c \\
\alpha_4 + z - z_c & \text{for } z_a \leq z \leq z_c
\end{cases} \\
\psi &\rightarrow \psi_\infty(z) = \begin{cases} 
\frac{1}{4} \alpha_1(z-z_u)^2 + \frac{1}{4} \alpha_2(z-z_u)^3 + \frac{1}{4} \alpha_3(z-z_u)^4 & \text{for } z \leq z_c \\
\alpha_5 + \alpha_4(z-z_c) + \frac{1}{2}(z-z_c)^2 & \text{for } z_a \leq z \leq z_c
\end{cases}
\end{align*}
\]

as \( x \rightarrow \infty \). \hspace{1cm} (37)

where \( \alpha_1 = 1 + bh^2, \alpha_2 = bh, \alpha_3 = b/3, \alpha_4 = -h - bh^2/3, \alpha_5 = h^2(1 + bh^2/2)/2, b = R/(2\psi_m) \), and \( h = z_a - z_c \). Solving for the remote flow at \( x \rightarrow -\infty \) and \( 1 \geq z \geq 1 - z_a \), we obtain

\[
\begin{align*}
\mathbf{u} &\rightarrow u_\infty(z) = u_e + \beta_1 \xi + \beta_2 \xi^2 + \beta_3 \xi^3 & \text{as } x \rightarrow -\infty \\
\psi &\rightarrow \psi_\infty(z) = \psi_m + u_e \xi + \frac{1}{2} \beta_1 \xi^2 + \frac{1}{3} \beta_2 \xi^3 + \frac{1}{4} \beta_3 \xi^4 & \text{as } x \rightarrow -\infty
\end{align*}
\]

(39) \hspace{1cm} (40)

where \( \xi = z - 1 + z_a, \beta_1 = 1 + bh_1^2, \beta_2 = bh_1, \beta_3 = b/3, h_1 = 1 - z_a - z_c \), and \( u_e < 0 \) is the outflow velocity along the interface between IIu and IIId. We choose \( R, z_a \) and \( z_c \) as external parameters. The internal parameters \( z_u \) (the steering level), \( u_e \) and \( \psi_m \) can be solved from the following three constraints:

\[
\begin{align}
\psi_\infty(1) &= \psi_\infty(1) \\
\psi_m &= \psi_\infty(z_a) \\
u_e^2 &= u_\infty^2(z_a) + Rh_1^2
\end{align}
\]

(41) \hspace{1cm} (42) \hspace{1cm} (43)

where functions \( \psi_\infty( ) \), \( \psi_\infty( ) \) and \( \psi_\infty( ) \) are defined in Eqs. (37), (38) and (40), respectively. Obviously Eq. (41) is due to the mass continuity or specifically due to continuity of the streamline along the boundaries. Equation (42) is the definition of \( \psi_m \). The energetic constraint (43) is obtained from the Bernoulli equation along the streamline from \( (\infty, z_a) \) to \( (-\infty, 1 - z_a) \). Since this streamline is along the interface and the updraught and downdraught are coupled antisymmetrically, pressure perturbations at \( (\infty, z_a) \) and \( (-\infty, 1 - z_a) \) are cancelled in the derivation of (43).

The interface dynamic condition is (with \( \Delta \theta_o = 0 \))

\[
\Delta(\nabla^2) = R[E(z - z_c) - E(1 + z_c - z)]
\]

(44)

where

\[
E(\xi) = \begin{cases} 
\frac{1}{2} \xi^2 & \text{for } \xi > 0 \\
0 & \text{for } \xi \leq 0.
\end{cases}
\]

Equation (36) with the boundary conditions

\[
\begin{align*}
\psi &= \psi_\infty(1) & \text{on } z = 0, 1 \\
\psi &= \psi_m & \text{along the interface between IIu and IIId}
\end{align*}
\]

(45) \hspace{1cm} (46)

and Eqs. (38), (40) and (44), and the similar equation and boundary conditions for the downdraught branch give a complete set of equations for the free internal boundary problem. Here the boundary conditions (38) and (40) are equivalent to

\[
\frac{\partial \psi}{\partial x} \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.
\]

(47)
Figure 7. Solutions for the steady 2-D mixed jump and steering-level model:

(a) $R = 0.2$, $z_c = 0.3$ and $z_s = 0.1$  
(b) $R = 1.0$, $z_c = 0.3$ and $z_s = 0.1$

(c) $R = 4.0$, $z_c = 0.3$ and $z_s = 0.1$  
(d) $R = 1.0$, $z_c = 0.1$ and $z_s = 0.1$

(e) $R = 1.0$, $z_c = 0.5$ and $z_s = 0.1$  
(f) $R = 1.0$, $z_c = 0.3$ and $z_s = 0.08$

(g) $R = 1.0$, $z_c = 0.3$ and $z_s = 0.2$. The contour intervals for $\psi$ are 0.01.
For the numerical computation, (47) is more convenient than (38) and (40). The detailed method of numerical solution is described in appendix C.

