A two-dimensional model of mesoscale frontogenesis in the ocean

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SUMMARY

A two-dimensional primitive equation model conserving potential vorticity is used to simulate the effect of a horizontal barotropic deformation field on a pre-existing isopycnic potential vorticity gradient. The novel feature of the prognostic model is that it accurately describes the development of the density and velocity distributions up to local Rossby number 0.5 and beyond. The model coordinates are based in the vertical on quasi-isopycnic vertical coordinates and in the horizontal on geostrophic coordinates. The model is integrated forward in time for three days creating a density front and a jet of 0.4 m s⁻¹ in the surface layers. Cross-jet velocities are one order of magnitude smaller. Conservation of potential vorticity leads to a modulation of spacing between isopycnals as both the cyclonic and anticyclonic vorticity associated with the horizontal shear of the accelerating jet increase. Vortex stretching and compression sustain a strong vertical circulation. The paper shows how the jet kinetic energy is limited by the amount of available potential energy in a catchment area defined by the extent of the region of confluence. The jet becomes shallower as frontogenesis approaches this limit.

The model is also applicable to the development of permanent streams in the large-scale circulation of the upper ocean, by slow (order one month) frontogenesis due to gyre-scale deformation acting on an isopycnic potential vorticity gradient in the seasonal thermocline.

The vigorous upwelling on the anticyclonic side of the jet raises nutrients through the euphotic zone, increasing primary productivity. This explains observed mesoscale modulations of biomass.

1. INTRODUCTION

This paper is concerned with the local dynamics of mesoscale fronts in the ocean, with particular emphasis on the seasonal thermocline. We describe a primitive equation model designed to predict the structure of such fronts in detail as the relative vorticity, $\zeta$, becomes comparable to the Coriolis frequency, $f$, i.e. as the Rossby number reaches order one. The model is designed to conserve isopycnic potential vorticity (IPV). The most unusual feature of the model is a distribution of grid points that changes at every time step, in a quasi-Lagrangian way. The grid points float up and down with isopycnals, and are advected horizontally with the barotropic convergence forcing frontogenesis. The time step used to solve the prognostic equations varies accordingly. The scheme achieves high spatial resolution with computational economy. In view of this novel method, we describe the model equations, boundary conditions, and structure in some detail before presenting the results for a particular choice of initial conditions. Our first task, however, is to set the scene by discussing the general problem of mesoscale fronts. We start by defining mesoscale in the context of the spectrum of oceanic turbulence.

The wavenumber spectrum of motion in the ocean extends over more than eight decades, from megametres to millimetres. It can be divided into four wavebands: gyre scale, synoptic scale, mesoscale and microscale (Woods 1980). The synoptic scale spans about one decade in this spectrum between scales of $\sim$20 km and $\sim$200 km. This waveband has a spectral peak in absolute energy (Freeland et al. 1975) associated with

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transient eddies powered mainly by barotropic and baroclinic instability of the permanent circulation. Space and velocity scales for synoptic-scale eddies are of order 100 km and 0.1 m s\(^{-1}\). Thus, the Rossby number is less than 0.1 and quasi-geostrophic theory provides a reasonably accurate description of eddy dynamics. Gyre-scale motion, including the permanent circulation in the ocean basins (controlled by the Sverdrup dynamics) has lower wavenumber. The microscale waveband, comprising three-dimensional turbulence which overturns density surfaces, occurs at the high wavenumber end of the spectrum. This paper is concerned with the mesoscale waveband embedded between the synoptic scale and the microscale. Mesoscale motion is characterized by r.m.s. vorticity of order \(f\) and local horizontal anisotropy, i.e. the kinetic energy is concentrated into jets rather than eddies. In the thermocline it covers about five decades of the spectrum between the synoptic scale and the microscale.

Mesoscale kinetic energy occurs in both permanent and transient forms. The defining characteristic is the width of the jet: its length can be over a megametre in the permanent case, or less than 100 km in the transient case. Examples of permanent mesoscale jets are the Sargasso Sea front studied by Voorhis and Hersey (1964) and Katz (1969), the subtropical fronts in the Pacific described by Roden (1975), the Azores current (Gould 1985), and the two branches of the North Atlantic current which cross the mid-Atlantic ridge at circa 47°N and 52°N respectively (Krauss 1986). These permanent streams are distorted by meanders and eddies, but the latter can in principle be modelled as developing from a hypothetical linear mesoscale jet. The challenge is to discover the form of that hypothetical initial condition, because its form influences the character of the observed meanders and eddies.

The transient motions in the synoptic- and meso-scale wavebands exhibit some of the features of two-dimensional turbulence, in which the vertical component of relative vorticity is conserved (Taylor 1915; Fjørtoft 1953). Batchelor (1969) has shown that the cascade of enstrophy (variance of vorticity) to high wavenumber in two-dimensional turbulence is effected by the formation of sharp fronts where the horizontal vorticity gradient increases locally until the scale becomes sufficiently small for molecular viscosity to dissipate the vorticity. The flow in the ocean thermocline is to a good approximation isopycnic, so isopycnic potential vorticity

\[
\text{IPV} = (\xi + f)/H (\Delta \rho/\rho)
\]

is conserved. Here \(H\) is the spacing between two isopycnic surfaces of density \(\rho \pm \Delta \rho/2\) and \(\xi\) and \(f\) are the components of relative and planetary vorticity.

This conservation law can be exploited to develop a theory of oceanic synoptic- and meso-scale turbulence in which changes in vertical vorticity \((\xi)\) are related to changes in the vertical spacing \((H)\) of isopycnals. If \(H = H_0\) when \(\xi = 0\) and \(H = H_0 + H'\) when \(\xi \neq 0\), then \(H'/H_0 = \xi/f = Ro\), where \(Ro\) is the local Rossby number.

If the Rossby number is small compared with unity, then the variation of \(H'\) can be computed from the vorticity of the geostrophic flow. Using the resulting quasi-geostrophic potential vorticity equation it is possible to develop theories of geostrophic turbulence (Charney 1971; Rhines 1979) in which an enstrophy cascade to small scale is effected by the formation of sharpening fronts at which the isopycnic gradient of isopycnic potential vorticity \((\text{IPVG})\) has a local maximum. At low Rossby number, the \(\text{IPVG}\) is described by the corresponding gradient in \(f/H_0\). As the \(\text{IPVG}\) increases, so does this \(f/H_0\) gradient, with the result that the baroclinicity increases. The vertical geostrophic shear increases accordingly, leading to a geostrophic jet flowing along the IPV front. As the front sharpens, the jet becomes narrower and faster, and the magnitude of the horizontal shear on either
side of the jet maximum increases. The magnitude of the relative vorticity of this jet shear eventually approaches \( f \), and the quasi-geostrophic potential vorticity equation can no longer be used accurately to predict the further evolution of the front.

Thus, geostrophic turbulence theory indicates how the enstrophy cascade starts in the synoptic-scale waveband, but does not allow us to follow it through the mesoscale waveband where the cascade is determined by frontal processes with Rossby numbers of order unity. Hoskins and Bretherton (1972), hereafter referred to as HB, attempted to resolve the problem with the help of the semi-geostrophic equations in which the jet shear vorticity is assumed to be geostrophic, while the cross-jet flow is not. Notable extensions of their work were provided by Cullen (1983) and Cullen and Purser (1984). Direct attempts to simulate two-dimensional turbulence with the primitive equations using rectangular grids (e.g. Lilly 1973) reveal the essential frontogenetic nature of the enstrophy cascade, but are frustrated by the limited bandwidth achievable in available computers. The problem is worse when one has to simulate the vertical structure in the baroclinic isopycnic turbulence occurring in the ocean. Nevertheless, such studies of the local dynamics of mesoscale fronts have high priority because of their central role in geophysical turbulence. Although the theories have been developed mainly in the context of atmospheric motion, they are equally applicable to oceanic motion (Monin and Ozmidov 1984). Indeed, the need to understand the local dynamics of mesoscale turbulence is more important in the ocean where the spectral bandwidth covers five decades, than in the atmosphere where it is less than three decades.

