A study of topographically induced multiple equilibria and low-frequency variability. I: Idealized topography

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(Received 7 January 1988; revised 26 July 1988)

SUMMARY

It is shown that both the steady and the vacillatory states of a two-layer, quasi-geostrophic, forced dissipative topographic model with a high spectral resolution can be truly intransitive. There exists only an odd number of equilibria in this geophysical system, as predicted by Benjamin's general theorem for the Navier-Stokes equation. The overall findings constitute a generalization of previous results for a barotropic model.

The analyses specifically establish for the case of a single-wave topography that, in response to a barotropic forcing, there exist: (1) a low threshold value of the topographic height required for the existence of hysteresis; (2) a higher threshold value for the occurrence of weakly vacillating states with a high frequency; (3) a detuning effect of the nonlinearity due to an increase of the equivalent resonance frequency; (4) a baroclinization effect of an asymmetric friction; and (5) a robustness of the multiple equilibria for an additional weak baroclinic forcing.

Under near-resonance conditions of the topographic waves for the case of a simple multi-wave topography, as defined according to the linear dynamics, none of the equilibria is stable. The strong wave–wave interaction in such a situation sustains a dynamically different pronounced vacillation with a dominant long period in the order of tens of days. The existence of multiple vacillations with distinctly different structure and period is established. Outside that range of parametric conditions, there is only one stable equilibrium state.

It is demonstrated that the time-mean wavy flow of a decidedly nonlinear time-dependent response can be reproduced, in a high degree of resemblance, by tuning the dissipation parameters in a counterpart linearized model that makes use of the 'known' mean zonal flow. It would be misleading, however, to conclude, on the basis of such an agreement, that the linear dynamics really accounts for the actual time-mean wavy flow. This example serves to reveal the potential dangers in interpreting an observed mean flow as consisting of linear planetary waves on a pre-determined zonally symmetric basic state.

1. INTRODUCTION

According to the observational studies, the structure and dynamical character of atmospheric blocking and low-frequency fluctuations might stem from a strong interaction between the time-mean flow and the synoptic time scale eddies (e.g. Dole 1983; Hoskins et al. 1983; Colucci 1985). There are, most likely, multiple causes for this class of phenomena. One distinct possibility, first suggested by Namias (1962), is that an anomalous condition of either the land or the sea surface variables, such as soil moisture, snow cover or sea surface temperature, could induce an 'anomalous' response with the observed characteristics. Some recent diagnostic as well as GCM calculations have provided evidence in support of that view. The internal nonlinear dynamical process in the atmosphere by itself may naturally give rise to such response. Simmons et al. (1983) have suggested that the low-frequency fluctuations with a time scale of tens of days might be simply associated with certain linear unstable modes of the climatological-mean upper tropospheric wavy flow (e.g. 300 mb). Another specific suggestion is based on the notion of topographically induced multiple equilibria or, more generally, multiple weather regimes in the presence of only a steady external forcing (Hart 1979; Charney and DeVore 1979; Reinhold and Pierrehumbert 1982). A variation of this notion is that of local nonlinear resonance (e.g. Pierrehumbert and Malguazzi 1984). The sea–air interaction may also play a special role in sustaining the intraseasonal variability, particularly in the tropical circulation (e.g. Lau and Peng 1987; Emanuel 1987). In contrast, Lindzen (1986) argues that most of the nonlinear processes mentioned above may not be pertinent to the atmospheric low-frequency variability.
A number of papers have reported observational and model evidence supporting the notion of topographically induced multiple equilibria and low-frequency variability (e.g. Charney et al. 1981; Yoden 1983; Källen 1983; Legras and Ghil 1985; Hansen and Sutera 1986). Yoden (1985) has verified the existence of multiple stable equilibria in a barotropic model with 110 modes in the presence of a weak but finite internal dissipation. But some aspects of the results in low-order models could be artifacts (Tung and Rosenthal 1985; Cehelsky and Tung 1987). It is therefore imperative to verify that the solution for the most strongly nonlinear condition in a given model analysis is no longer sensitive to a further increase of the spectral truncation.