(c) Model results

Examples of internal flow solutions are shown in Figs. 7(a)–(g). These solutions support our previous qualitative analyses and indicate the existence of upshear-sloping convection in a mixed jump and steering-level model. By choosing the Richardson number $R$, the LCL $z_c$ and the depth of the jump outflow $z_a$ as the three independent parameters, the dependence of the interface slope on these parameters was examined. The results are plotted in Figs. 8(a) and (b), where $k$ measures the interface slope (with respect to the horizontal coordinate $x$) and the interface is found approximately linear over the depth of $b$ (see Eq. (C1) of appendix C). As $k$ increased beyond the range of Fig. 8, it was found that (C1) became a poor approximation or even invalid. (In this case, either the mean square root of the error of Eq. (47) became large or the mean value of the error over the upper or lower half part of the interface could not be adjusted to become small through changing $k$ and $b$.) However, qualitatively it was observed that the interface slope $k$ increased continuously as the jump outflow became shallower ($z_a \leq 0.08$), the LCL became lower ($z_c \leq 0.1$), or $R$ increased ($>4$). In the limiting case of $z_a \rightarrow 0$ ($R > 0$) our mixed model degenerates into one of the steering-level type and the interface becomes downshear-sloping. Thus the general features can be drawn as follows: the interface slope $k$ increases (towards vertical or even downshear-sloping) as (i) $z_a$ decreases, (ii) $z_c$ decreases, or (iii) $R$ increases (except for $z_c \geq 0.5$). These features corroborate the analyses in section 3(a) and thereby can be explained by those analyses.

The dashed curves in Fig. 8(a) are obtained for the modified cases where the thermal vorticity effect is excluded by setting $R = 0$ in (36) but $R \neq 0$ in (44). These modified cases are similar to the exact cases where $R = 0$ but $\Delta \theta_a \neq 0$. For the latter, the r.h.s. of (44) is replaced by the r.h.s. of (11), so the thermal vorticity effect cannot be precisely extracted from the comparison between these and the above unmodified cases. Thus, to extract the thermal vorticity effect precisely, we use the modified cases. The dashed curves in Fig. 8(a) indicate that the interface will become more vertical, i.e. less upshear,
Figure 8. Interface slope ($k > 0$ is upshear sloping) against $R$: (a) with $z_a = 0.1$ and different values of $z_i$; (b) with $z_e = 0.3$, and different values of $z_a$. The dashed curves in (a) are for the modified cases with thermal vorticity effect neglected, i.e., $R = 0$ in (36).

If the thermal vorticity effect is absent. Thus, as with the steering-level model, the thermal vorticity effect here is also in favour of producing an upshear-sloping interface. However, unlike the steering-level model, the differences between the dashed and solid curves in Fig. 8(a) indicate that the thermal vorticity effect can be crucial in maintaining an upshear-sloping structure if $R$ is large and/or the LCL is low. Obviously, the thermal vorticity
effect is more effective in the mixed jump and steering-level model than in the previous steering-level model. The reason for this can be understood as follows. First we note from Eqs. (5) and (36) that the two models will have the same vorticity z-distribution along the interface if their heating profiles are the same, i.e. $z_c = 0$ in Eq. (36). With the same thermal vorticity increment, the moist updraught in the mixed (or steering-level) model turns anticlockwise (or clockwise) in the upper levels, so the updraught speed along the upper-level interface increases rapidly (or slowly) due to the vorticity constraint and tends to produce a strong (or weak) Bernoulli pressure deficit balancing the Bernoulli pressure surplus due to the thermal energy effect more (or less) effectively. In a more realistic non-steady model such as that of Thorpe et al. (1982), the Bernoulli pressure deficit produced by the thermal vorticity effect might be important in offsetting the Bernoulli pressure surplus due to the thermal energy effect thus preventing an early collapse of the upshear-sloping interface.

