Spin vectors and rates of change of wind direction

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SUMMARY

Established formulae for various rates of change of horizontal wind direction may be unified by a treatment in terms of the spin vector, S, of a vector field M. The spin vector measures the rate of change of direction of M, in the appropriate osculating plane, and may be defined with respect to any infinitesimal displacement in space and time. When M is a horizontal vector field, the osculating plane is horizontal, and S immediately gives the rate of change of direction relative to a fixed horizontal axis. The formulae involved are those of Hide and of Bryden for the rate of change of geostrophic wind direction with height, and of Burk and Staley and of Lecluyse and Neumann for the Lagrangian rate of change of wind direction. When M is a 3-dimensional vector field, the orientation of the osculating plane is not in general constant and the Cartesian components of S do not represent the rate of change of direction in the most concise terms. The Lagrangian spin vector of the 3-dimensional velocity field may nevertheless be considered as a basic kinematic quantity, on a par with the rates of change of kinetic energy, vorticity and divergence.

1. INTRODUCTION

Various expressions for rates of change of horizontal flow direction have been presented in the meteorological and oceanographical literature. Hide (1971) and Bryden (1976) give formulae for the rate of change of geostrophic flow direction with an axial coordinate and with height, while Burk and Staley (1979) and Lecluyse and Neumann (1986) consider the Lagrangian rate of change of horizontal flow direction without restriction to geostrophic (or indeed hydrostatic) dynamics.

It is shown here that the above results may be unified through a spin vector treatment. Spin vectors may be defined for any vector field but arise most naturally in the differential geometry of twisted curves in 3-dimensional space. This topic is considered briefly in section 2, together with a generalization of the spin vector concept which is necessary for most meteorological and oceanographical applications. In section 3, the expression of Burk and Staley and of Lecluyse and Neumann is obtained, as are semi-geostrophic and quasi-geostrophic approximations to it, by exploiting a general spin vector formula. The expression of Hide (1971) and Bryden (1976) is obtained from this formula in section 4. In section 5 the 3-dimensional Lagrangian spin vector of non-hydrostatic flow is considered. Concluding remarks are contained in section 6.

Apart from its discussion of the 3-dimensional spin vector, this short paper is concerned solely with the unification of familiar formulae for rates of change of wind direction. The reader who is happy to accept the familiar formulae piecemeal, and who is not interested in questions of underlying structure, is likely therefore to find little of importance here. It is felt, however, that such questions of structure are not entirely academic, since their resolution shows the individual formulae in a new perspective and offers conceptual simplification.

2. SPIN VECTORS OF TANGENTS, BINORMALS AND VECTOR FIELDS

Suppose that M is a vector which is a function of a single parameter q. The vector

\[ S(M) = \left( \mathbf{M} \times \frac{d\mathbf{M}}{dq} \right) / \mathbf{M}^2 \]  

1535
is known as the *spin vector* of $\mathbf{M}$ (with respect to $q$). The magnitude of $\mathbf{S}$ is equal to the angular rate of change of direction of $\mathbf{M}$, and $\mathbf{S}$ is directed along the axis of the associated infinitesimal rotation. Proof of these results is elementary; see, for example, Wardle (1965).

Equation (1) finds one of its most important applications in the differential geometry of twisted curves—a subject which is clearly relevant to fluid dynamics because particle trajectories are in general twisted curves. The following outline is a simple adaptation of text-book material on differential geometry and vector analysis (Wardle 1965; Weatherburn 1965). Figure 1 summarizes nomenclature and spatial orientation.

Consider a twisted curve represented by a vector function $\mathbf{r}$ of a convenient parameter, and—with fluid dynamical trajectories in mind—choose time $t$ as the parameter. Then $d\mathbf{r}/dt = \mathbf{U}$ = velocity of a particle describing the trajectory, and $d^2\mathbf{r}/dt^2 = d\mathbf{U}/dt$ (for which the equation of motion provides an expression).