The objective of part I of this investigation is to delineate the nature of the different possible types of topographically induced equilibrated response in the context of a quasigeostrophic two-layer model with idealized topography. It is obvious that as long as an idealized topography is used in an analysis, the model flows would not be directly comparable to the atmospheric flow regardless of the choice for the values of the model parameters. A more serious effort towards deducing some more realistic response will be made in part II, in which the actual topography of the northern hemisphere will be used. We will first ascertain the existence of topographically induced multiple equilibria and/or equilibrated states with a low-frequency variability in a simple forced baroclinic model that has a single-wave topography. An extension of that analysis is to explore the existence of distinctively different types of possible response in such a geophysical system with a multi-wave, but still highly idealized, topography. A sufficient spectral resolution will be used in each case to ensure that whatever results one gets are not artifacts of a low-order system. Near a resonance condition, as defined according to the linear theory, it is found that the nonlinear model may have several equilibria of which none, or one, or more than one, may be stable, dependent upon the height and shape of the topography. This overall finding is perhaps not too surprising since the topography dictates the degree of nonlinearity.

One naturally wonders if, in analogy to multiple equilibria, there might also exist multiple vacillations in relevant geophysical fluid systems. It intuitively appears to be entirely possible. After all, the latter (multiplicity of periodic attractors) is just a mathematical generalization of the former (multiplicity in fixed point attractors). It should be perhaps emphasized that by multiple vacillations we mean distinctively different vacillations for a given external condition. As such, we are not referring to the well-known different forms of vacillation encountered under different parametric conditions. It is at least known that the cubic nonlinear logistic map

\[ x_{n+1} = (a - 1)x_n - ax_n^3 \]  

(1)

does have two periodic attractors for the parameter range \(2.0 < a < 4.0\) (Knobloch and Weiss 1987). Because of their possible relevance to atmospheric flows, it warrants a search for the existence of multiple vacillations in this topographic model. It will be seen that multiple vacillations indeed exist in this basic model for the case of multi-wave topography.

The model is described in section 2. Section 3 outlines the two complementary methods of analysis employed in this study. The reasoning behind the choice of the parameter values is discussed in section 4. The various results for the case of a single-wave topography are then presented in the six subsections of section 5. We will investigate the dependence of the equilibrated response and its stability property upon the strength of a barotropic forcing, the height of the topography, the asymmetry in the friction and the role of an additional baroclinic forcing. The results for the case of a simple multi-wave topography are presented in section 6. We will demonstrate that not only may
equilibrated states with low-frequency fluctuations exist, but also they can have multiplicity in such a system. Section 7 reports an additional analysis intended to reveal the fallacy of interpreting the time-mean flow of a nonlinear system in terms of a corresponding linearized model. The paper ends with some concluding remarks.

2. Model

Let us consider a two-layer quasi-geostrophic model with two forms of dissipation. One form is introduced through the Ekman layer dynamics as represented by \( r_j \nabla^2 \psi_j \) (\( j = 1, 2 \)) where \( \psi_j \) are the nondimensional Q–G streamfunctions. The coefficients \( r_j \) are expected to be of \( O(10^{-1}) \) since they are related to the Rossby number \( Ro \) and the Ekman number \( E_j \) of the two layers by \( r_j = (\sqrt{E_j})/Ro \). The other form of dissipation is an internal friction represented as a biharmonic diffusion, \( \kappa \nabla^6 \psi_j \) (\( j = 1, 2 \)) with \( \kappa/r_j \sim 10^{-3} \). This is intended to serve as a sink for the enstrophy cascade towards the high wavenumber components. The nondimensional potential vorticity equations can then be written as

\[
\begin{align*}
\partial q_1/\partial t + J(\psi_1, q_1) &= -(r_1 \nabla^2 + \kappa \nabla^6)\psi_1 \\
\partial q_2/\partial t + J(\psi_2, q_2 + h) &= -(r_2 \nabla^2 + \kappa \nabla^6)\psi_2
\end{align*}
\]

(2) (3)

where \( q_1 = \nabla^2 \psi_1 + \beta y - F(\psi_1 - \psi_2) \) and \( q_2 + h = \nabla^2 \psi_2 + \beta y + F(\psi_1 - \psi_2) + h \) are the Q–G potential vorticities of the flow in the two layers. \( J \) stands for the Jacobian operator, \( y \) the meridional coordinate and \( h(x, y) \) a topography. Distance, velocity and time are to be measured in units of \( L_\ast, U_\ast \) and \( L_\ast/U_\ast \) respectively. \( L_\ast \) is chosen so that the nondimensional width of the domain is equal to \( \pi \). The ratio of the meridional to zonal length scales of the fundamental wave mode is denoted by \( \gamma \). The length of the domain is then \( 2\pi/\gamma \). The height of the topography is measured in units of \( 2D\, Ro \) where \( D \) is the mean dimensional thickness of each layer. Thus \( h(x, y) \) should be at most of \( O(1) \).