4. DISCUSSION

It is interesting to compare our mixed model with the idealized model proposed in Fig. 12 of Thorpe et al. (1982). The models contain similar jump updraughts in combination with an overturning updraught. But in their model there was no jump downdraught, the interface was between the jump updraught and overturning downdraught. However, in the conceptual model summarized in their Fig. 15 there was a rotor or vortex on the downdraught side of the low-level interface. Since their idealized and conceptual models were based on the time-average structure of numerical simulation of 2-D viscous deep convection, which is obviously different from our inviscid model in many aspects, the above difference in low-level downdraught structure may be understood as follows. First, we note that our model is antisymmetric, so the jump downdraught has to exist as long as the jump updraught exists. If the flow is assumed to be non-antisymmetric with a strong low-level inflow for the moist updraught, then according to the analysis in section 2(c) the jump downdraught may become weak or even vanish. On the other hand, if we take a different look at the instantaneous flow field in their Figs. 4(a) and (b), we may observe that there was an early downdraught split near the ground in their Fig. 4(a), but the jump downdraught did not form later in their Fig. 4(b). As this downdraught branch confronted the updraught inflow, it overturned and formed a rotor, which in turn cut off the feeding flow from the downdraught. Note that both the updraught inflow above the rotor and the ground underneath the rotor move fast towards the rear of the storm, so the pressure gradient between the updraught inflow (with low pressure) and the downdraught overturning flow (with high pressure) could be effectively balanced by the friction on the rotor. However, this frictional effect near the surface and interface is absent in our inviscid model. With this understanding, we believe that the quasi-steady convection produced in their model is partially reproduced in our model. Apparently, in order to obtain steady 2-D convection similar to that of Thorpe et al. in all major aspects, we need to improve our current model by including at least (a) non-antisymmetry and (b) viscosity. This will be considered in the future.

Obviously the model results of Thorpe et al. are more realistic than ours. But our model has the merit of being simple; it may provide a further insight into the internal flow regime of 2-D quasi-steady convection. According to Thorpe et al., to maintain a steady 2-D convection both a shallow downdraught and non-constant shear are necessary, specifically the upper-level shear should be weak while the low-level shear should be strong. Now, these conditions may be related to the dependence of the interface slope on the external parameters obtained in this paper as follows. (i) First we know that the
maintenance of 2-D quasi-steady convection calls for the updraft to keep upshear-sloping. By our model results, this in turn requires the updraft outflow to split into two branches, i.e. a jump outflow and an overturning outflow as it is obstructed by the upper boundary or a stable layer. If the upper-level shear is strong and in the same (or opposite) direction to the low-level shear in a pre-storm environment, then it will be difficult to form a jump (or overturning) updraft outflow. This may explain why the upper-level shear should be weak in the pre-storm environment. (ii) Furthermore, we have seen that the interface in our model tends to be vertical or downshear-sloping when the heating and cooling layers \(1 - z_c\) become deep. This might be related to the necessity of a shallow downdraught because in real storms the downdraught depth is actually controlled by the cooling layer. (iii) The dependence of the interface slope on the Richardson number (proportional to the ratio between CAPE and the inflow kinetic energy) indicates that to maintain an upshear-sloping updraft the inflow has to be strong. When the downdraught jump outflow is realistically resisted by surface friction and eddy viscosity, the above condition is favoured by strong low-level shears in the ambient flow.

The above results may also have a connection with the observed differences between the tropical fast propagating squall lines and mid-latitude steering-level-type squall lines. For the tropical squall lines, the LCL is low, CAPE, and thus the Richardson number, are large, so neither steering-level-type convection nor the mixed jump and steering-level model can develop favourably because the moist updraft tends to become downshear. This kind of environment may be in favour of the fast-propagating 3-D jumping convection suggested by Moncrieff (1981). For the mid-latitude squall lines, the LCL is high and the Richardson number is not too large, so it is in favour of an upshear mixed jump and steering-level convection. In this case, the forward propagating gust front outflow is a crucial factor for the maintenance of the upshear-sloping structure.