Suppose that $\mathbf{U} = U\mathbf{i}$, where $U = |\mathbf{U}| \neq 0$. Then $\mathbf{i}$ is the unit *tangent* to the trajectory. Suppose further that $|d\mathbf{i}/dt| \neq 0$. The vector $d\mathbf{i}/dt$ is then directed at right angles to $\mathbf{i}$ (because $|\mathbf{i}| = 1$) and lies in the plane containing $\mathbf{U}$ and $d\mathbf{U}/dt$. This plane is the *osculating plane* to the trajectory. Clearly

$$ \frac{d\mathbf{i}}{dt} = \mathbf{p} \frac{d\alpha}{dt} \tag{2} $$

where $\mathbf{p}$ is a unit vector lying in the osculating plane and directed perpendicular to $\mathbf{i}$, and $\alpha$ is the angle between $\mathbf{i}$ and a chosen direction $\mathbf{c}$ which lies in the osculating plane. $\mathbf{p}$ is the unit *principal normal* to the trajectory, $\alpha$ being measured anticlockwise from $\mathbf{c}$; see Fig. 1. The spin vector of the unit tangent $\mathbf{i}$ is

$$ \mathbf{S}(\mathbf{i}) = \mathbf{i} \times \frac{d\mathbf{i}}{dt} = \mathbf{b} \frac{d\alpha}{dt} \tag{3} $$

where $\mathbf{b} = \mathbf{i} \times \mathbf{p}$ is the unit *binormal* to the trajectory. It is important to note that

$$ \mathbf{S}(\mathbf{U}) = \left( \mathbf{U} \times \frac{d\mathbf{U}}{dt} \right) / U^2 = \mathbf{i} \times \frac{d\mathbf{i}}{dt} = \mathbf{S}(\mathbf{i}). \tag{4} $$

Also

$$ \frac{d\alpha}{dt} = U/R \tag{5} $$

where $R$ is the radius of curvature of the trajectory.

Since it is perpendicular to the osculating plane, the binormal specifies the orientation of that plane. It can be shown that

$$ \mathbf{S}(\mathbf{b}) \equiv \mathbf{b} \times \frac{d\mathbf{b}}{dt} = Ti \tag{6} $$

where

$$ T = \frac{R^2}{U^5} \left[ \mathbf{U}, \frac{d\mathbf{U}}{dt}, \frac{d^2\mathbf{U}}{dt^2} \right] \tag{7} $$

the square brackets denoting the triple scalar product of the enclosed vectors. The quantity $T$ is the *torsion* of the trajectory. It gives the rate of turning (here defined with respect to time) of the binormal about the unit tangent. From Eqs. (6) and (7) it is apparent that the orientation of the osculating plane is not in general constant along 3-dimensional trajectories. This is an intuitively reasonable result, and an important one.
If we were interested only in rates of change of flow direction along material trajectories, the above discussion would provide an adequate framework for analysis. However, we may be concerned with rates of change of direction of other vectors along other trajectories, with local changes, or with rates of change with respect to distance only. To analyse such situations it is helpful to generalize the spin vector concept.

Consider the change of direction of a general 3-dimensional vector field $\mathbf{M} = \mathbf{M}(x, y, z, t)$ which results from infinitesimal changes $\delta \mathbf{r} = (\delta x, \delta y, \delta z)$, $\delta t$ in the (Cartesian) space and time coordinates. Clearly

$$\delta \mathbf{M} = \delta \mathbf{r} \cdot \nabla \mathbf{M} + \delta t(\partial \mathbf{M}/\partial t)$$

(8)

(in which $\nabla$ is the 3-dimensional gradient operator). The vector

$$\left(\mathbf{M} \times \delta \mathbf{M}\right)/M^2$$

(9)

has magnitude equal to the change in direction of $\mathbf{M}$ and is directed along the axis of the associated rotation. Now any infinitesimal displacement $(\delta \mathbf{r}, \delta t)$ can be expressed in terms of an infinitesimal change $\delta \tau$ in a single parameter $\tau$ as

$$(\delta \mathbf{r}, \delta t) = \delta \tau (\mathbf{l}, \lambda)$$

(10)

where $\mathbf{l}$ is a vector and $\lambda$ a scalar, both chosen according to the particular displacement. Then

$$S(\mathbf{M}) = \left(\mathbf{M} \times \frac{d\mathbf{M}}{d\tau}\right)/M^2$$

(11)

defines the spin vector $S(\mathbf{M})$ of the vector field with respect to changes in the parameter $\tau$. $S(\mathbf{M})$ evidently has all the properties of the spin vector defined by Eq. (1).
Some examples may help to clarify things. An instantaneous physical displacement \( \delta r = m \delta \tau \) (where \( m \) is the unit vector in the direction of \( \delta r \), and \( \delta r = |\delta r| \)) has \( \delta r = \delta \tau \), and

\[
(\mathbf{l}, \mathbf{\lambda}) = (m, 0) \\
\frac{d}{d\tau} = m \cdot \nabla.
\]