Furthermore, the Froude number, \( F \), is \( f^2 \, L_\ast^2/[gD \delta \rho/\rho] \) where \( g \delta \rho/\rho \) is the reduced gravity and \( f \) the mean Coriolis parameter. The beta parameter, \( \beta \), is \( L_\ast^2 \beta/\pi U_\ast \). Both are likely to have values on the range 1 to 10.

The forcing in the studies cited earlier is typically introduced as a Newtonian heating in terms of a single zonal spectral component of a background temperature \( (\theta^\ast) \) in conjunction with a relaxation coefficient. Alternatively, we may consider a forcing in a form that is often used in nonlinear instability studies, namely through the introduction of a steady zonal background flow \( U_j \). The corresponding streamfunction is

\[
\Psi_j = -U_j y \quad \text{for} \quad j = 1, 2
\]

(4)

where \( U_j \) are constants. Such a flow (4) is a solution of (2) and (3) only for the special case of a zonally symmetric topography. In general, the term \( J(\Psi_2, h) \) is non-zero. Since the topographic term \( J(\psi_2, h) \) arises from the vertical velocity induced at the topographic boundary in a quasi-geostrophic system, it plays the role of a mechanical forcing of the system under consideration. It is analogous to the vorticity forcing introduced in the lower layer by Källen (1983). The flow would necessarily depart from \( \Psi_j \). The energy of this system is removed through the frictional process and the mountain torque associated with the departure field of \( \psi_j \) from \( \Psi_2 \).

The zonal background flow \( U_j \) may be equivalently introduced in terms of a barotropic part, \( V = (U_1 + U_2)/2 \), and a baroclinic part, \( U = (U_1 - U_2)/2 \). The analysis is then performed in terms of barotropic and baroclinic components, \( \psi \) and \( \theta \), of the departure
field of the flow defined by

\[
\begin{align*}
\psi_1 &= - (V + U)y + (\psi + \theta) \\
\psi_2 &= - (V - U)y + (\psi - \theta).
\end{align*}
\]

Upon substituting (5) into (2) and (3), we can rewrite the governing equations as

\[
(\partial / \partial t + V \partial / \partial x)(\nabla^2 \psi + h/2) + J(\psi, \nabla^2 \psi + \beta y + h/2) + J(\theta - Uy, \nabla^2 \theta - h/2) = -(r_\ast \nabla^2 + \kappa \nabla^6) \psi - r_\ast \nabla^2 \theta
\]

\[
(\partial / \partial t + V \partial / \partial x)(\nabla^2 \theta - 2F \theta - h/2) + J(\theta - Uy, \nabla^2 \psi + \beta y + 2F \psi + h/2) + 
J(\psi, \nabla^2 \theta - h/2) = -(r_\ast \nabla^2 + \kappa \nabla^6) \theta - r_\ast \nabla^2 \psi
\]

where \( r_\ast = (r_1 + r_2)/2; \quad r_\ast \ast = (r_1 - r_2)/2. \) The domain under consideration is \( 0 \leq x \leq 2\pi/\gamma, \ 0 \leq y \leq \pi. \) Since we are not trying to simulate any particular geographical feature, \( \gamma \) will be set to unity in this investigation. A more realistic value of the aspect ratio \( \gamma \) will be used in part II when the actual topography is under consideration. The appropriate boundary conditions are, with \( \xi \) for both \( \psi \) and \( \theta, \)

\[
\xi(0, y, t) = \xi(2\pi \gamma^{-1}, y, t)
\]

\[
\xi_x = 0; \quad \int_0^{2\pi \gamma^{-1}} \xi dx = 0 \quad \text{at} \quad y = 0, \pi.
\]

The solution as well as the topography may be represented in terms of the basis functions of the Laplace operator satisfying the boundary conditions (8). They are then expressed as

\[
\xi = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \xi_{m,n} S_{m,n} + \sum_{k=1}^{K} \xi_{0,k} C_{0,k}
\]

with \( \xi \) being \( \psi, \theta \) or \( h. \) The basis functions are the same as those used in Mak (1985):

\[
S_{m,n} = \sqrt{\gamma/\pi} \cdot e^{in\psi} \sin(ny); \quad C_{0,k} = \sqrt{\gamma/\pi} \cos(ky).
\]

\( \psi_{0,k} \) and \( \theta_{0,k} \) are real variables, whereas \( \psi_{m,n} \) and \( \theta_{m,n} \) are complex variables. The use of (9) and (10) in conjunction with (6) and (7) then leads to a set of \((4MN + 2K)\) coupled ordinary differential equations of real variables.