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Appendix A

Method of analytical solutions for the steering-level model

There are three particular solutions of \(\psi^{(1)}\) corresponding to the three forcing terms on the r.h.s. of Eq. (17): \(\psi^{(1)}_{\psi_1}, \psi^{(1)}_{\psi_2}\) and \(\psi^{(1)}_{\psi_3}\). Obviously \(\psi^{(1)}_{\psi_1} = \psi^{(1)}(\xi)\), where \(\psi^{(1)}(\xi)\) is given in Eq. (19). Substituting (14) and (21) into the last two forcing terms of (17) gives

\[
\frac{2}{\frac{d \Gamma^{(0)}}{d \xi}} \frac{d^2 \psi^{(0)}}{d \xi^2} + \frac{d^2 \Gamma^{(0)}}{d \xi^2} \frac{d \psi^{(0)}}{d \xi} = \\
-\pi^2 \sum_{m=1}^{\text{odd}} \sum_{n=1}^{\text{odd}} n a_n \alpha_n e^{-n \pi \xi} \sin(n + m) \pi \xi - (n - m) \sin(n - m) \pi \xi 
\]

(A.1)

where \(\alpha_n\) is given by Eq. (23). Thus, the particular solutions are
\[ \psi^{(1)}_{\xi_2} + \psi^{(1)}_{\xi_3} = \sum_{m=1}^{\text{odd}} \sum_{n=1}^{\text{odd}} n a_n \alpha_m \left[ \sin(n + m) \pi \xi + \sin(n - m) \pi \xi \right] e^{-n \pi \xi}. \]

(A2)

These particular solutions satisfy the boundary condition of Eq. (18) only on the rigid boundaries \( \xi = 0 \) and 1 but not on the interface \( \xi = 0 \). To satisfy the latter, the corresponding homogeneous solutions are

\[ \psi^{(1)}_{\xi_1} = -B \sum_{m=1}^{\infty} f_m e^{-m \pi \xi} \sin m \pi \xi \]

(A3)

\[ \psi^{(1)}_{\xi_2} + \psi^{(1)}_{\xi_3} = \sum_{m=1}^{\text{odd}} \sum_{k=1}^{\text{even}} (k - m) a_{k-m} \alpha_m e^{-k \pi \xi} \sin k \pi \xi - \]

\[ - \sum_{m=1}^{\text{odd}} \sum_{k=1}^{\text{even}} (k + m) a_{k+m} \alpha_m e^{-k \pi \xi} \sin k \pi \xi \]

\[ = - \sum_{k=2}^{\infty} \sum_{m=1}^{\text{even}} [(k + m) a_{k+m} + |k - m| a_{k-m}] \alpha_m e^{-k \pi \xi} \sin k \pi \xi \]

(A4)

where \( f_m \) is given by (24). In (A4) the following rules are used for the exchange of the summations:

\[ \sum_{m=1}^{\text{odd}} \sum_{k=1}^{\text{even}} (k, m) = \sum_{k=2}^{\text{even}} \sum_{m=1}^{\text{even}} (k, m) \]

(A5)

\[ \sum_{m=1}^{\text{odd}} \sum_{k=1}^{\text{even}} (k, m) = \sum_{m=1}^{\text{odd}} \left( \sum_{k=1}^{\text{even}} + \sum_{k=1}^{\text{odd}} + \sum_{k=1}^{\text{even}} \right)(k, m) \]

\[ = \sum_{m=1}^{\text{odd}} \sum_{k=2}^{\text{even}} [(-k, m) + (k, m)] + \sum_{m=1}^{\text{even}} \sum_{k=1}^{\text{even}} (k, m) \]

\[ = \sum_{k=2}^{\text{even}} \sum_{m=1}^{\text{even}} [(-k, m) + (k, m)] + \sum_{k=2}^{\text{even}} \sum_{m=1}^{\text{even}} (k, m) \]

(A6)

where \( \Sigma_{k=0}^{\infty} (k, m) = 0 \) because \( (k, m) \) contains \( \sin k \pi \xi \). Replacing \( (k, m) \) with \( (m, n) \) in (A4) and substituting the expansion of \( \psi^{(1)}(\xi) \) and (A2)–(A4) into

\[ \psi^{(1)} = \psi^{(1)}_{\xi_1} + \psi^{(1)}_{\xi_2} + \psi^{(1)}_{\xi_3} + \psi^{(1)}_{h_1} + \psi^{(1)}_{h_2} + \psi^{(1)}_{h_3} \]

gives Eq. (22), i.e. the solution of (17)–(18).

