AN ADDITIONAL BIBLIOGRAPHICAL SKETCH ON THE DEVELOPMENT OF ERTEL'S POTENTIAL VORTICITY THEOREM

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1. INTRODUCTION

I note with interest the recent paper by Hoskins et al. (1985) concerning the use of Ertel's potential vorticity. Gill (1982), Hide (1983), Kahlig (1970), Pedlosky (1987), Pichler (1966) and many other theoreticians and synoptic meteorologists beginning with Reed (1955) and Reed and Danielsen (1959) have used the fundamental Ertel theorem.

On the other hand, publications on theoretical meteorology and related geophysical topics have tended to overlook Ertel's other work. It is fitting that this oversight be remedied. Therefore, a short bibliographical review of Ertel's papers may be of interest.

2. EARLY STAGES OF ERTEL'S RESEARCH

Hans Ertel engaged at an early stage in hydrodynamical problems and their application in meteorology, geophysics, physical hydrography and oceanography. At the beginning of his scientific career he gave an elegant proof of Bjerknes' circulation theorem (Ertel 1933). In the following years he was mostly interested in meteorological research (e.g. the theory of atmospheric turbulence, pressure waves, extended forecasting, tropopause waves, cyclone movement, eddy viscosity, etc.).

He worked on problems of hydrodynamical research, e.g. transformation of the basic equations and variational principles of hydrodynamics. In 1939 he reduced the equations of motion of an atmosphere with an arbitrary equation of state to Hamilton's general dynamic principle. He published two books: Methods and problems in dynamical meteorology (1938, reprinted 1972) and Elements of operational calculus (1940).

In 1941 Ertel studied the equations of atmospheric circulation in connection with the pressure field. He gave an equation which is similar to those used more recently for numerical forecasts of the pressure field; however, the significance of his equation was not understood at that time (1941a). In addition, he dealt with theoretical questions of weather prediction. In his paper 'The impossibility of an exact weather forecast based on pressure charts for small portions of the Earth' (1941b) he showed mathematically that it is not possible to obtain an unambiguous solution to the problem of calculating the pressure (or wind) field for time $t > 0$ even for a homogeneous and incompressible atmosphere, on the basis of initial conditions at $t = 0$ in a given portion of the earth, since the boundary conditions depend on unknown factors outside the area. Subsequently (1948a) Ertel developed a method for approximating the future displacements of air masses. For the purpose of short-range weather forecasting, a system of equations was developed analytically to answer, as a first approximation, the question of which air masses will arrive at a given locality at some future time. The solution of this system of equations results from a process of successive approximation, whose convergence conditions are determined. For practical application, Ertel gave a simple graphical method. In 1948 too, Ertel discussed in detail the fundamental problems of weather prediction from the standpoint of theoretical meteorology (1948b). In all of these works Ertel employed the results of geophysical hydrodynamic research.

3. ASPECTS OF THE DEVELOPMENT OF ERTEL'S THEOREM AND HYDRODYNAMICAL RESEARCH

In 1942, Ertel presented his general hydrodynamic vortex theorem, and he showed that the laws of classical hydrodynamics can be substantially generalized. In the same year he demonstrated that Bjerknes' theorem is a special case of his own vortex theorem (see Ertel 1942d; Gill 1982). After 1942, Ertel presented several hydrodynamical vortex theorems which through specifying a free scalar or vector field function allow simple derivations of the known vortex theorems. In consequence of these vortex theorems he developed further hydrodynamical conservation laws (see Table 1; Ertel 1960, 1964, 1965b, c; Gill 1982).
During 1949 Ertel and Carl-Gustav Rossby published a new conservation theorem derived from the Lagrangian form of hydrodynamical equations (Ertel 1952, 1955a; Ertel and Rossby 1949a, b).

For steady circulation-preserving motion, a measure of convection in terms of stream tubes, vortex tubes, and the cells into which they dissect space was published by Ertel and Köhler (1949). A very elegant theorem on circulating motions was published by Ertel in 1950: Truesdell (1958) has described it in detail for English readers. The Ertel theorem on circulating motions states that in a steady, rotational circulation-preserving motion whose streamlines are closed curves intersecting a certain surface orthogonally, the circulation of the streamlines is a function of their energy only. Hence it is the same for all streamlines lying upon the same Lamb surface (these surfaces were introduced by Lamb 1878; see Truesdell 1958).

In 1955 Ertel presented a canonical algorithm for the reduction of the known hydrodynamical vortex theorems in differential form (Ertel 1955a). In the same year he developed another hydrodynamic vortex theorem (1955b). By specifying a scalar function it gives Bjerknes' circulation theorem.