Two idealized forms of topography are examined in this paper. They are prescribed in terms of the spectral coefficients as follows:

(A) \( h_{1,1} = \delta \neq 0, \quad h_{i,j} = 0 \quad \text{for} \quad i \neq 1, j \neq 1 \)

(B) \( h_{1,1} = h_{1,2} = h_{2,1} = \delta \neq 0, \quad \text{with all other} \ h_{i,j} \ \text{being zero}. \)

Topography (A) stands for the extreme form of the idealized topography. It has been used in many previous model studies. We use it just as a simplest setting for investigating the effects of other physical factors. Topography (B) is chosen as a first attempt to address the nonlinear dynamics of a flow over a multi-wave topography in the simplest possible manner. The object is to establish the existence of different types of possible equilibrated states without pretending that such states would have genuine counterparts in the real atmosphere. We will see that the response even for such simple forms of topography with a moderate height can be quite complex and intriguing. We will be in a better position to investigate the case of a 'realistic' topography after we have acquainted ourselves with these relatively simple situations.
It is found, by making some trial computations with different degrees of spectral truncation, that using $M = N = K = 9$ would be adequate for establishing convergence of the solution for topographic height up to order unity (i.e. $\delta \sim 1$) and dissipation coefficients $r_j \approx 0.1$. The results reported here are obtained with such a spectral resolution. The computational task amounts to solving a set of 342 coupled nonlinear ordinary differential equations for 342 real dependent variables, $\{X_j\}$. They are symbolically referred to as

$$dX_j/dt = F_j(X_1, X_2, \ldots, X_J) \quad \text{for} \quad j = 1, 2, \ldots, J, \text{with } J = 342$$

for convenience. Each $F_j$ is an algebraic expression with quadratic nonlinearity which represents the wave–zonal and wave–wave interactions. A lower truncation would be sufficient for the cases of a lower topography.

3. METHODS OF ANALYSIS

Relatively complete information about the dynamics of the response $\{X_j\}$ at large time can be deduced by applying two complementary methods of analysis. One method is to integrate (12) forward in time numerically with a standard second-order predictor–corrector algorithm. We will do so with a time step of 0.01 non-dimensional time units ($\approx 15$ minutes) which is sufficiently small for the range of conditions under consideration. This method, however, has obvious limitations. While it enables us to straightforwardly determine a stable equilibrium state in a particular run, it does not allow us to ascertain the existence of unstable equilibria. Another drawback is that this method can be quite computationally time consuming, particularly in those situations where the system evolves towards a certain unknown equilibrated state.

We may alternatively seek to determine the steady state solution $\{\bar{X}_j\}$ directly by solving the following set of nonlinear algebraic equations

$$F_j(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_J) = 0, \quad \text{for} \quad j = 1, 2, \ldots, J.$$  \hfill (13)

This of course can only be done iteratively. The so-called Powell (1970) hybrid method is used. Provided that the initial guess is sufficiently close to an equilibrium state, one can determine the latter after a small number of iterations. When we apply the algebraic method, the solution obtained with the time integration method for a particular parameter setting far from a resonance condition may be conveniently used as an initial guess for the case of an adjacent value of the scanning parameter. This method makes it feasible for us to establish the variation of each of the multiple equilibria with the key parameters.

A desirable follow-up analysis is to determine the linear stability property of each equilibrium state. Defining a perturbation $\{X'_j\}$ by

$$X_j = \bar{X}_j + X'_j \quad \text{where} \quad \bar{X}_j \gg X'_j$$  \hfill (14)

we obtain

$$dX'_j/dt = \sum_{k=1}^{J} \left. \frac{\partial F_j}{\partial X_k} \right|_{\bar{X}_j} X'_k \quad \text{for} \quad j = 1, 2, \ldots, J.$$  \hfill (15)

We next reduce (15) to an eigenvalue problem by seeking a solution for $X'_j$ in the form of $e^{\alpha t}$. All this can be done quite straightforwardly. By scanning through all the eigenvalues (342 of them in the case of $M = N = K = 9$), we can then tell whether the equilibrium state under consideration is linearly stable or not. When it is unstable, the magnitude of the maximum eigenvalue would further reveal whether the equilibrium is strongly or
weakly unstable. It might be added that since the equilibria, which constitute nonzonal flow fields, are generated by specific topographic forcing and the equations for $X'$ involve the topography as well, there is no ambiguity in the interpretation of such instability results (Andrews 1984).