To satisfy the interface dynamic condition (20), we substitute (14) and (22) into \( w^{(0)} = -\partial \psi^{(1)}/\partial \xi + O(\varepsilon) \) and \( w^{(1)} = -\partial \psi^{(1)}/\partial \xi + O(\varepsilon) \) respectively, and then into the l.h.s. of (20):

\[ \Delta(w^{(0)}w^{(1)}) = \text{the antisymmetrical part of } 2(w^{(0)}w^{(1)})|_{\xi=0} \]

(A7)

where \( M_{kn} \) and \( d^n_k \) are given by Eqs. (26)–(27) and rules for the exchanges of summations similar to (A5)–(A6) are used. Projecting the r.h.s. of (20) onto the Fourier space \{\cos k \pi \xi\} and comparing with (A7) gives Eq. (25), i.e. the equation for the Fourier coefficients of the interface geometry.
Figure A1. The estimated domain of \((R, \Delta \theta_0)\) for which solutions exist. The solid (dashed) boundaries correspond to \(\gamma_0 = \gamma_1 = 1.0\) (not 0.8) in Eqs. (A9) and (A11).

The domain of \((R, \Delta \theta_0)\) for which solutions exist can be estimated as in M78. For \(R + \Delta \theta_0 < 0\), the interface is upshear, so the upper bound of \(\Delta(v^2)\) can be estimated as

\[
\Delta(v^2) \leq u^2|z = 0, u_0 = \gamma_0 u_0^2(0) = \gamma_0 (\alpha_1 z_u - \alpha_2 z_u^2 + \alpha_3 z_u^3)^2
\]

(A8)

where \(\alpha_1, \alpha_2\) and \(\alpha_3\) are given in Eq. (6). Substituting (A8) into (12) with \(z = 0\) gives

\[
R + \Delta \theta_0 \geq -\gamma_0 (z_u + \frac{1}{2} b z_u^2)^2.
\]

(A9)

For \(R + \Delta \theta_0 > 0\), the upper bound of \(\Delta(v^2)\) is estimated at \(z = 1\) as

\[
\Delta(v^2) \leq u^2|z = 0, u_1 = \gamma_1 u_1^2(1) = \gamma_1 (\alpha_1 h_u^2 + \alpha_2 h_u^2 + \alpha_3 h_u^3)
\]

(A10)

where \(h_u = 1 - z_u\). Substituting (A10) into (12) with \(z = 1\) gives

\[
R + \Delta \theta_0 \leq \gamma_1 [1 - z_u + \frac{1}{2} b (1 - z_u^3)]^2.
\]

(A11)

Since both \(z_u\) and \(b\) are functions of \(R\) defined implicitly by (7)–(8), (A9) and (A11) specify a domain of \((R, \Delta \theta_0)\) in Fig. A1 for the existence of solutions. If the kinetic energy at the lower (upper) corner of the updraught equals the inflow (outflow) kinetic energy on \(z = 0\) (\(z = 1\)) for the case of \(R + \Delta \theta_0 < 0\) (\(>0\)), then \(\gamma_0 = \gamma_1 = 1\). The solid and dashed boundaries of the domains in Fig. A1 correspond to \(\gamma_0 = \gamma_1 = 1\) and \(\gamma_0 = \gamma_1 = 0.8\) respectively. For two particular cases of either \(R = 0\) or \(\Delta \theta_0 = 0\), we obtain Eqs. (34a) and (34b) with \(\gamma_0 = \gamma_1 = 1\). As shown by both the analytical and numerical solutions, \(\gamma_0\) and \(\gamma_1\) may be close to 1 but never equal to 1. For \(\Delta \theta_0 = 0\) and \(R > 0.75\), it is found that the interface condition cannot be satisfied by the numerical solution of (17)–(18) no matter how the interface is adjusted (see appendix B). This may suggest that \(\gamma_0 = \gamma_1 = 0.8\), which gives the dashed boundaries in Figs. A1 and (34c). Furthermore, \(\Delta \theta_0\) is new in our model compared with M78, so we need to know the global range of \(R\) for all possible values of \(\Delta \theta_0\) in which solutions exist. From Eqs. (6)–(8) we obtain

\[
b = 3(z_u - \frac{1}{2})(1 - z_u^3)^{-1}
\]

\[
\psi_m = \frac{1}{2} z_u^2 + \frac{1}{2} b z_u^4 \quad \text{and} \quad R = 2b \psi_m.
\]

(A12)

Thus, we may choose \(z_u\) as a free parameter and obtain \(b\), \(\psi_m\) and \(R\) explicitly. If \(0 < z_u < 0.365\), then \(0 > R \geq -0.23\) and there are two steering levels associated with a secondary upper-level reverse cell. When \(z_u\) increases from 0.365 to \(4^{-1/3}\), there is only one steering-level \(z_u\), and \(R\) increases from \(-0.23\) to \(\infty\). If \(4^{-1/3} < z_u < 1\), then \(\infty > R > 0\).
\( \psi_m \) becomes negative (indicating reverse circulation) and there are again two steering levels with a secondary low-level reverse cell (after \( z_u > 0.7 \)). Therefore, there is no solution for \( R < -0.23 \). Two solutions exist for \( R \geq -0.23 \), but only one of them has a single steering level (0.365 < \( z_u < 4^{-1/3} \)).