There is ample evidence of mesoscale fronts in the ocean, but their dynamics have not yet been investigated in detail because of the difficulty of measuring the components of mesoscale potential vorticity. But there exists literature documenting observations of fronts in the traditional sense, which means observations of strong gradients of temperature and/or salinity on intermediate (meso-) scales. A strong T-S front in the Mediterranean Sea near Malta has been described by Briscoe et al. (1974) and Johannessen et al. (1977). In addition to temperature and salinity measurements they estimated the frontal jet velocity, \( \sim 0.4 \, \text{m} \, \text{s}^{-1} \), by geostrophic calculations. At the resolution of the observations the relative vorticity derived from the jet shear did not exceed values of \( 0.2f \). Temperature and salinity sections showed evidence for vertical circulations in the frontal zone. A detailed study dealing with vertical circulation patterns at oceanic fronts in the same area has been published by Woods et al. (1977). They revealed strong vertical displacements of isotherms and isotherm folding which they assumed to be the consequence of isopycnic up- and down-welling. Mesoscale modulation of spacing between isopycnals (i.e. one component of potential enstrophy) was first mapped at a front in the tropical thermocline during GATE (Woods and Minnett 1979; Leach et al. 1985). The first study bringing together observations of T-S structures, currents, relative vorticity and isopycnal spacing modulations in a well-defined frontal area at the North Atlantic Polar Front has been published by Bauer et al. (1985), Leach (1986) and Fischer et al. (1988).

The classical mechanism for frontogenesis is the action of a horizontal deformation field on a scalar gradient (Bergeron 1928). That mechanism is equally relevant to the formation of mesoscale fronts in the ocean, provided a method is found to take account of the fact that the density of seawater depends on two components, temperature and salinity. MacVean and Woods (1980, referred to as MW) did that by dividing the thermohaline field into a dynamically active component, baroclinicity, and a dynamically passive component, thermoclinicity. The effect of a deformation field on the former is to create a jet, the structure of which is modified by vortex stretching and compression as the Rossby number approaches unity (Sawyer 1956): this is called dynamic fronto-
genesis. The effect of a deformation field on the passive component (thermoclinicity) is to sharpen the scalar gradient without accelerating the flow: this is called kinematic frontogenesis. In this paper, which is concerned with modelling dynamic frontogenesis, we use isopycnic potential vorticity gradient as the dynamically active component. That has two advantages. Firstly, diabatic processes in the thermocline are so weak that isopycnic potential vorticity can be treated as a conservative scalar property of seawater on the synoptic time scale, as isentropic potential vorticity can in the atmosphere (Rossby 1940; Hoskins et al. 1985). Secondly, the use of iPVG simplifies comparison with mesoscale observations, because it is less contaminated by internal wave noise than is mesoscale baroclinicity.

Summarizing, dynamic frontogenesis is the action of a deformation field on iPVG and leads to an iPVG front, with an associated jet. Kinematic frontogenesis is sharpening of a passive scalar gradient such as thermoclinicity (i.e. the isopycnic gradient of temperature) by a deformation field acting in concert with the ageostrophic motion produced by dynamic frontogenesis; it leads to a thermohaline front. Isopycnic gradients of iPVG and temperature in the thermocline are created by horizontal gradients of the correlation of mixed layer density with depth and temperature respectively, during the vernal retreat of the mixed layer (Woods and Barkmann 1986a). Both are always present in the thermocline (Bauer and Woods 1984; Stammer and Woods 1987), thus the front produced by a deformation field comprises local maxima in both iPVG and isopycnic temperature gradient, but one or the other may be more prominent at a particular location, depending on the relative strength of the initial large-scale gradients. The correlation between iPVG and T has been measured by Fischer et al. (1988).

MW modelled mesoscale frontogenesis in the ocean, using HB's semi-ageostrophic equations, in which the jet and therefore the vorticity is assumed to be in geostrophic balance. The MW model accurately simulated many of the observed features of fronts in the thermocline, especially in the early stages of frontogenesis, but gave an unrealistic density field as the Rossby number approached unity, after three days in a deformation field characteristic of oceanic eddies.

Our goal was to design a prognostic model using the fully nonlinear primitive equations, thereby avoiding the need to assume that the jet is in geostrophic balance. The special feature of the model is a coordinate scheme that moves horizontally with the deformation field and vertically with isopycnals (within certain constraints described later). This quasi-Lagrangian distribution of coordinates ensured high resolution at the sharpening front and conservation of potential vorticity to sufficient accuracy. The model is described in detail below. Sections 2 and 3 present the equations of motion and boundary conditions in the quasi-Lagrangian coordinate scheme; sections 4 and 5 describe the structure of the model and the initial conditions for the particular run selected for presentation in this paper; the results of that run are presented and diagnosed in section 6. The implications of those results for mesoscale turbulence in the ocean are discussed in section 7.

2. VERTICAL MODEL COORDINATE

A set of nonlinear equations representing the flow field in generalized vertical coordinates has been derived by Bleck (1978). The horizontal momentum equations are

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{u} + \mathbf{s} \frac{\partial \mathbf{u}}{\partial \mathbf{s}} - \mathbf{f}_0 &= -\alpha \left( \frac{\partial \rho}{\partial \mathbf{x}} \right)_x - \left( \frac{\partial \Phi}{\partial \mathbf{x}} \right)_x,
\end{align*}
\]  

(1a)
and
\[ (\frac{\partial v}{\partial t}) + v \cdot \nabla_s v + \dot{s} \frac{\partial v}{\partial s} + f u = -\alpha \left( \frac{\partial p}{\partial y} \right)_s - \left( \frac{\partial \Phi}{\partial y} \right)_s \] (1b)

where \( s \) is a generalized vertical coordinate, \( \dot{s} \) means \( ds/dt \), \( x \) and \( y \) are the horizontal coordinates (measured not along the \( s \) surface, but in the projection onto a horizontal surface), and \( v = \mathbf{i}u + \mathbf{j}v \) is the horizontal velocity vector (in the strict sense). The subscript \( s \) is a reminder that partial differentiations are carried out with \( s \) held constant, so \( \nabla_s = \mathbf{i}(\partial/\partial x)_s + \mathbf{j}(\partial/\partial y)_s \). Partial differentiation with respect to \( s \) is carried out with \( x \), \( y \) held constant, i.e. in the direction of gravity. The other symbols have the conventional meaning: \( t \) (time), \( f \) (Coriolis parameter), \( \alpha \) (specific volume), \( p \) (pressure) and \( \Phi \) (geopotential).

The hydrostatic approximation is expressed as
\[ \frac{\partial \Phi}{\partial s} = -\alpha \frac{\partial p}{\partial s}. \] (1c)

The continuity equation is
\[ \frac{\partial}{\partial t} \left( \frac{\partial p}{\partial s} \right) + \nabla_s \cdot \left( v \frac{\partial p}{\partial s} \right) + \frac{\partial}{\partial s} \left( \dot{s} \frac{\partial p}{\partial s} \right) = 0. \] (2)

The thermodynamic equation, excluding compressibility, diffusion and other diabatic effects is
\[ \dot{\alpha} = \left( \frac{\partial \alpha}{\partial t} \right)_s + v \cdot \nabla_s \alpha + \left( \dot{s} \frac{\partial p}{\partial s} \right) \frac{\partial \alpha}{\partial p} = 0. \] (3)

3. Horizontal Model Coordinate

Following HB, frontogenisis in this model is driven by an externally imposed, steady, depth-independent deformation field which is defined by the streamfunction \( \psi = -\gamma xy \), where \( \gamma \) is the deformation rate. The deformation velocities in the \( x \) and \( y \) directions are \( u_D = \partial \psi/\partial y = -\gamma x \) and \( v_D = -\partial \psi/\partial x = \gamma y \).