4. PRELIMINARY CONSIDERATION

We begin by making use of the linear theory for this forced circulation to help identify the range of parameter values in which the nonlinear process should be important. Such a range must be near a condition for resonance which occurs when the forcing frequency matches that of one of the normal modes. When there is no topography and background flow this model has two inviscid normal modes for each pair of wavenumbers $(k, l)$ corresponding to a barotropic Rossby wave and a baroclinic Rossby wave. Their respective zonal phase speeds are

$$C_{**} = -\beta/(k^2 + l^2), \quad C_* = -\beta/(k^2 + l^2 + 2F).$$

(16)

It follows that a steady background flow $V$ with $U = 0$ over a topographic wave component with a wavenumber $(k, l)$ in this system would resonantly excite: (1) a barotropic Rossby wave if $V = -C_{**}$, and (2) a baroclinic Rossby wave if $V = -C_*$. These two values can be quite far apart for low wavenumbers because of the dependence on the Froude number. Hansen and Sutera (1986) have reported observational evidence of two distinctive statistical flow regimes that exhibit a baroclinic vertical structure. It may then be relevant for us to first focus on a range of $V$ in the neighbourhood of $V = V_* = -C_*$. For the type (A) topography defined in (11), we expect that the wave field would consist of a dominant baroclinic Rossby wave component, $\theta_{1,1}$ and some of its spectral neighbours. The latter are indirectly excited as a result of the nonlinear cascade process. It is useful to recall the algebraic conditions for having non-zero values for the interaction coefficients associated with the basis functions under consideration. The interaction between two wave modes with wavenumbers $(m, n)$ and $(-m, \bar{n})$ would generate a zonal mode with wavenumbers $(0, l)$, only if

$$l + n + \bar{n} = \text{odd integer}.$$  

(17)

The interaction between a wave mode $(i, n)$ and a zonal mode $(0, \bar{l})$ would generate a wave mode $(i, j)$ provided that

$$j + n + \bar{l} = \text{odd integer}.$$  

(18)

The interaction between two wave modes $(m, n)$ and $(\tilde{m}, \tilde{n})$ for $|m| \neq |\tilde{m}|$, would generate a third wave mode $(i, j)$ where

$$i = m \pm \tilde{m} \quad j = |n \pm \tilde{n}|.$$  

(19)

These are respectively the conditions for having a non-zero value for the interaction coefficients $D_{\mu \nu}^1$, $D_{\mu}^1$, and $A^{ij}_{\mu \nu \theta}$ given in Mak (1985). The interaction coefficients then enable us to predict in advance which subset of spectral modes could be possibly excited. For example, since the $(k, l) = (1, 1)$ wave is directly driven through the topographic effect, only the following subset of modes would be excited: $(0, 1), (0, 3), (0, 5), \ldots, (1, 1), (1, 3), (1, 5), \ldots, (2, 2), (2, 4), (2, 6), \ldots, (3, 1), (3, 3), (3, 5), \ldots$, etc. They constitute a family of modes with the $(1, 1)$ wave as the head member of the family. The interaction between the $(1, 1)$ wave and the $(0, 1)$ zonal mode is strongest in this case, and the wave–wave interaction is secondary. How strong these modes are would depend upon the degree of nonlinearity. The latter should be stronger for a higher topography and a weaker viscosity.
For the type (B) topography, the directly forced waves have wavenumbers 
\((k, l) = (1, 1), (1, 2), (2, 1)\). Likewise, we may think of the wave modes in terms of three 
families of modes, each of which is associated with one of these directly forced waves. 
All spectral modes can be excited in this case through the cascade process. If the 
resonance conditions in terms of \(V\) for the three topographic waves turn out to be close 
together, the directly forced waves would interact strongly. These are cross-family 
interactions. In this case, the wave–wave interaction may be stronger than the wave– 
zonal flow interaction. The total wave field could be quite complex and even its qualitative 
features cannot be readily anticipated in advance. We will see that the results turn out 
to be quite intriguing and yet readily understandable from first principles.

5. RESULTS FOR A SINGLE-WAVE TOPOGRAPHY (TYPE (A))

Let us begin by examining the response to perhaps the simplest possible form of 
forcing in conjunction with the simplest form of topography. The former is a barotropic 
forcing \((V \neq 0, U = 0)\) and the latter is a single-wave topography (type (A)). It suffices 
to consider a vertically symmetric friction with a moderate value \((r_1 = r_2 = 0.1\) and 
\(K = 10^{-4}\)) and \(\beta = 10\) and \(F = 2.8\) as in Mak (1985). Resonant excitation of a baro-
tropic Rossby wave mode, \(\psi_{1,1}\), requires a surface flow, \(\bar{u}_2 = (-C_{\psi\psi})_{\text{dim}} = 5U_\ast\). With 
\(U_\ast \sim 5\) m s\(^{-1}\), this would be rather strong. On the other hand, resonant excitation 
of the baroclinic Rossby wave mode, \(\theta_{1,1}\), only requires a much weaker surface flow, 
\(\bar{u} = (-C_{\psi\theta})_{\text{dim}} = 1.3U_\ast\). Thus, we will focus upon the range \(0.5 \leq V \leq 2.5\) in 
the following analysis. The effect of an additional baroclinic forcing will be considered in 
subsection 5(f).