APPENDIX B

**Finite element solution for the steering-level model**

First we estimate the interface geometry \( x = \Gamma(z) \) from the analytical results (21), (23) and (29) for given parameters \( R \) and \( \Delta \theta_o \). Then we can construct a set of finite elements compatible with the interface shape (see Figs. 2(b) and 4(b), (d)) and solve Eqs. (5)–(7) numerically by using the methods described later. This numerical solution is further checked with the interface dynamic condition (12) or (13) and the integral mean \( (m_1) \) and mean square root \( (m_2) \) of the relative error of the interface dynamic condition is calculated:

\[
m_1 = 2 \int_0^{1/2} \Delta(\tilde{\psi}^2 - PE - P_o)dz/(\tilde{\psi}_{\text{max}}^2) \quad (B1)
\]

\[
m_2 = \left\{ \int_0^1 \left[ \Delta(\tilde{\psi}^2 - PE - P_o) \right]^2 dz \right\}^{1/2} / (\tilde{\psi}_{\text{max}}^2) \quad (B2)
\]

where \( \tilde{\psi}_{\text{max}} = \max_{\delta} \tilde{\psi} \{ \Gamma(z), z \} \), i.e. the maximum \( \tilde{\psi} \) along the interface. Obviously, \( m_1 \) will change sign if the integration interval in (B1) is replaced by \([1, 1]\). Thus, \( m_1 \) measures the overall torque on the interface due to the unbalanced Bernoulli pressure across the estimated interface while \( m_2 \) measures the detailed and local imbalance of the Bernoulli pressure across the interface.

Based on the physical understanding of the flow properties associated with the interface slope (see section 2(c)), we can adjust the overall slope of the interface and bring \( m_1 \) down to zero. This was done by multiplying an adjustable coefficient \( \gamma \) to Eq. (21), i.e.

\[
x = \gamma \Gamma^{(1)}(z). \quad (B3)
\]

When \( m_1 > 0 \) (or \( < 0 \)), \( \gamma \) should be adjusted such that the interface turns towards the downshear (or upshear) direction. Practically we stopped the adjustment as \( |m_1| \leq 0.001 \). We found that as \( |m_1| \) was adjusted to become small, \( m_2 \) also became small for the cases shown in Fig. 5. As \( |m_1| \leq 0.001 \), we obtain \( m_2 \leq 0.008 \) for most cases (except for \( R \) and/or \( \Delta \theta_o < 0 \) but \( R + \Delta \theta_o \neq 0 \)) and \( m_2 < 0.015 \) for all cases in Fig. 5.

For an initially estimated or subsequently adjusted interface geometry as shown by Eq. (B3), system (5)–(8) was solved by using the finite element method. In order to allow for more accurate representation of the shape of the free interfacial boundary, biquadratic finite elements with curved sides were employed (cf. Figs. 2(b) and 4(b), (d)). The Galerkin approach was utilized in the finite element formulation so that the element shape functions were identical with the weighting functions in making the weighted residual vanish.

A local coordinate system \( (\xi, \eta) \) defined in each element was exploited for convenience because the working units are individual elements in the finite element method. This local coordinate system was normalized in the sense that \( \xi = \pm 1 \) and \( \eta = \pm 1 \) gave the four sides of an element. The concept of the isoparametric elements was used such that the global coordinates \( x \) and \( z \) were expanded in the same shape functions for
expanding the dependent variable $\psi$. In this manner, spatial derivatives with respect to the global coordinates were expressed as derivatives with respect to the local coordinates and the inverse of the Jacobian matrix $\partial(x, z)/\partial(\xi, \eta)$. The integrals resulting from the Galerkin procedure were numerically evaluated using the Gaussian quadrature formula which has an accuracy of $2n - 1$ where $n$ is the number of Gaussian points in one dimension.