The model box is compressed in the \( x \) direction and stretched in the \( y \) direction by the deformation field. Since we are only interested in the secondary motion induced by the deformation process, it seems appropriate to transform the model equations into a coordinate system which is material with respect to the deformation velocity field. This is equivalent to separating the velocity components at every point into
\[ u = u_D + u' \] (4a)
\[ v = v_D + v' \] (4b)

and then seeking equations for \( (u', v') \). Substituting (4) into the momentun equation (1) yields
\[ \left( \frac{\partial u'}{\partial t} \right)_s + u_D \frac{\partial u_D}{\partial x} + u' \frac{\partial u_D}{\partial x} + u_D \left( \frac{\partial u'}{\partial x} \right)_s + \\
+ u' \left( \frac{\partial u'}{\partial x} \right)_s + v_D \left( \frac{\partial u'}{\partial y} \right)_s + v' \left( \frac{\partial u'}{\partial y} \right)_s + \dot{s} \frac{\partial u'}{\partial s} - f v_D - f v' = \\
-\alpha \frac{\partial p_D}{\partial x} - \frac{\partial \Phi_D}{\partial x} - \alpha \left( \frac{\partial p}{\partial x} \right)_s - \left( \frac{\partial \Phi}{\partial x} \right)_s \] (5a)
\[ \left( \frac{\partial \nu'}{\partial t} \right)_s + v_D \frac{\partial \nu_D}{\partial y} + \nu' \frac{\partial \nu_D}{\partial y} + v_D \left( \frac{\partial \nu'}{\partial y} \right)_s + \nu' \left( \frac{\partial \nu'}{\partial y} \right)_s + u_D \left( \frac{\partial \nu'}{\partial x} \right)_s + s \frac{\partial \nu'}{\partial s} + fu_D + fu' = \]

\[ - \alpha \frac{\partial p_D}{\partial y} - \frac{\partial \Phi_D}{\partial y} - \alpha \left( \frac{\partial p_c}{\partial y} \right)_s - \left( \frac{\partial \Phi_c}{\partial y} \right)_s \].

(5b)

Here we make use of the fact that \( u_D \) is independent of \( y, s \) and \( v_D \) independent of \( x, s \). On the right-hand side of (5) the gradients of pressure and geopotential have been split into parts \((p_D, \Phi_D)\), which sustain the deformation velocities \((u_D, v_D)\), and parts \((p_c, \Phi_c)\), which drive the motion described by \((u', v')\). Thus, substituting \((u_D, v_D)\) for \((u, v)\) in (1) yields

\[ u_D \frac{\partial u_D}{\partial x} - f v_D = - \alpha \frac{\partial p_D}{\partial x} - \frac{\partial \Phi_D}{\partial x} \]

(6a)

\[ v_D \frac{\partial v_D}{\partial y} + f u_D = - \alpha \frac{\partial p_D}{\partial y} - \frac{\partial \Phi_D}{\partial y} \]

(6b)

Subtracting (6) from (5) yields

\[ \left( \frac{\partial u'}{\partial t} \right)_s + u' \frac{\partial u_D}{\partial x} + u_D \left( \frac{\partial u'}{\partial x} \right)_s + u' \left( \frac{\partial u'}{\partial x} \right)_s + \]

\[ + v_D \left( \frac{\partial u'}{\partial y} \right)_s + u' \left( \frac{\partial u'}{\partial y} \right)_s + s \frac{\partial u'}{\partial s} - fu' = - \left( \frac{\partial M}{\partial x} \right)_x \]

(7a)

\[ \left( \frac{\partial v'}{\partial t} \right)_s + v' \frac{\partial v_D}{\partial y} + v_D \left( \frac{\partial v'}{\partial y} \right)_s + v' \left( \frac{\partial v'}{\partial y} \right)_s + \]

\[ + u_D \left( \frac{\partial v'}{\partial x} \right)_s + u' \left( \frac{\partial v'}{\partial x} \right)_s + s \frac{\partial v'}{\partial s} + fu' = - \left( \frac{\partial M}{\partial y} \right)_y \]

(7b)

where

\[ M(x, y) = \alpha p_c + \Phi_c \]

(8)

is the ‘Montgomery potential’ introduced by Montgomery (1937) for isopycnic analysis. Because we are only interested in secondary motion \( u'(x), v'(x) \) in a \( y \)-independent baroclinic mass field the \( y \) derivatives vanish in (7):

\[ \left( \frac{\partial u'}{\partial t} \right)_s + u_D \left( \frac{\partial u'}{\partial x} \right)_s + u' \left( \frac{\partial u_D}{\partial x} \right)_s + \frac{\partial u'}{\partial s} - fu' = - \left( \frac{\partial M}{\partial x} \right)_x \]

(9a)

\[ \left( \frac{\partial v'}{\partial t} \right)_s + v_D \left( \frac{\partial v'}{\partial y} \right)_s + v' \left( \frac{\partial v_D}{\partial y} \right)_s + \frac{\partial v'}{\partial s} + fu' = 0. \]

(9b)

We now define a variable \( \hat{x} \) which acts as a Lagrangian coordinate for fluid elements riding with the \( u_D \) field. Since these parcels move from their initial position \( x = x_0 \) to
\[ x = x_0 e^{-yt} \] during the time interval \( t \), their coordinate \( \dot{x} \), which is synonymous with their starting position, is given by

\[ \dot{x} = xe^{-yt}. \] (10)

Differentiating (10) with respect to time yields a relationship between the velocity components \( \dot{u} = d\dot{x}/dt \) and \( u = dx/dt \):

\[ u = \dot{u}e^{-yt} + u_D. \] (11)

Spatial derivatives in the \( x \) and \( \dot{x} \) system at time \( t \) are related through

\[ \partial/\partial \dot{x} = e^{-yt} \partial/\partial x \] (12)

while (11) applied to the identity

\[ \partial/\partial t_x + \dot{u}\partial/\partial \dot{x} = \partial/\partial t_x + u\partial/\partial x \] (13)

yields an expression relating the local time derivatives in the \( x \) and \( \dot{x} \) system:

\[ \partial/\partial t_x = \partial/\partial t_x + u_D\partial/\partial x. \] (14)

After combining the first two terms in (9a) and (9b) with the help of (14) and some regrouping of the remaining terms the transformed momentum equations assume the form

\[ \left( \frac{\partial u'}{\partial t} \right)_{\dot{x}} + \frac{\partial}{\partial x} \left( \frac{u'^2 + v'^2}{2} \right) - v' \left( \frac{\partial v'}{\partial x} + f \right) + u' \left( \frac{\partial u_D}{\partial x} + \dot{s} \frac{\partial u'}{\partial s} \right) = - \frac{\partial M}{\partial x} \] (15a)

\[ \left( \frac{\partial v'}{\partial t} \right)_{\dot{x}} + u' \left( \frac{\partial v'}{\partial x} + f \right) + v' \left( \frac{\partial v_D}{\partial y} + \dot{s} \frac{\partial v'}{\partial s} \right) = 0. \] (15b)

To keep the equations simple, we dropped the subscript \( s \) and expressed horizontal derivatives in terms of \( x \) rather than \( \dot{x} \) to eliminate the factor \( e^{-yt} \) seen in (12). The hydrostatic equation (1c) remains unchanged. To transform the continuity equation (2) we once again make use of (14). Since \( \partial p/\partial s \) is independent of \( y \), the equation reduces to

\[ \frac{\partial}{\partial t_x} \left( \frac{\partial p}{\partial s} \right) + u' \frac{\partial p}{\partial s} + \frac{\partial}{\partial s} \left( \dot{s} \frac{\partial p}{\partial s} \right) = 0. \] (16)

In the same manner the thermodynamic equation (3) simplifies to

\[ \dot{\alpha} = \left( \frac{\partial \alpha}{\partial t} \right)_{\dot{x}} + u' \frac{\partial \alpha}{\partial x} + \left( \dot{s} \frac{\partial \alpha}{\partial s} \right) \frac{\partial \alpha}{\partial p} = 0. \] (17)

Equations (15), (16) and (17) together with the hydrostatic equation (1c) written in the form

\[ \partial M/\partial s = p \partial \alpha/\partial s \] (18)

form the desired set of equations in \( (\dot{x}, s) \) coordinates. Except for the two terms \( u'\partial u_D/\partial x = -yv' \) and \( v'\partial v_D/\partial y = yv' \) in (15), the equations for \( (u', v') \) in the \( \dot{x} \) system are formally identical to the equations for \( (u, v) \) in the regular system. This makes it easy to adapt existing model code (Bleck and Boudra 1981) to the present task. Note that spatial derivatives in (15), (16) and (17) are evaluated in \( x \), rather than \( \dot{x} \), coordinates.