(12)

In particular, instantaneous vertical displacements have \( \delta \tau = \delta z \), \( \mathbf{M} = (0, 0, 1) \) and so \( \frac{d}{d\tau} = \partial/\partial z \). A displacement corresponding to the motion in time \( \delta t \) in the velocity field \( \mathbf{A} \) has \( \delta \tau = \delta t \) and

\[
(\mathbf{l}, \mathbf{\lambda}) = (\mathbf{A}, 1) \\
\frac{d}{d\tau} = (\partial/\partial t + \mathbf{A} \cdot \nabla).
\]

(13)

\( A \) is the chosen advection flow. It may or may not be the actual flow \( U \) in a particular application. Local changes with respect to time have the prescription (13) with \( A = 0 \).

These examples show that \( \tau, \mathbf{l} \) and \( \mathbf{\lambda} \) in (10) need not each have the same dimensions in all cases. It is necessary only that \( \tau l \) have the dimensions of length, and \( \tau \lambda \) the dimensions of time.

In this way a spin vector of the field \( \mathbf{M} \) may be defined with respect to any specified infinitesimal displacement, whether it be an instantaneous physical displacement, a displacement along real or imagined particle trajectories, or a local time change. It is clear that the notions of osculating plane and torsion remain well-defined and useful. The osculating plane is defined by the vectors \( \mathbf{M} \) and \( \mathbf{M} + \delta \mathbf{M} \) and is normal to the spin vector \( \mathbf{S}(\mathbf{M}) \), while the torsion represents the rate of rotation of \( \mathbf{S}(\mathbf{M}) \) about the direction of \( \mathbf{M} \).

In sections 3 and 4, Eq. (11) is applied to the cases noted in section 1. In each of these, \( \mathbf{M} \) is a velocity field having horizontal components only. An important simplification then occurs. For if \( \mathbf{k} \cdot \mathbf{M} \) and \( \mathbf{k} \cdot \delta \mathbf{M} \) both vanish (\( \mathbf{k} \) being unit vector in the vertical) the associated spin vectors are purely vertical; the osculating plane is always horizontal and the torsion is zero. This is so although: (i) \( \mathbf{M} \) in general has vertical extent and variation; (ii) vertical displacements may be involved in the calculation of \( \delta \mathbf{M} \); and (iii) particle trajectories themselves may have non-zero torsion. When \( \mathbf{M} \) has horizontal components only

\[
\mathbf{S}(\mathbf{M}) = \left( \mathbf{M} \times \frac{d\mathbf{M}}{d\tau} \right) / \mathbf{M}^2 = k d\alpha/d\tau
\]

(14)

where \( \alpha \) is the angle (measured anticlockwise) between \( \mathbf{M} \) and some fixed direction \( \mathbf{c} \) in the horizontal plane. We choose \( \mathbf{c} \) to be the zonal (\( x \)) direction. The case in which \( \mathbf{M} \) is the 3-dimensional velocity field \( U \) is examined in section 5.

3. LAGRANGIAN RATE OF CHANGE OF HORIZONTAL WIND DIRECTION

Suppose that

\[
\frac{d}{d\tau} = (\partial/\partial t + \mathbf{A} \cdot \nabla)
\]

(15)

where \( \mathbf{A} \) is a 2- or 3-dimensional velocity field, and that we deal with the equation of horizontal motion in the (possibly approximated) form

\[
(\partial/\partial t + \mathbf{A} \cdot \nabla)\mathbf{M} = - (1/\rho) \nabla H P - f k \times \mathbf{V} + \mathbf{F}.
\]

(16)
The notation used on the right-hand side of Eq. (16) follows that of Lecluyse and Neumann (1986), except for the use of \( \mathbf{V} \) (rather than \( \mathbf{U} \)) to represent the horizontal wind vector. A Cartesian coordinate system rotating with angular velocity \( \frac{\mathbf{f}}{k} \) is assumed. Three different cases of Eqs. (15) and (16) are of interest.