His later work dealt with several hydrodynamical questions and applications in meteorology, geophysics, hydrology and oceanography (e.g. oceanic circulation dynamics, wind-driven currents, boundary layer flow, hydrodynamics of rivers, ocean wave force). Important results of this work include the physical vorticity theorem and the commutation formulae, and their various applications (see Friedman 1978; Friedman and Schutz 1978; Katz and Lynden-Bell 1982; Meiss and Horton 1983; Thomson and Stewart 1977; Truesdell 1951a, b, 1958).

In 1969 Ertel studied the substantial variation of vortices within a finite time interval. He showed that the differential equation of vortices for an autobarotropic state (one in which pressure is a function of density only, see, e.g., Gill (1982) page 228) has the form of a homogeneous differential equation of first order. Cauchy's analytic description of the individual variation can be regarded as a special case of this equation. Other results of Ertel's later work, e.g. the corresponding representations of the equation of continuity and his theorem of 1969, can be used for several meteorological problems. From his final meteorological research we mention the 'Relationship of ageostrophic wind deviation to the impulse moment' (1969a), his study of 'Vorticity as horizontal

<table>
<thead>
<tr>
<th>Year</th>
<th>Topic</th>
<th>References</th>
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</thead>
<tbody>
<tr>
<td>1939</td>
<td>Variation principle of hydrodynamics</td>
<td>Ertel 1939</td>
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<tr>
<td>1939</td>
<td>General variation principle of the atmospheric dynamic</td>
<td>Ertel 1939</td>
</tr>
<tr>
<td>1941</td>
<td>Ertel's general program for weather forecasting to 1948</td>
<td>Ertel 1941a-b, 1944, 1948a-b</td>
</tr>
<tr>
<td>1948</td>
<td>Method for approximating the future displacements of air masses</td>
<td>Ertel 1948a</td>
</tr>
<tr>
<td>1949</td>
<td>Ertel–Rossby convection theorem</td>
<td>Ertel and Rossby 1949a, b</td>
</tr>
<tr>
<td>1949</td>
<td>Ertel–Köhler's description of steady convection</td>
<td>Ertel and Köhler 1949</td>
</tr>
<tr>
<td>1950</td>
<td>Ertel's theorem on circulation motions</td>
<td>Ertel 1950; Truesdell 1958</td>
</tr>
<tr>
<td>1952</td>
<td>Ertel's potential theorem</td>
<td>Ertel 1952, 1954</td>
</tr>
<tr>
<td>1955</td>
<td>Ertel's general hydrodynamical theorem (including the Helmholz vorticity theorem)</td>
<td>Ertel 1955b</td>
</tr>
<tr>
<td>1960</td>
<td>General representation of the equation of continuity</td>
<td>Ertel 1960, 1965b</td>
</tr>
<tr>
<td>1962</td>
<td>Hydrodynamical equations of motion and vortex equations in corresponding forms</td>
<td>Ertel 1962</td>
</tr>
<tr>
<td>1964</td>
<td>Ertel's commutation formulae</td>
<td>Ertel 1964/65c; Kahlig 1970</td>
</tr>
<tr>
<td>1969</td>
<td>Differential equations of fluid dynamics for an autobarotropic vertical flow</td>
<td>Ertel 1969a</td>
</tr>
<tr>
<td>1970</td>
<td>Flow theorem of meteorology and hydrology</td>
<td>Ertel and Cadez 1970</td>
</tr>
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shear along geodetic lines flowing over a mountain ridge' (1967). 'Relationship between the frictional force and the wind in the Gauss–Argand diagram' (1969b) and 'Relationship between kinematic parameters of horizontal flow fields in the atmosphere' (1970a), and his last paper, on the inertial motions in the atmosphere and the ocean. In 1970 he published a useful note on 'Transformation of the differential form of the Weber hydrodynamic equation in relation to the Earth's rotation'. It showed that by including the Coriolis term, Weber's differential equations of hydrodynamics are transformed into a system that can be applied to meteorology and hydrography. The extended Weberian equations are readily seen to have a direct relationship with the circulation theorem of Bjerknes and variation principles of atmospheric dynamics derived by Ertel.

4. CONCLUSION

Ertel's fundamental work in theoretical meteorology and the application of fluid dynamics to meteorological and geophysical problems, culminating in his general vortex theorem, provided a key element in the transition from classical to modern meteorology and geophysical hydrodynamics. His results meanwhile have found manifold application in all parts of meteorological and geophysical science.

It is fitting that the neglect of his work be remedied. Therefore it would be most valuable if an English translation of his general work could be made available. In the meantime, the following selected reference list may be helpful for scientists working in the field of theoretical meteorology and geophysics, including the application of Ertel's theorem.

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