We shall finish this section by deriving the vorticity and potential vorticity equations in the \( \dot{x} \) system for the special case where \( \dot{s} = 0 \). The vorticity equation is obtained by differentiating (15b) with respect to \( \dot{x} \) (note that differentiation with respect to \( x \) would
not allow us to interchange the order of spatial and temporal differentiations in the first term):

$$\frac{\partial}{\partial t} \left( \frac{\partial u'}{\partial x} \right) + \frac{\partial}{\partial x} \left[ u' \left( \frac{\partial v'}{\partial x} + f \right) + v' \frac{\partial v_D}{\partial y} \right] = 0. \quad (19)$$

It appears desirable to retain consistency with (15)–(17) and express the derivative $\partial / \partial x$ in terms of $\partial / \partial \hat{x}$. Use of (12) allows us to write the first term of (19) as

$$\frac{\partial}{\partial t} \left( e^{-\gamma} \frac{\partial v'}{\partial x} \right) = \left[ -\gamma \frac{\partial v'}{\partial \hat{x}} + \frac{\partial}{\partial t} \left( \frac{\partial v'}{\partial \hat{x}} \right) \right] e^{-\gamma}. \quad (20)$$

The factor $e^{-\gamma}$ can be removed from the entire equation after applying (12) to the second term in (19). Since $\partial v_D / \partial y = \gamma$, two terms cancel and we are left with

$$\frac{\partial}{\partial t} \left( \frac{\partial v'}{\partial x} + f \right) + \frac{\partial}{\partial x} \left[ u' \left( \frac{\partial v'}{\partial x} + f \right) \right] = 0. \quad (21)$$

In preparation for deriving the potential vorticity equation we write the term $(\partial v' / \partial x + f)$ in (21) as $Q \partial p / \partial s$ where

$$Q = \frac{\partial v'/\partial x + f}{\partial p / \partial s}$$

is the potential vorticity in $s$ coordinates. This leads to

$$\frac{\partial p}{\partial s} \left( \frac{\partial Q}{\partial t} \right) + Q \frac{\partial}{\partial t} \left( \frac{\partial p}{\partial s} \right) + \frac{\partial}{\partial x} \left[ u' \frac{\partial p}{\partial s} Q \right] = 0. \quad (22)$$

From this we subtract the continuity equation (16) multiplied by $Q$. The remaining terms form the desired potential vorticity equation

$$\frac{\partial Q}{\partial t} + u' \frac{\partial Q}{\partial x} = 0. \quad (23)$$

Thus, in grid regions where $s = 0$ the quantity $Q$ is conserved while being advected by the $u'$ field.

The finite difference equations of the posed problem are presented in the appendix.

4. NUMERICAL IMPLEMENTATION

Our two-dimensional model domain is a vertical plane located parallel to the $x$ axis in the deformation field and bounded by rigid walls at $\hat{x} = \pm L_0/2$. The boundary conditions at the walls (which move with the $u_0$ flow) are

$$u' = 0 \quad \text{and} \quad \partial v' / \partial x = 0 \quad \text{(free slip).}$$

The surface and bottom boundary conditions are those for material surfaces, i.e. $\delta \partial p / \partial s = 0$ at $p = 0$ Pa and $p = 1000 \times 10^4$ Pa. No diabatic heating takes place.

In the horizontal, our model domain is subdivided into 256 equally spaced intervals $\Delta x(\hat{r})$. In the vertical, 10 layers are separated by surfaces $s =$ constant. The thickness $\Delta p$ of these layers may vary with $x$ and/or time. The surfaces $s =$ constant are referred to as 'coordinate surfaces' or 's surfaces'. Surface 1 is the sea surface, surface 11 the bottom.
The layers between the surfaces are called ‘coordinate layers’ or ‘s layers’ (see Fig. 1). The grid increment \( \Delta x \) is time dependent, advecting with the deformation field,

\[
\Delta x(t) = \Delta x_0 e^{-\gamma t}
\]

where \( \Delta x_0 = \Delta x(t=t_0) \) was set to 3.125 km. The initial cross-front scale is \( L_0 = 800 \) km. In order to satisfy the linear computational stability criterion the time step must vary in proportion to the grid scale, i.e.

\[
\Delta t(t) = \Delta t_0 e^{-\gamma t}
\]

where \( \Delta t_0 = \Delta t(t=t_0) = 150 \) seconds.

To specify the initial conditions we have to define a functional dependence between \( \alpha \) and \( \rho \) in every grid column. Though the vertical coordinate is not restricted to be isopycnic in general we have found it convenient to assign every layer a specific volume independent of \( x \) at \( t = t_0 \).

At every time step the surfaces are checked for minimum separation. If the distance between two surfaces falls below \( \Delta \rho_0 \) (we have chosen the minimum layer thickness to be approximately 3 m, so \( \Delta \rho_0 = 3 \times 10^4 \text{Pa} \) water is permitted to penetrate the surfaces to maintain minimum separation. This is needed to prevent the coordinate surfaces from intersecting the sea surface or, in a more formal sense, to allow division by \( \delta_s \rho \) in (29) and (31). Thus, density within a coordinate layer can change in space and time. A competing algorithm tries to restore isopycnic conditions, again through vertical mass exchange, in places where \( \alpha \) deviates from its nominal value. This is to ensure conservation of isopycnic potential vorticity wherever possible. A complete description of this mechanism is given in appendix C of Bleck and Boudra (1981).

In order to get the above mechanism off to a smooth start we define isopycnic \( s \) layers of thickness \( \Delta \rho_0 \) or larger as initial condition. Consequently, there does not exist any horizontal density gradient at the sea surface at \( t = 0 \) (as in Charney (1971), but unlike MW). Integrating the model directly from that initial condition implies that no horizontal density gradient can develop at the surface because \( \zeta \) cannot become less than \( -f \). That may be regarded as an undesirable restriction on the simulation of oceanic fronts, which often have a significant surface density gradient.
To test the sensitivity of our simulations to the presence of a surface density gradient we have developed a scheme where we remove the top 10 m of water in small increments every time step during the first 24 hours of a model run. Several surface areas are then forced to permit passage of denser water from below (because of the restriction of minimum layer thickness mentioned above) and a horizontal density gradient can develop in the surface layer. During this time period strong vertical motion \( \frac{\partial p}{\partial s} \) relative to the uppermost surface areas occurs. The removal of the top ten metres is completed before the Rossby number reaches 0.1, i.e. it takes place during the kinematic phase of frontogenesis, before significant vortex stretching occurs. It does therefore not influence the dynamic frontogenesis, except in the sense that it generates more realistic initial conditions. We find that the surface water removal process only leads to a minor strengthening (about 10%) of the frontal jet. This is because the surface density gradient is less important than the isopycnic potential vorticity gradient for the frontogenesis simulation described in this paper. It has the undesirable side effect of changing the potential vorticity initial conditions in an uncontrolled manner. We therefore chose not to use it in the simulations discussed below.

Two kinds of waves are eliminated by the model: fast-moving barotropic gravity waves are filtered out by a rigid-lid approximation (see Bleck and Boudra 1981, appendix D). To damp small grid-scale 'numerical waves' eddy viscosity terms have been added to the right-hand sides of equations (7a, b):

\[
\left( \frac{\partial p}{\partial s} \right)^{-1} \nabla_s \cdot \left[ \frac{\partial p}{\partial s} \left( -\frac{1}{2} h^2 \nabla_s u' \right) \right]
\]

\[
\left( \frac{\partial p}{\partial s} \right)^{-1} \nabla_s \cdot \left[ \frac{\partial p}{\partial s} \left( -\frac{1}{2} h^2 \nabla_s v' \right) \right].
\]

We have chosen this bi-harmonic type of eddy viscosity because the damping is much more effective at short length scales than in the mono-harmonic formulation. The eddy viscosity, \( \nu \), is 0.4 times the absolute value of the total deformation of the horizontal motion field multiplied by \( \Delta x^2 \) (see Bleck and Boudra 1981), and \( h \) is set to 2500 m.

A centred time-differencing (leapfrog) scheme is used with respect to all terms in the prognostic equations, except that viscosity terms are integrated forward in time to avoid linear computational instability. We apply the upstream differencing anti-diffusive scheme of Smolarkiewicz (1983) to the flux term \( u' \delta_s \alpha \) in the thermodynamic equation (3). This minimizes numerical diffusion and prohibits overshooting of density.