(i) General horizontal motion. With the choices \( \mathbf{A} = \mathbf{U}, \mathbf{M} = \mathbf{V} \), Eq. (16) becomes the exact equation of horizontal motion. Use of Eq. (14) then gives

\[
V^2 \frac{d\alpha}{dt} = -fV \cdot \mathbf{V}_a + k \cdot (\mathbf{V} \times \mathbf{F}).
\]  

(17)

Here \( d/ dt = (\partial/ \partial t + \mathbf{U} \cdot \nabla) \) is the material derivative, and \( \mathbf{V}_a = \mathbf{V} - \mathbf{V}_g \) is the ageostrophic flow, the geostrophic flow \( \mathbf{V}_g \) being defined as usual by

\[
\mathbf{V}_g = \left( \frac{1}{\rho f} \right) k \times \nabla_h p.
\]  

(18)

Equation (17) was given by Lecluyse and Neumann (1986), and an equivalent form by Burk and Staley (1979). It can be derived by using the \( x, y \) components of Eq. (16) in conjunction with \( v = u \tan \alpha \) (where \( u, v \) are the components of \( \mathbf{V} \) in the \( x, y \) directions) but a vector treatment is more concise and makes immediately clear the geometric relations between \( \mathbf{V} \) and \( \mathbf{V}_a \), and between \( \mathbf{V} \) and \( \mathbf{F} \), which are associated with a Lagrangian rate of change of flow direction. Equation (17) shows that neither ageostrophic flow perpendicular to \( \mathbf{V} \) nor frictional forces parallel to \( \mathbf{V} \) are associated with changes in \( \alpha \) following the 3-dimensional motion. The importance of frictional forces having components perpendicular to \( \mathbf{V} \) was one of Lecluyse and Neumann's main concerns; the occurrence of such forces has also been noted recently by Arya (1985).

Qualitative physical interpretation of the term \( -fV \cdot \mathbf{V}_a \) in Eq. (17) is straightforward. For example, if the flow is westerly and hypergeostrophic then the Coriolis force outweighs the pressure gradient force and an equatorward deflection of the flow will occur.

Equation (17) gives the rate of change of the direction of horizontal motion of a particle. Let \( V \) be the horizontal speed of flow. Then the specific kinetic energy equation

\[
V \frac{dV}{dt} = \nabla \cdot \mathbf{V} - \frac{1}{\rho} \nabla \cdot \nabla_h p
\]  

(19)

gives the rate of change of \( V \). Since they arise from the operations \( \mathbf{V} \times \) and \( \nabla \cdot \) applied to the equation of horizontal motion, Eqs. (17) and (19) constitute a complementary pair of relations in the same way as the (vertical) vorticity and (horizontal) divergence equations do. Together they define the rate of change of \( \mathbf{V} \) in a hodographic representation. Such a representation is attractive because: (i) the rate of change of specific kinetic energy is described by one of the governing equations only; and (ii) Coriolis effects only enter the equation for \( d\alpha/dt \).

Equations (17) and (19) take on particularly simple forms when natural coordinates are adopted. Let \( \mathbf{j} \) be the unit vector perpendicular to the direction of motion \( \mathbf{i} \) such that \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \). If \( s \) and \( n \) are coordinates locally parallel to \( \mathbf{i} \) and \( \mathbf{j} \), and frictional forces are neglected, then Eqs. (17) and (19) become

\[
d\alpha/dt = -f \{1 + (1/\rho f V) \partial \rho/\partial n\}
\]  

(17a)

\[
dV/dt = -(1/\rho) \partial \rho/\partial s.
\]  

(19a)

Equations (17a) and (19a) are equivalent to forms given by Haltiner and Martin (1957; section 11.13) and Hoskins (1975; section 3). Nearly-geostrophic motion is typified by \( d\alpha/dt = V/r \ll f \), where \( r \) is the radius of curvature of the horizontal projection of particle
trajectories (Hoskins, op. cit.). In the case of purely ageostrophic motion, Eq. (17a) reduces to the familiar expression for the rate of anticyclonic turning of the wind vector in inertial oscillations (see Thompson 1978).