To illustrate how the Galerkin finite element equations were obtained, we first define the nine shape functions in $(\xi, \eta)$ as follows:

$$
N_1(\xi, \eta) = \frac{1}{2} \eta (\xi - 1)(\eta - 1), \quad N_2(\xi, \eta) = \frac{1}{2} \xi (\xi + 1)(\eta - 1), \\
N_3(\xi, \eta) = \frac{1}{2} \xi (\xi + 1)(\eta + 1), \quad N_4(\xi, \eta) = \frac{1}{2} \xi (\xi - 1)(\eta + 1), \\
N_5(\xi, \eta) = \frac{1}{2} \eta (\eta - 1)(1 - \xi^2), \quad N_6(\xi, \eta) = \frac{1}{2} \xi (\xi + 1)(1 - \eta^2), \\
N_7(\xi, \eta) = \frac{1}{2} \eta (\eta + 1)(1 - \xi^2), \quad N_8(\xi, \eta) = \frac{1}{2} \xi (\xi - 1)(1 - \eta^2), \\
N_9(\xi, \eta) = (1 - \xi^2)(1 - \eta^2).
$$

It is seen that these shape functions are non-zero only in a small span of the nodes they are attached to. Mathematically, this is reflected by the fact that

$$
N_i(\xi, \eta) = \begin{cases} 1 & \text{at node } i \\ 0 & \text{at other nodes.} \end{cases}
$$

The dependent variable $\psi$ and the independent variables $x$ and $z$ can all be expanded in the nine shape functions in each element as follows:

$$
x = \sum_{i=1}^{9} \hat{x}_i N_i(x, z) \quad \text{or} \quad \sum_{i=1}^{9} \hat{x}_i N_i(\xi, \eta) \\
z = \sum_{i=1}^{9} \hat{z}_i N_i(x, z) \quad \text{or} \quad \sum_{i=1}^{9} \hat{z}_i N_i(\xi, \eta) \\
\psi = \sum_{i=1}^{9} \hat{\psi}_i N_i(x, z) \quad \text{or} \quad \sum_{i=1}^{9} \hat{\psi}_i N_i(\xi, \eta).
$$

Substitute these expressions into Eq. (5) and make the weighted residual vanish with the shape functions $\{N_i\}$ also used as the weighting functions:

$$
\int_{\Omega_e} \left[ N_i \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} - F(x, z) \right) \right] dx \, dz = 0.
$$

Here $\Omega_e$ represents the domain of an element and $F(x, z)$ is the r.h.s. of Eq. (5). Now, if we perform integration by parts to the second derivative terms, we get
\[
\int_{\Gamma_e} N_i \frac{\partial \psi}{\partial n} \, dl = \\
\int \int_{\Omega_e} \left\{ \psi_i \left( \frac{\partial N_j(x, z)}{\partial x} \frac{\partial N_i(x, z)}{\partial x} + \frac{\partial N_i(x, z)}{\partial z} \frac{\partial N_j(x, z)}{\partial z} \right) + N_j(x, z) F(x, z) \right\} \, dx \, dz
\]

where \( \Gamma_e \) represents the boundary of the element. Rewrite this equation in the \((\xi, \eta)\) coordinate system:

\[
\int \int_{\Omega_e} \left\{ \psi_i \frac{\partial N_i(\xi, \eta)}{\partial x} \frac{\partial N_j(\xi, \eta)}{\partial x} + \frac{\partial N_i(\xi, \eta)}{\partial z} \frac{\partial N_j(\xi, \eta)}{\partial z} + N_j(\xi, \eta) F(\xi, \eta) \right\} |J| \, d\xi \, d\eta = \\
\int_{\Gamma_e} N_i(\xi, \eta) \frac{\partial \psi}{\partial \eta} \, dl.
\]

Terms \( \partial N_i(\xi, \eta) / \partial x \) and \( \partial N_i(\xi, \eta) / \partial z \) are found from

\[
\partial N_i(\xi, \eta) / \partial (x, z) = J^{-1} \partial N_i(\xi, \eta) / \partial (\xi, \eta)
\]

where \( J^{-1} \) is the inverse matrix of the Jacobian matrix

\[
J = \begin{pmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta}
\end{pmatrix}.
\]

The resultant integrals can be evaluated numerically, and the element-level Galerkin equations are then assembled to form the global algebraic system. Solution of the global algebraic system will yield nodal values of \( \psi \) for the entire discretized domain. After this is done, velocity components \( w = -\partial \psi / \partial x \) and \( u = \partial \psi / \partial z \) along the free boundary can be found from

\[
\frac{\partial \psi}{\partial x} = \sum_{i=1}^{9} \hat{\psi}_i \frac{\partial N_i}{\partial x} \quad \text{and} \quad \frac{\partial \psi}{\partial z} = \sum_{i=1}^{9} \hat{\psi}_i \frac{\partial N_i}{\partial z}.
\]

These values will be used to check the interface dynamic condition (12).