(a) Distinctly different branches of equilibria

The first set of computations is made by the time integration method for a sig-
nificant topographic height \(\delta = 1.0\) with different values of \(V\) but with the same initial 
condition. An initial state of a weak arbitrary circulation is used in these calculations 
\(\psi_{m,n} = \theta_{m,n} = 0.01; \psi_{0,k} = \theta_{0,k} = 0\) for all \(m, n\) and \(k\). The solution at large time in 
each parameter setting is a steady state with \(\theta_{1,1}\) being the dominant wave as expected. 
The results of this set of computations already reveal the existence of two distinctively 
different branches of solution. One branch is characterized by a strong wave field which 
increases with \(V\) up to \(V \sim 1.5\). The other branch is a much weaker wave field of which 
the intensity increases with decreasing values of \(V\) for \(V > 1.5\). The two branches do not 
meet at \(V \sim 1.5\). Associated with the strong wave field, there is a large zonal flow 
correction, mostly due to the barotropic gravest meridional mode \(\psi_{0,1}\). The latter is a 
consequence of the wave–zonal flow interaction. It follows that the strong wave state has 
a weak net zonal flow. Conversely, the weak wave state has a strong net zonal flow. 
These two branches of solution are referred to as the upper branch and lower branch 
respectively.

The next set of computations confirms that these two branches overlap over a 
substantial range of \(V\). Specifically, an upper branch equilibrium can be found at \(V = 
1.6\) if we use the steady state solution for \(V = 1.5\) as the initial state, and this procedure 
is repeated for progressively higher values of \(V\). Figure 1(a) shows the variation of the 
amplitude of \(\theta_{1,1}\) with \(V\) obtained in the two sets of computation. Figure 1(b) shows the 
corresponding phase angle. The negative values of this phase angle represent a phase lag 
relative to the topography such that the pressure at a point on the western side of a ridge 
is larger than that at the counter point on the eastern side. That gives rise to an net 
mountain torque. The two stable equilibria constitute a hysteresis loop. Such a result of
hysteresis is therefore genuine and is not an artifact of spectral truncation. This is in agreement with the barotropic model results of Yoden (1985) and Rambaldi and Mo (1984).

A qualitatively different feature of the upper branch equilibrium emerges when \( V \) reaches 1.75. Each spectral component in the equilibrated state now has a small oscillating component about the mean. The oscillation is found to be stronger for larger values of \( V \) up to \( V = 2.05 \). Figure 2 shows such a manifestation in the evolution of \( |\psi_{1,2}| \) for \( V = 1.9 \) at large time. The initial state of this run is the equilibrated state for \( V = 1.8 \). This periodic solution is reminiscent of Yoden's (1985) stable periodic solution. The transient eddy activity found by Holloway and Eert (1987) in their 'blocked' regime in a very high resolution barotropic model with a topography that consists of a series of meridional ridges and a broadband spectrum of random relief seems to be essentially
Figure 2. Evolution of the amplitude of the wave components $\psi_{1,j}$, $j = 1$ to 6 for $V = 1.9$, $U = 0.0$, $\delta = 1$, $r_1 = r_2 = 0.1$, illustrating the emergence of a weakly vacillatory state.

associated with this type of oscillation. The amplitude of the fluctuation in both results is small compared with the mean value. We will show in subsection 5(d)(iii) that the emergence of this oscillation is associated with a Hopf bifurcation. When $V$ is increased beyond 2.05, the wave field collapses and the system quickly evolves to the lower branch equilibrium state.

(b) Threshold parametric value for multiple equilibria

The next set of computations is made for the purpose of determining the threshold value of the topographic height $\delta$ that can sustain multiple equilibria. Since the multiple equilibria owe their existence to the nonlinear effect, they should degenerate to a single equilibrium if the topographic height $\delta$ is sufficiently low. Figure 3 shows a threshold value of $\delta$ of about 0.18 for $r_1 = r_2 = 0.1$ and thus confirms that anticipation. The dimensional threshold value of the topography is then $0.36Ro \, D \sim 200 \, \text{m}$ for $Ro \sim 0.1$ and $D \sim 5 \, \text{km}$. Since this is a rather small value, it is not a very stringent requirement. The threshold value of $\delta$ is naturally higher for a greater damping. It is also reasonable to find that the range of values of $V$ for multiple equilibria to exist is broader for a higher mountain. Moreover, we see from Fig. 3 that the weakly oscillating state discussed in the last subsection would emerge only when the topographic height exceeds a second threshold value ($\delta \sim 0.65$).
(c) **Effect of nonlinearity on the resonance condition**