Short-period fluctuations in the velocity field are damped by a three-point (0.25–0.5–0.25) time-smoother applied to the prognostic variables \((u', v')\) (Asselin 1972). A weaker (0.01–0.98–0.01) time-smoother is applied to the \( \delta_s p \) field.

Static instability, caused occasionally during upwelling episodes by spurious horizontal advection of \( \alpha \), is eliminated by a simple convective adjustment mechanism which smears out the \( \alpha \) profile whenever \( \partial \alpha / \partial p \geq 0 \) is diagnosed.

5. INITIAL CONDITIONS

The initial density field is shown in Fig. 2. The horizontal slope of the pressure field on the \( k \)th surface (which is an isopycnic at \( t = 0 \)) is

\[
p_k(x) = p_k(-L_0/2) + [\tanh(cxL_0^{-1}) + 1] \Delta p_{\text{max}}/2.
\]

\( k \) runs from 2 to 10. Surfaces 1 and 11 are defined to be horizontal. \( \Delta p_{\text{max}} \), the maximum
horizontal pressure variation on every \( s \) surface between \( x = -L_o/2 \) and \( x = L_o/2 \) is set to \( 60 \times 10^4 \) Pa. Our value for \( \gamma \), which controls the ‘sharpness’ of the hyperbolic tangent function, is 14. The pressure values \( p_k \) at \( x = -L_o/2 \) for \( s \) surfaces 1 to 11 and the average \( s \) layer densities \( \sigma_{is} = \alpha_{s}^{-1} - 1000 \) for \( s \) layers 1.5 to 10.5 are summarized in Table 1. (The numbering system for coordinate layers has been chosen to indicate the relation between layers and levels.) Thus, some of the \( \sigma \) surfaces in Fig. 2 correspond to \( s \) surfaces and others do not. Especially the surface \( \sigma = 25.9 \) kg m\(^{-3}\) has not been defined explicitly in the model as an \( s \) surface, but we verified that adding this surface to the model had no effect on the solution. In addition to the model parameters defined in section 4 we use a Coriolis frequency \( f = 10^{-4} \) s\(^{-1}\) (latitude 43°). The deformation rate \( \gamma = 10^{-5} \) s\(^{-1}\) is the same as that used by MW. The initial velocity field is assumed to be in geostrophic balance with the density field. This can be justified by the fact that the Rossby number initially is less than 0.01. The small initial Rossby number implies negligible horizontal variations of relative vorticity.

Since the spacing between any two isopycnals is independent of \( x \), there exists no IPVG in layers 2.5 to 9.5 (the relative vorticity of the initial jet being negligible). On the other hand, a significant IPV difference of 3.7 radian/(Gm s) over 200 km exists in the uppermost layer between the sea surface and the \( \sigma = 26.1 \) kg m\(^{-3}\) surface. If instead we calculate the difference over the total extent of the seasonal thermocline (which may be the depth range between the sea surface and the isopycnal \( \sigma = 27.0 \) kg m\(^{-3}\)) it reduces to about 0.1 radian/(Gm s). The bottom layer IPV (because of the flat seabed) is one order of magnitude less.

TABLE 1. INITIAL Mass FIELD PARAMETERS

<table>
<thead>
<tr>
<th>( k ) (level index)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k ) (layer index)</td>
<td>1.5</td>
<td>2.5</td>
<td>3.5</td>
<td>4.5</td>
<td>5.5</td>
<td>6.5</td>
<td>7.5</td>
<td>8.5</td>
<td>9.5</td>
<td>10.5</td>
<td>11</td>
</tr>
<tr>
<td>( p_x(-L_o/2) \times 10^4 ) Pa</td>
<td>0</td>
<td>5</td>
<td>11</td>
<td>17</td>
<td>25</td>
<td>37</td>
<td>60</td>
<td>100</td>
<td>300</td>
<td>500</td>
<td>1000</td>
</tr>
<tr>
<td>( \sigma_{is} ) kg m(^{-3})</td>
<td>25.9</td>
<td>26.1</td>
<td>26.3</td>
<td>26.5</td>
<td>26.7</td>
<td>26.9</td>
<td>27.1</td>
<td>27.3</td>
<td>27.5</td>
<td>27.7</td>
<td></td>
</tr>
</tbody>
</table>
The choice of initial conditions may influence the model results significantly. The crucial parameter controlling the dynamics of a fluid is the IPVG (Hoskins et al. 1985). We decided to select an initial density field representing an initial potential vorticity distribution close to that used in MW. The difference in IPVG across the front is of the same order of magnitude as the large-scale horizontal potential vorticity gradient, which can be deduced from maps of isopycnal potential vorticity distributions in the North Atlantic derived from atlas data (Stammer and Woods 1987). In the northern part of the North Atlantic during August these maps give a north–south IPVG difference on \( \sigma_i = 26 \text{kg m}^{-3} \) close to 1.4 radians/(Gm s) over a distance of 1 Mm. The magnitude of this gradient is about ten times larger than our IPVG in the seasonal thermocline and that in MW. Because the IPVG has been derived from atlas IPVG differences over a distance of 1 Mm, local IPVG maxima (which are one order of magnitude larger at permanent streams such as the Azores current) have been smoothed out. Thus, the initial IPVG used in our model may be rather low for the seasonal thermocline, characteristic of the slow recirculation regime of gyres. The sensitivity of model results to different initial IPVG distributions will be the topic of future model experiments.

6. Results

(a) Structure of the front

We integrated the model over three days. We compare cross-sections of \( \sigma_i \) on day 0 and day 3 (Fig. 3). Three features are obvious. First, the depth of isopycnals has changed. To the left of the dotted line, which indicates the location of no vertical displacement, depth has increased on all isopycnals at fixed horizontal positions \( x/L \). Maximum changes in depth are approximately 15 m in three days, which implies an average downwelling velocity of \( \approx 5 \text{ m d}^{-1} \). At the right-hand side of the no-vertical-displacement line an upwelling speed of \( \approx 5 \text{ m d}^{-1} \) has led to a decrease of depth on all isopycnals. Second, the initial hyperbolic tangent shape of every isopycnal has been distorted. This can only be due to ageostrophic motion. Third, the vertical spacing between pairs of isopycnals has changed. At \( t = 0 \) the spacing between two isopycnals in layers 2.5 to 9.5 has been constant for all \( x \) by definition. Figure 3 shows that the layer thickness has increased in the left part of the model domain and decreased in the right part. This is also a sign of ageostrophic mass transport in the negative \( x \) direction.

Figure 3. Comparison of density fields on day 0 and day 3 indicating up- and down-welling areas (arrows), the result of vortex stretching induced by growing relative vorticity.
The baroclinicity, $b$, defined as the slope of an isopycnal, has been calculated for $\sigma_i$ surfaces 26.2 and 27.2 kg m$^{-3}$ located at mean depths of \approx 40 and \approx 170 m respectively (Fig. 4). At $\sigma_i = 26.2$ kg m$^{-3}$ the maximum baroclinicity growth rate is rapid in the beginning of the model run but decreases with time. Furthermore the baroclinicity maximum shifts to the left. This shift can be observed on the lower $\sigma_i$ surface, too, but here the baroclinicity, which had an initial value the same as that of the upper surface, grows at a much lower rate and reaches only 60% of the maximum value of the upper isopycnal on day 3.

Figure 4. Temporal changes of baroclinicity, $b$: (a) close to the sea surface ($\sigma_i = 26.2$ kg m$^{-3}$); (b) at the bottom of the thermocline ($\sigma_i = 27.2$ kg m$^{-3}$).
Figures 5(a), (b) show cross-sections of the $v'$ field. Within three days the maximum jet speed has increased from 0.08 m s$^{-1}$ to 0.39 m s$^{-1}$. The flow is directed into the paper in the upper layers and out of it below about 500 m, but the deep flow is weak. Peak values are $\approx -0.05$ m s$^{-1}$. The deviation of the jet axis from the vertical increases with time. The jet axis can be obtained by connecting the maximum $v'$ values in every layer. It has a slope of $\approx 10^{-2}$ on day 3 in the upper 50 m. The jet width has decreased. The distance between the surface outcrops of the 0.05 ms$^{-1}$ isochore shrinks from $\approx 60$ km on day 0 to $\approx 40$ km on day 3.