**Appendix C**

*Method of numerical solution for the mixed model*

To solve (36) with (38), (40)–(46) numerically, the interface shape (between regions IIa and IIb) should be estimated first. After this, \( \psi \) can be solved by successive relaxation of (36) subject to (45)–(47). From this solution the interface dynamic condition (44) should be checked and the interface shape can be adjusted based on the internal flow properties discussed in the previous sections. In principle, this procedure can be repeated until (44) is satisfied to any required accuracy. In practice, however, it is difficult for (44) to reach a high accuracy. First we note that \( \psi^2 \) in (44) can be calculated only by one-side differencing, so to ensure its accuracy (\( \Delta x, \Delta z \)) should be small enough. On the other hand, in difference form the accuracy of \( v = j \times \nabla \psi \) is lower than \( \psi \) by at least a factor of \( \max(1/\Delta x, 1/\Delta z) \) and will become even worse if the interface has to be adjusted away from the grid points. To remedy the latter problem, we introduce the following coordinate transformation: \( (x, z) \rightarrow (\xi, z) \), where the curve \( z = -\xi(x) \) in \((x, z)\) space is described by the estimated interface. In \((\xi, z)\) space the interface is always \( z = -\xi \) (independent of the interface adjustment procedure), while the infinite layer in \((x, z)\) space is transformed into a rectangular domain \((z_a < \xi < 1 - z_a, 0 \leq z \leq 1)\) in \((\xi, z)\)
space. By choosing $\Delta \xi = \Delta z$, the interface is exactly expressed by the grid points on the diagonal line $z = -\xi$. In these new coordinates, the two differential terms in (36) and (44) change their forms, i.e.

$$\frac{\partial^2}{\partial x^2} \rightarrow \frac{\partial^2}{\partial \xi^2} - \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi}$$

where $\xi = dx/d\xi$ and $\xi = d^2x/d\xi^2$. Thus when the interface is adjusted, only the functional coefficients in the above two differential terms are subject to change. This treatment greatly simplifies the interface adjustment procedure and prevents the interface from deviating from the grid points.

In discrete form, the number of degrees of freedom for the interface adjustment can be as large as the number of the internal grid points along the $\xi$ coordinate. In practice, it is almost impossible to adjust too many independent parameters as the interface shape is being searched. On the other hand, we note that the right hand side of Eq. (44) is a smooth function, and so the variation of $v$ along the interface and the interface shape should also be smooth. Thus, if the interface can be approximated by a smooth piecewise—analytical function with a few independent parameters, then the adjustment procedure can be greatly simplified. Having tested several different formulations, we found that for a moderately wide range of parameters $R$, $z_c$, and $z_a$ the interface can be fairly well approximated by

$$z = -\xi(x) = \begin{cases} -kx & \text{for } |x| \leq b/k, \\ -\left\{ b + \frac{1}{\lambda} \arctan[\lambda(k|x| - b)] \right\} \text{sgn}(x) & \text{for } |x| > b/k \end{cases}$$

(1)

where $\lambda = \pi/(1 - 2z_a - 2b)$. There are only two independent parameters in (1) which control the shape of the interface: $k$ controls the interface slope and $b$ ($0 \leq b \leq 0.5 - z_a$) controls the overall distribution of the interface curvature. The interface was adjusted by changing these two parameters until the dynamic interface condition (44) was best satisfied by the following two criteria: (i) the integral mean value of the relative error of (44) over the upper (or lower) half of the interface, i.e. between $z_a < z < 0.5$ (or $0.5 \leq z < 1 - z_a$) (see Eq. (1B) also) is zero; (ii) the integral mean square root of the relative error of (44) over the whole interface ($z_a < z < 1 - z_a$) (see (B2) also) reaches the minimum. The results in Figs. 7 and 8 were obtained with this method and the integral mean square roots of the relative errors of (44) over the interface were less than 0.01 for most cases and less than 0.05 for all cases in Figs. 7(a)–(g) and 8(a) and (b).

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