(ii) Semi-geostrophic and quasi-geostrophic approximations. With the choices $A = U$, $M = V_g$, Eq. (16) becomes the semi-geostrophic equation of horizontal motion (Hoskins 1975). Use of Eq. (14) then gives

$$V_g^2 \frac{d\alpha}{dt} = -fV_g \cdot V_a + k \cdot (V_g \times F).$$

(20)

With the choices $A = M = V_g$, Eq. (16) becomes the quasi-geostrophic equation of horizontal motion (see, for example, Gill 1982). From Eq. (14)

$$V_g^2 \left( \frac{\partial}{\partial t} + V_g \cdot \nabla \right) \alpha = -fV_g \cdot V_a + k \cdot (V_g \times F).$$

(21)

In both (20) and (21), $\alpha$ corresponds to the direction of the geostrophic flow $V_g$.

The semi-geostrophic and quasi-geostrophic expressions (20) and (21) differ only in the advection terms appearing on their left-hand sides. Both differ from Eq. (17) in the appearance of: (i) $V_g^2$ instead of $V^2$ on their left-hand sides; (ii) $V_g \times F$ instead of $V \times F$; and (iii) $V_g \cdot V_a$ instead of $V \cdot V_a$. In terms of $V_g = |V_g|$, the corresponding kinetic energy equations are

$$V_g \frac{dV_g}{dt} = V_g \cdot F - \frac{1}{\rho} V \cdot \nabla \Pi p$$

(22)

$$V_g \left( \frac{\partial}{\partial t} + V_g \cdot \nabla \right) V_g = V_g \cdot F - \frac{1}{\rho} V \cdot \nabla \Pi p$$

(23)

both of which contain the same ageostrophic flow term as the general form (19).

4. RATE OF CHANGE OF GEOSTROPHIC WIND DIRECTION WITH HEIGHT

In the case $d/d\tau = \partial/\partial z$, $M = V_g$, Eq. (14) gives

$$V_g^2 \frac{\partial \alpha}{\partial z} = k \cdot \left\{ V_g \times \frac{\partial V_g}{\partial z} \right\}$$

(24)

for the rate of change of geostrophic flow direction with height. Upon use of the thermal wind equation, Eq. (24) reduces to

$$V_g^2 \frac{\partial \alpha}{\partial z} = (g/f) \theta V_g \cdot \nabla \theta$$

(25)

which relates $\partial \alpha/\partial z$ to the geostrophic advection of potential temperature $\theta$. Equation (25) follows from the definition of $V_g$ (Eq. (18)) assuming only the hydrostatic approximation.

Relations equivalent to Eq. (25) are well known, of course, and have been derived and discussed by Hide (1971) and Bryden (1976), amongst others. (Miller and Thompson (1942) gave a similar form—and attributed it to C.-G. Rossby.) Use of the geostrophic approximation in the thermodynamic equation enables $\partial \alpha/\partial z$ to be related through Eq. (25) to the vertical motion $w$ in steady, adiabatic flow (see, for example, Hide 1971). Rhines (1986) gives references to the many oceanographic studies which have exploited the resulting relation. He also notes that Eq. (25) may be derived easily by forming $V_g \times \partial V_g/\partial z$, which is essentially the spin vector approach.

(Hide (1971) in fact treats a case somewhat different from that studied here. In examining generalizations of the Proudman–Taylor theorem, he considers rates of change with respect to an axial coordinate parallel to the frame rotation vector $\Omega$, and adopts an extended definition of geostrophy.)
5. **Lagrangian spin vectors in non-hydrostatic flow**