A meaningful question to ask at this point is: Why does the upper branch equilibrium ‘lean’ towards the larger values of $V$ with respect to the value of linear resonance $V_*$ as seen in Fig. 1? The dynamical nature of this feature may be understood as follows. Linear resonance occurs when we ‘tune’ the system with a forcing frequency equal to that of one of its normal modes (natural frequencies). But the frequency of a finite amplitude wave also varies with its amplitude. We now show that the effective frequency of this topographically induced wave increases with its amplitude, as most readily shown for a weakly nonlinear case. For such a case, we may truncate the large set of equations to a subset consisting of only the following four equations that govern the dominant zonal and wave components $(X_1, X_2, X_3, X_4) = (\psi_0, \theta_0, \psi_1, \theta_1, \delta)$.

\[
\begin{align*}
  dX_1/dt &= a_{1,1}X_1 + a_{1,2} \text{ Im}((X_3 - X_4)\delta) \\
  dX_2/dt &= a_{2,1}X_2 + a_{2,2} \text{ Im}((X_3 - X_4)\delta) + a_{2,3} \text{ Im}(X_3 X_4) \\
  dX_3/dt &= a_{3,1}X_3 + a_{3,2}(X_1 - X_2)\delta + a_{3,3}(X_1 X_2 + X_1 X_2) + a_{3,4} \delta \\
  dX_4/dt &= a_{4,1}X_4 + a_{4,2}(X_1 - X_2)\delta + a_{4,3}X_3 X_2 + a_{4,4}X_4 X_1 + a_{4,5} \delta
\end{align*}
\]

where the coefficients $a_{m,n}$ are known constants. Suffice to note that, with $\kappa = 0$, $r_1 = r_2 = r$, $\Delta = 8\gamma^3/(3\pi^2)$ and $\lambda_{m,n} = (\gamma m)^2 + n^2$, we have

\[
\begin{align*}
  a_{3,3} &= -i(\lambda_{1,1} - \lambda_{0,1})\Delta/\lambda_{1,1} \\
  a_{4,1} &= -i[\beta/(2F + \lambda_{1,1}) - V] - r\lambda_{1,1}/(2F + \lambda_{1,1}) \\
  a_{4,3} &= -i(\lambda_{1,1} - \lambda_{0,1} - 2F)\Delta/(2F + \lambda_{1,1}) \\
  a_{4,4} &= -i(\lambda_{1,1} - \lambda_{0,1} + 2F)\Delta/(2F + \lambda_{1,1}) \\
  a_{4,5} &= -i\gamma[2(2F + \lambda_{1,1})].
\end{align*}
\]
One can readily derive from (20)
\[ d^2X_4/dt^2 = [(a_{4,1})^2 + (a_{4,4}X_1)^2 + a_{4,3}a_{3,3}(X_2)^2]X_4 + a_{4,1}a_{4,5}\delta + \text{remaining terms}. \]  
(22)

The quantity in the square brackets is negative definite in the limit of \( r = 0 \). The effective resonant frequency is the square root of the coefficient of \( X_4 \) in (22). Such frequency may increase significantly due to the nonlinear feedback associated with a finite zonal flow correction \( (X_1, X_2) \). Consequently, even if the system is externally driven at the resonant condition, it becomes progressively 'detuned' as soon as the wave grows. It follows that a stronger response occurs when the system is driven with a super-resonant frequency \( (V > V^*) \) as seen in Fig. 1(a). A similar detuning effect is common in other nonlinear systems (see Pippard 1985).

\( d \) Complementary results obtained with the algebraic method

We next present the variation of the equilibria as a function of \( V \) determined with the algebraic method, and the results of their linear instability property.

(i) Equilibria for a weakly nonlinear case. Let us begin by briefly examining the result with a weak nonlinearity associated with a low topography, say \( \delta = 0.2 \). Figure 4 shows the variation of \( \psi_{0,1}, |\theta_{1,1}|, \) and the phase angle of \( \theta_{1,1} \) with \( V \). As one might anticipate on the basis of Fig. 3, there are two slightly overlapping branches of stable equilibria (indicated by the solid curves). They are connected by an unstable branch of equilibria (indicated by the dashed curve) which can be identified as the wave drag instability since the eigenvalue computed with (15) has a real positive value. Moreover, the ridge of the strong wave state is located at about a quarter wavelength upstream of the topographic ridge \( (~90^\circ \text{ lag}) \). The ridge of the low-wave state in turn is located at half of a wavelength upstream of the topographic ridge \( (~180^\circ \text{ lag}) \). These results are to be expected in light of other studies cited earlier and can be actually reproduced with a much lower-order system.