The jet is also getting shallower with time. Initially, the top four layers contain less than 60% of the total kinetic energy. At $t = 0.4$ d, exactly 60% of the total kinetic energy

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![Figure 5](image_url)  

**Figure 5.** (a) and (b): Within three days the jet speed $v'$ accelerates by about a factor of five. Thick lines are isochoros, broken lines indicate $\sigma$, surfaces. (c): Progressive shallowing of the jet revealed by the 60% kinetic energy level.
is confined to the top four layers. At \( t = 3.6 \text{d} \) the above is true for the top three layers, and at \( t = 4.4 \text{d} \) for the top two layers (Fig. 5(c)).

Figure 6 shows the ageostrophic \( u' \) field. Its main features are motion to the left in the upper layers (extreme values are \( \approx -47 \text{ mm s}^{-1} \) on day 3) and relatively weak motion to the right below with peak values of \( \approx 7 \text{ mm s}^{-1} \).

![Figure 6](image)

Figure 6. The cross-jet speed \( u' \) (thick lines) is directed to the left (negative) during the entire model run. Note that the deformation velocity has been neglected; it has maximum values of \( 0.3 \text{ m s}^{-1} \) at the sidewalls. Broken lines are \( \sigma \) surfaces.

The vertical speed, \( w \), is diagnosed by the model at every time step. To eliminate inertial wave effects we averaged it over one inertial period \( \tau = 17.45 \text{h} \). The results are shown in Fig. 7, which confirms that during the entire model run there is upward motion to the right and downward motion to the left of the jet axis, looking downstream. Maximum upwelling speeds in the upper 150 m are \( \approx 6 \text{ m d}^{-1} \). Extreme downwelling speeds here are \( \approx -7 \text{ m d}^{-1} \). Maximum absolute upwelling and downwelling speeds are found below 150 m where the density gradient is much weaker, but the relative vorticity is not negligible. Local maxima of \( \approx \pm 10 \text{ m d}^{-1} \) can be found at \( \approx 200 \text{ m} \).

![Figure 7](image)

Figure 7. The vertical speed \( \overline{w'} \) averaged over one inertial period. The stippled area indicates upwelling.
From the $v'$ distribution we can deduce fields of relative vorticity $\zeta = \partial v' / \partial x$ (the deformation field is vorticity-free). Figure 8 shows contours of $\zeta/f$, which is equivalent to a local Rossby number $Ro$, on day 3. Before day 1, $Ro$ is very small because of the weak jet and the larger cross-front scale. Within three days, strong cyclonic vorticity has developed close to the surface. Peak values of $Ro \approx 0.5$ are found. The anticyclonic cell is less intense. The minimum value found here is $Ro \approx -0.2$. (If the model is initialized with a more concentrated baroclinic zone, corresponding to a stronger initial jet of $0.18 \text{ m s}^{-1}$, the Rossby number on day 3 exceeds unity. Maximum jet speed remains near $0.5 \text{ m s}^{-1}$.) The cyclonic and anticyclonic cells are not symmetric: the horizontal and vertical extent of the anticyclonic cell is larger than the extent of the cyclonic cell.

![Figure 8](image-url)

Figure 8. The local Rossby number $Ro$ lies between extremes of $-0.5$ and $-0.2$. The stippled area indicates anticyclonic vorticity, broken lines are $\sigma$, surfaces.

Conservation of potential vorticity requires a balance between the relative vorticity and spacing between isopycnals (i.e. the layer thickness). The temporal change of the thickness of layer 2-5 can be seen in Fig. 9. Two features are obvious. During the three-day run the spacing increases continuously on the cyclonic side of the jet and decreases on the anticyclonic side. The location of maximum spacing shifts to the left with time.

![Figure 9](image-url)

Figure 9. Temporal changes of layer thickness $H$ (layer 2-5) compensating for the strengthening of relative vorticities.
(b) Local dynamics

A physical interpretation of the model results can best be done from the viewpoint of conservation of potential vorticity of individual vortex tubes between isopycnic surfaces. Figure 10(a) shows a sketch of the model box at the beginning of the deformation process.

![Figure 10](image)

Figure 10. Sketch of the model dynamics: (a) at the beginning of the deformation; (b) after continued deformation. $g_1$, $g_2$ and $g_3$ are isopycnic, $Q_1$ and $Q_2$ mark water columns.

The baroclinicity represented by the slope of the isopycnals $\rho_1$, $\rho_2$ and $\rho_3$ is small, so the resulting baroclinic jet indicated by the arrows is relatively weak. Two water columns are marked by $Q_1$ and $Q_2$. Note the difference between the two columns: $Q_1$ is located between two isopycnals with different spacing on the cyclonic and anticyclonic side of the jet. The relative vorticity is negligible at this time, so the unequal spacing implies a cross-front potential vorticity gradient in this layer. In contrast to that, $Q_2$ is located between two isopycnals having equal spacing everywhere which means that the potential vorticity gradient is zero in that layer. Figure 10(b) is a sketch of the model box at a later time: the box has been compressed by the horizontal deformation field increasing the baroclinicity (Fig. 4) and accelerating the jet (Fig. 5). The kinetic energy increase must be accompanied by a decrease in available potential energy, i.e., by an ageostrophic mass flux from the anticyclonic to the cyclonic side along isopycnals. At the same time, the decrease of the cross-front scale and acceleration of the jet generate two cells of relative vorticity with different sign at the two sides of the jet (Fig. 8). Individual conservation
of potential vorticity now requires an increase of the spacing between isopycnals on the
cyclonic (stretching of vortex tubes) and a decrease (contraction) on the anticyclonic side
close to the jet. This modulates the cross-frontal mass flux and determines the details of
the up- and down-welling (Fig. 7). The vertically integrated mass flux divergence leads
to an increase of the absolute vertical velocity with depth. This forced up-/down-ward
flow opposes the increase of baroclinicity due to the deformation especially in deeper
layers (Fig. 4(b)). Consequently the strong temporal changes of relative vorticity are con-
fined to the upper layers which requires a stronger ageostrophic mass flux there (Fig. 6).
This is the reason for the tilt of the jet axis. Conservation of potential vorticity allows
an unlimited growth of cyclonic vorticity but the anticyclonic vorticity growth is limited
to $-f$. Hence the intensities of the cyclonic and anticyclonic vorticity cells differ and the
jet velocity distribution is asymmetric.

So far we have not given an answer to the question of what happens if the deformation
process continues beyond day 3. The answer, in short, is that the jet stops intensifying.
Since this behaviour marks a significant deviation from the previous analytic solutions to
the frontogenesis problem (HB; MW; Cullen 1983), it deserves further investigation.
The possible cause that comes to mind first is viscosity, which is not present in the analytic
models. However, Fig. 11 confirms that the limitation in jet intensity is not due to
destruction of kinetic energy by viscous dissipation. Throughout the first three days the
term expressing the conversion from potential to kinetic energy (baroclinic conversion)
is several orders of magnitude larger than the dissipation term. The energy conversion
process decreases beyond day 3.8 not because of viscosity, but because mesoscale
dynamics cannot extract any more energy from the reservoir of available potential energy
(APE). This does not imply that the final state of the frontogenesis process is characterized
by horizontal isopycnals: some residual baroclinicity is still required to geostrophically
balance the frontal jet.

![Graph](image)

Figure 11. The baroclinic conversion term is always several orders of magnitude larger than the viscous
dissipation. Thus dissipation does not alter the jet's acceleration significantly.

The next question to ask is whether limitations in grid resolution, domain size, or
the choice of the deformation rate prevent the unlimited build-up of jet seen in the
analytic models. We have repeated our experiment with half the original grid size and
have found the end result to be insensitive to this change. Repeating the experiment with
a deformation rate $\gamma = 10^{-6}s^{-1}$ and running the model over 40 days (kinematically
equivalent to four days with $\gamma = 10^{-5} \text{s}^{-1}$) gave the same frontal structure and jet limitation on day 40. In both cases the jet reaches a final width and peak velocity within a finite time.