In non-hydrostatic flow the Lagrangian spin vector

\[ S(U) \equiv (U \times dU/dt)/U^2 \tag{26} \]

gives the rate of turning, in the osculating plane, of the direction of the 3-dimensional flow \( U \) (see Eqs. (3) and (4)). The components of \( S \) relative to fixed axes \( Oxyz \) are equal to the components of the Lagrangian rate of turning of \( U \) relative to \( Oxyz \). However, this representation is unhelpful in at least two respects. First, the components so defined are not equal to the three spin vectors obtained by considering the motion (and accelerations) projected successively onto the three coordinate planes. This situation arises because \( U^2 \), the squared magnitude of the 3-dimensional flow, occurs in the denominator of Eq. (26). Second, since only two angles are needed in order to define a flow direction relative to \( Oxyz \), the three spin vector components do not give the most concise description of rates of change of flow direction. There are various ways of extracting the desired information. The most direct is to introduce the azimuth \( \alpha \) and elevation \( \beta \) of the velocity vector \( U \), defined as indicated in Fig. 2, so that \( U \) may be expressed as

\[ U = U(\cos \beta \cos \alpha, \cos \beta \sin \alpha, \sin \beta) \tag{27} \]

where \( U = |U| \). It is then readily shown that

\[ \frac{d\alpha}{dt} = \frac{U^2}{V^2} k \cdot S(U) = \{k \cdot (V \times dV/dt)\}/V^2 \tag{28} \]

and

\[ d\beta/dt = n \cdot S(U) \tag{29} \]

where

\[ n = -k \times U/V. \tag{30} \]

(Here—as in section 3—\( V \) is the horizontal part of \( U \), and \( V = |V| \).)

![Figure 2. The azimuth \( \alpha \) and elevation \( \beta \) of the velocity vector \( U \).](image-url)
Equations (28) and (29) show how the rates of change of azimuth and elevation of \( U \) are related to the components of the spin vector \( S(U) \) in the vertical and in the direction normal to the vertical plane containing the instantaneous motion. The second equality in Eq. (28) is consistent with Eq. (14). In terms of \( V \) and the vertical component \( W \), Eq. (29) reduces to

\[
U^2 \frac{d\beta}{dt} = V \frac{dW}{dt} - W \frac{dV}{dt}
\]

which may be more easily obtained by differentiating \( W = V \tan \beta \) and noting that \( U^2 = V^2 + W^2 \).

No particularly startling reduction occurs when the equations of horizontal and vertical motion are used to substitute for \( \frac{dV}{dt} \) and \( \frac{dW}{dt} \) in Eq. (31)—there is no simplification like that involving the Coriolis terms in the derivation of Eq. (17) for \( \frac{d\alpha}{dt} \). Nevertheless, the resulting equation for \( \frac{d\beta}{dt} \) could be useful in the interpretation of material trajectories.

The 3-dimensional Lagrangian spin vector seems to be important mainly in conceptual respects. \( U \times \frac{dU}{dt} \) and \( U \cdot \frac{dU}{dt} \) are complementary quantities exactly as in the essentially 2-dimensional problem considered in section 3(i): \( U \times \frac{dU}{dt} \) is a vector carrying information about the rate of change of flow direction, while \( U \cdot \frac{dU}{dt} \) is a measure of the rate of change of flow speed. Through the physical interpretation of \( S \), the quantity

\[
U^2 S = U \times \frac{dU}{dt}
\]

becomes a recognizable dynamical entity amongst the many possible combinations of the velocity, acceleration, vorticity, energy and divergence fields and their rates of change.

6. Concluding Remarks

The aims of this note have been to bring out the underlying spin vector structure of various expressions for rates of change of horizontal flow direction, and to draw attention to certain properties of the 3-dimensional Lagrangian spin vector. A number of incidental suggestions seem worth following up.

Some properties of quasi-geostrophy (Eq. (25), for example) relate directly—and simply—to the speed and direction of the geostrophic flow. The behaviour of the entire quasi-geostrophic model in such a hodographic form should be investigated. Note that a natural coordinate system is not envisaged: the horizontal velocity field would be represented by speed \( V \) and direction \( \alpha \) as functions of spatial coordinates (and time) as usual. The hydrostatic primitive equations might also be susceptible to such a representation; some potential advantages were noted in section 3.

The Cartesian components of the 3-dimensional Lagrangian spin vector \( S(U) \) do not give a concise description of rates of change of non-hydrostatic flow direction. However, \( S(U) \) itself is a basic kinematic quantity whose dynamic properties deserve further study. Together with other differential geometric quantities, it might also prove useful in the analysis and interpretation of 3-dimensional particle trajectories.

Finally, it may be noted that only spin vectors of velocity fields have been considered here. Spin vectors of other vector fields are of general theoretical interest and might repay investigation.

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