Domain size, on the other hand, turns out to have a profound influence on the numerical solution. In Fig. 12(a) we show the effect of doubling the domain size on the baroclinic zone width $L_b$ attained during the time integration. $L_b$ is defined to be the horizontal extension between the outcrops of the 0.01 m s$^{-1}$ isoloc. The thin lines indicate the time dependence of the domain sizes related to initial sizes $L_o = 1600$ and 800 km whereas the thick lines represent the baroclinic zone widths. It is evident that $L_b$ shrinks at a lesser rate than $L$, thus they approach each other asymptotically. For later times $L_b$ now is limited by $L$ as an upper bound. Figure 12(b) shows, for domain sizes of $L_o = 1600, 800, 400$ and 200 km, the relative change in layer thickness adjacent to the sidewalls as a function of model time, and Fig. 12(c) the maximum jet speed. The correspondence between the instant when the jet speed ceases to increase linearly in time and the instant when the modulation of layer thickness brought about by the secondary circulation reaches the sidewall is rather striking. It suggests that artificially limiting the size of the potential energy 'catchment area' exerts a powerful influence on the frontogenesis process—to the point where a 1600 km-wide strip of fluid is needed at initial time to permit jet growth to the rather modest velocity of 0.5 m s$^{-1}$.

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**Figure 12.** Effects of repeated doubling of the initial domain size $L_o$. (a) The instant when the domain size $L$ and the baroclinic zone width $L_b$ approach asymptotically is shifted forward in time. Only the runs with $L_o = 800$ km and $L_o = 1600$ km are shown. (b) The relative thickness change $H/H(t_o)$ at the left sidewall $x = -L(t)/2$. Early significant vortex stretching is prohibited. (c) The maximum jet speed $v_{\text{max}}$ attained during the model run increases.
7. DISCUSSION

The results presented above show that vigorous frontogenesis occurs when a deformation field acts on an isopycnic gradient of potential vorticity in the seasonal thermocline. The outcome is independent of the rate of deformation, provided it is sustained until a major portion of the APE between the sidewalls has been converted to jet kinetic energy. For a typical synoptic-scale deformation rate \(10^{-8} \text{s}^{-1}\) and an 800 km-wide strip of fluid participating in the deformation process this is achieved in about three days, which is much less than the circulation time of quasi-geostrophic eddies. For a typical gyre-scale deformation rate of \(10^{-8} \text{s}^{-1}\) frontogenesis is completed in 30 days, which is much shorter than the lifetime of the seasonal thermocline. The model results are not unduly sensitive to the existence of a surface density contrast. We have verified this in additional runs where we removed the top 10 metres of water, as described in section 4. The only significant result was a 10% increase in final jet speed, but the basic dynamics did not change. The initial conditions used to illustrate our model comprised an 800 km-wide IPVG with a magnitude that is typical of the weakest found in the seasonal thermocline. We conclude that sharp IPV fronts and jets will be a universal feature of the seasonal thermocline.

Where the jets are forced by gyre-scale confluence, they extend over great distances. They are called 'streams' in the general circulation. Examples in the North Atlantic Ocean include the branches of the North Atlantic drift (Krauss 1986) and the Azores current (Gould 1985). The local dynamics of extensive near-surface fronts in the subtropics, which have previously been modelled in terms of Ekman convergence (Cushman-Roisin 1981; de Ruijter 1983; de Szoeke 1980), also depends on the large-scale deformation of IPVG in the seasonal thermocline. Roden (1975) observes strong thermoclinicity and weak baroclinicity in these fronts and explains this as being due to horizontal gradients of temperature and salinity created in the mixed layer by climatological variations in air–sea interactions and subsequently converged by Ekman transport. This presents a problem, because horizontal variations of temperature and salinity in the mixed layer are normally accompanied by horizontal variations of the depth of isopycnals subducted into the seasonal thermocline during the spring ascent of the mixed layer (Woods and Barkmann 1986a) which would imply that seasonal mixed-layer dynamics creates baroclinicity over distances exhibiting climatological temperature and salinity variations. Our model results suggest a way to reconcile this apparent conflict. The initial baroclinicity created by mixed-layer dynamics is reduced by vortex stretching during dynamic frontogenesis, but leaving strong thermoclinicity and haloclinicity. The location of such streams in the seasonal boundary layer depends on the axis of confluence associated with upper ocean currents. Their strength varies regionally with the magnitude of the IPVG established during the vernal retreat of the mixed layer (Woods 1985; Woods and Barkmann 1986a,b). Such streams are not predicted by the present-day oceanic general circulation models (e.g. Sarmiento and Bryan 1982) because they do not resolve IPVG in the seasonal thermocline.

Synoptic-scale deformation extends over a much shorter distance, normally less than 100 km, but is much more powerful than gyre-scale deformation. Thus the ambient APE is rapidly converted into jet kinetic energy. This acceleration accompanied by vigorous vertical circulation characterizes mesoscale frontogenesis in the context of oceanic weather systems. Upwelling on the anticyclonic side of these jets raises nutrients through the euphotic zone, increasing primary productivity. The resulting mesoscale modulation of phytoplankton biomass is seen in satellite images of ocean colour (Gower et al. 1980).
FRONTOGENESIS IN THE OCEAN

Synoptic-scale motion occurs throughout the ocean (Robinson 1983), and the \( \text{IPVG} \) in the seasonal thermocline is almost everywhere larger than the value chosen as initial conditions for our model. Consequently, the seasonal thermocline is everywhere veined by transient mesoscale jets up to a hundred kilometres long. These jets with Rossby number of order unity are believed to be responsible for the observed high level of enstrophy in the upper ocean. The variation in isopycnic layer thickness associated with this enstrophy constitutes a noise in estimates of potential vorticity based on the assumption that \( \xi = 0 \). This noise can be eliminated only by averaging large numbers of independent density profiles (Bauer et al. 1985) or by measuring relative vorticity (Leach 1986). Layer thickness modulation due to vortex stretching near high Rossby number jets makes an important contribution to the density fine structure in the thermocline. The vertical circulation during frontogenesis gives it a sloping coherence which has been observed by Leach et al. (1985).

The level of \( \text{IPVG} \) is sufficient everywhere in the seasonal thermocline for synoptic-scale deformation to produce vigorous mesoscale frontogenesis. At locations where gyre-scale confluence has already sharpened the \( \text{IPVG} \) into a frontal stream, cyclogenesis winds up the narrow front into a spiral, containing bands of relatively high and low \( \text{IPV} \) (Hoskins et al. 1985). Mesoscale frontogenesis at the boundaries between these bands is particularly vigorous, with intense vertical circulation, which can produce spectacular increases in phytoplankton biomass in a spiralling pattern around eddies (like the cloud bands associated with atmospheric cyclones).

The limitation of jet growth which we have found to result from not allowing the jet to draw its energy from infinitely far away has implications for the potential enstrophy cascade in the thermocline. If the jet remains two-dimensional it cannot become narrower than about 10 kilometres, and the centres of the cyclonic and anticyclonic shear zones remain over 5 km apart. The explanation for this fact is, that at high Rossby number the jet becomes asymmetric with further sharpening on the cyclonic side, but none on the anticyclonic side, with the result that upwelling (on the anticyclonic side) becomes very slow, while quite rapid downwelling can continue on the cyclonic side. The cascade has not penetrated very far into the mesoscale waveband before it stops itself by flattening the isopycnals; so the frontogenesis mechanism described here does not effect potential enstrophy dissipation inside the thermocline. Our model suggests that the potential enstrophy would be dissipated in the surface mixed layer as the jet rises into it. The other possibility is that the jet will become unstable and meander, allowing curvature vorticity to carry the cascade to higher wavenumber. The results of our investigation of that process will be the subject of a later paper.

8. Conclusions

We have presented results from a prognostic primitive equation model of frontogenesis that accurately describes the local dynamics as the Rossby number approaches unity. The model yields frontal structures that differ from those predicted by the model of MacVean and Woods (1980) based on the semi-geostrophic equations, in the sense that the front stops intensifying beyond a certain stage. Nevertheless, the basic mechanism of mesoscale frontogenesis, namely the ageostrophic circulation resulting from vortex stretching as the relative vorticity becomes comparable with the Coriolis frequency, is the same in both models.

As the Rossby number approaches unity, vortex stretching flattens isopycnals, despite the continuing tendency of the deformation field to increase baroclinicity. In the ocean, the region of confluence has limited extent, which also limits the initial available
potential energy convertible to kinetic energy of a mesoscale jet. The model has been used to investigate the consequences of this limitation with respect to the final shape of the jet by a series of runs in which the model sidewalls have been kept away at various distances from the initial baroclinic zone. The main result is, that the jet stops intensifying as soon as the catchment area of available potential energy coincides with the lateral extent of the region in which flattening of isopycnals, induced by vortex stretching, occurs. In addition, vortex stretching sets a limit on the minimum width of the jet, thereby truncating the cascade of potential enstrophy to high wavenumber and causes the jet to become progressively shallower. Other features of mesoscale frontogenesis that were described by the MacVeain–Woods model (e.g. the vigorous upwelling on the anticyclonic side of the jet) are confirmed and described accurately for the first time.

Tests with different initial conditions have shown that the final form of the mesoscale jet depends on the initial profile of the isopycnic gradient of potential vorticity (IPVG); a detailed sensitivity study will be published later. We have shown that the magnitude of IPVG in the seasonal thermocline is everywhere sufficient to fuel vigorous mesoscale frontogenesis. The location of mesoscale fronts therefore depends on that of the confluences which sharpen them, but the latter does not determine their eventual form. We considered two types of confluence. The first (synoptic-scale deformation) takes less than a week to form mesoscale jets up to one hundred kilometres long. The second (gyre-scale deformation) is a permanent feature of the large-scale circulation. It takes about a month to produce long (order megametres) narrow jets (or ‘streams’) with high Rossby number dynamics. Examples in the upper ocean include the branches of the North Atlantic current, the Azores current and the Sargasso Sea front. The failure of oceanic general circulation models to describe the observed occurrence of intense streams in the North Atlantic current is attributed to their present inability to resolve the IPVG in the thermocline.

APPENDIX

Finite difference equations

The finite difference equations for Eqs. (1), (2) and (3) have been derived by Bleck (1978, 1979) and Bleck and Boudra (1981) for a staggering scheme referred to by Arakawa and Lamb (1977) as the ‘C’ grid. Some elements of these equations can be traced back to Sadourny (1975). The finite difference momentum equations conserving potential vorticity and potential enstrophy whenever \( s = \alpha \) and \( \dot{s} = 0 \) are

\[
\left( \frac{\partial u}{\partial t} \right)_s + \delta_x \left[ \frac{u^x + p^{x^y}}{2} \right] - V^y \overline{Q}^y + (\delta_u \overline{p}^y)^{-1} \left( \frac{\partial p}{\partial s} \right) \delta_x u = -\overline{\alpha}^x \delta_x \overline{p}^y - \delta_x \overline{\Phi}^y \quad (24a)
\]

\[
\left( \frac{\partial v}{\partial t} \right)_s + \delta_y \left[ \frac{u^x + p^{x^y}}{2} \right] + \overline{U}^{xy} \overline{Q}^{xy} + (\delta_y \overline{p}^y)^{-1} \left( \frac{\partial p}{\partial s} \right) \delta_y v = -\overline{\alpha}^y \delta_y \overline{p}^y - \delta_y \overline{\Phi}^y \quad (24b)
\]

Here, \( (U, V) = (u \delta_x \overline{p}^y, v \delta_y \overline{p}^y) \) are the components of the mass flux vector and

\[
Q = (\delta_x v - \delta_y u + f)/\delta_y \overline{p}^y
\]

is the quantity which in isopycnic coordinates represents potential vorticity. The overbars denote an averaging process over two adjacent grid points (this averaging can be compounded in different directions as, for example, in \( \overline{p}^{x^y} \)) and the \( \delta_x, \delta_y, \delta_z \) operators are the single-grid-interval finite difference analogs of \( \partial/\partial x, \partial/\partial y, \partial/\partial z \). The right-hand sides of (24a) and (24b) can be rewritten by using the Montgomery potential.
Applying the 'product rule' (Bleck 1978, Eq. (17b)), making use of the distributivity of the $\delta_r$ operator and recognizing $\alpha p^x + \Phi^x$ as a finite difference form of $M$ we note that

$$-\overline{\alpha}^x \overline{\delta_r p^x} - \delta_x \overline{\Phi^x} = -\delta_x M + \overline{\rho^x} \delta_x \alpha$$  \hspace{1cm} (26a)

follows for the right-hand side of (24a). In the same manner the right-hand side of (24b) changes to

$$-\overline{\alpha}^y \delta_r \overline{\Phi^y} = -\delta_y \overline{\rho^y} \delta_y \alpha.$$  \hspace{1cm} (26b)

The finite difference hydrostatic equation expressed in terms of $M$ can be obtained by differentiating (8) with respect to $s$ and substituting (1c). The resulting equation

$$\delta_s M = \rho \delta_s \alpha$$ \hspace{1cm} (26c)

can easily be integrated from the bottom up to calculate $M$. Note that $M$ only changes where $\alpha$ changes, i.e., $M$ is constant in the vertical within isopycnic layers.

The finite difference continuity equation reads

$$\frac{\partial}{\partial t} \delta_s p + \delta_x U + \delta_y V + \delta_x \left( \delta_s \frac{\partial p}{\partial s} \right) = 0.$$ \hspace{1cm} (27)

Bleck and Boudra (1981) derived two versions of the finite difference thermodynamic equation. The first is able to model the conversion between potential and kinetic energy, the second conserves specific volume. We favoured the second, which is

$$\frac{\partial \alpha}{\partial t} + \frac{1}{\delta_s p} \left[ \overline{U \delta_s \alpha^x} + \overline{V \delta_y \alpha^y} + \left( \delta_s \frac{\partial p}{\partial s} \right) \delta_s \alpha \right] = 0.$$ \hspace{1cm} (28)

For our model use we shall modify Eqs. (24), (26), (27) and (28) in the following way:

— all finite difference derivatives with respect to $y$ become zero;
— averaging over $y$ can be omitted;
— averaging over $x$ must be replaced by averaging over $\bar{x}$;
— time derivatives $\partial/\partial t$ have to be replaced by $(\partial/\partial t)_s$;
— $u, v$ have to be replaced by $u', v'$;
— the finite difference form of the deformation terms $u' \partial u/\partial x$ and $v' \partial v/\partial y$ in (15a, b) must be added to the left-hand sides of (24a, b).

Then the momentum equations (24a, b) reduce to

$$\frac{\partial u'}{\partial \bar{x}} \frac{\partial}{\partial \bar{t}} + \delta_x \left( \frac{U^{2g} + v'}{2} \right) - V' Q + (\delta_x p')^{-1} \left( \delta_s \frac{\partial p}{\partial s} \right) \delta_x u' - \gamma u'' = -\delta_x M + \overline{\rho^x} \delta_x \alpha$$  \hspace{1cm} (29a)

$$\frac{\partial v'}{\partial \bar{x}} \frac{\partial}{\partial \bar{t}} + \delta_x \left( \frac{U^{2g} + v'}{2} \right) - V' Q + (\delta_x p')^{-1} \left( \delta_s \frac{\partial p}{\partial s} \right) \delta_x v' + \gamma v'' = 0 \hspace{1cm} (29b)

where $U' = u' \delta_r \overline{\rho^x}$ and $V' = v' \delta_r \overline{\rho^y}$ and $Q = (\delta_x v' + f)/\delta_r \overline{\rho^x}$. The continuity equation (27) simplifies to

$$\frac{\partial}{\partial \bar{t}} \delta_s p + \delta_x U' + \delta_s \left( \delta_s \frac{\partial p}{\partial s} \right) = 0$$ \hspace{1cm} (30)

and the thermodynamic equation (28) is now

$$\frac{\partial \alpha}{\partial \bar{t}} + \frac{1}{\delta_s p} \left[ U' \delta_s \alpha^x + \left( \delta_s \frac{\partial p}{\partial s} \right) \delta_s \alpha \right] = 0.$$

Equation (26c), for determining the Montgomery potential, remains unchanged.
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