(ii) Multiplicity of equilibria. An equilibrium state is expected to vary continuously with a parameter such as the background zonal flow \( V \). It follows that we should be able to depict the multiplicity of equilibria by a continuous curve showing the variation of each equilibrium state with \( V \). The topological implication of this is that there can be only an odd number of equilibria at a parameter setting in this system. This property has indeed been proved to be true for the Navier–Stokes equations, and thus for our system, by Benjamin (1976) on the basis of the Leray–Schauder degree theory. New equilibria therefore must emerge in pairs when the degree of nonlinearity is progressively increased. We have seen that there is one unique equilibrium state when the nonlinearity is negligibly weak (e.g. for \( \delta \leq 0.18 \), Fig. 3), and three equilibria exist under finite but moderate nonlinear conditions (e.g. for \( \delta = 0.2 \), Fig. 4). We will see that five equilibria would exist under a strongly nonlinear condition (e.g. \( \delta = 1.0 \), Fig. 5).

(iii) Equilibria for a strongly nonlinear case. A similar set of computations is made for a large topographic height, \( \delta = 1.0 \). Benjamin's theorem serves as guidance for identifying the underlying order among the equilibria obtained from a relatively small number of computations. This is useful especially when the values of a dominant component of the different equilibria are close to one another. A case in point is the response presented in Fig. 5(a) which depicts the variation of the equilibrium \( |\theta_{1,1}| \) with the background flow \( V \) for \( \delta = 1.0 \). It is seen that the stable upper branch equilibria extend to \( V = 1.71 \) and the stable lower branch equilibria extend to \( V = 1.55 \), in agreement with the corresponding result obtained with the time integration method (see Fig. 1(a)).
It is noted that the change from the stable upper branch equilibria to unstable equilibria corresponds to the emergence of a weakly oscillating component in those equilibrium states. The latter was first obtained with the time integration method. The largest eigenvalues of the instability analysis are a pair of complex values. The emergence of the oscillation is therefore identifiable with a Hopf bifurcation. It may be added that wave–wave interaction plays an essential role in this periodic state. For this reason, one would not detect the existence of such response with a single-wave system as in Charney and DeVore (1979), but could do so even with a lower order but multi-wave system as in Yoden (1983).

We must rather be specific if we wish to address the issue whether or not the equilibrium flow under consideration is nonlinearly stable. Obviously, we would not argue on the basis of what would happen to an arbitrarily large initial disturbance, for
in that event the system would asymptotically evolve to the lower branch steady state (weak wave/strong zonal flow). However, the issue of nonlinear stability would still be meaningful if the initial disturbance is finite but not large compared to a certain measure of the ‘size’ of the attractor basin. Let us denote the latter by $R$. Then, provided $\delta$ is small compared with $R$, for any $\delta$ there exists, according to our numerical time integration, an $\varepsilon$ such that $|\psi| < \delta$ at $t = 0 \rightarrow |\psi| < \varepsilon$ for all $t > 0$. It is on the basis of this restricted line of argument that we may conclude that the equilibrium state is nonlinearly stable in the third sense (McIntyre and Shepherd 1987). Only by inference may we state that this attractor is also nonlinearly stable in the second sense.

The two stable branches are linked by unstable equilibria. The branch of unstable equilibria itself is multi-valued. The curve of projection on $|\theta_{1,1}|$ as a function of $V$ has a complex appearance (Fig. 5a). However, the corresponding plots of the phase angle
of $\theta_{1,1}$ (Fig. 5(b)), the zonal barotropic component $\psi_{0,1}$ (Fig. 6(a)) and the zonal baroclinic component $\theta_{0,1}$ (Fig. 6(b)) reveal that the path in phase space linking the two stable branches is an essentially simple one. The phase angle of $\theta_{1,1}$ changes smoothly, although not monotonically, with $V$ from a small negative value to $-180$ degrees when $V$ increases from 0.5 to 2.5. The barotropic zonal flow correction $\psi_{0,1}$ in the equilibria is large and is to be expected as a consequence of the feedback effect of the finite amplitude topographically forced wave. The unstable equilibria near the stable upper branch and those near the stable lower branch for this case of symmetric friction are quite

Figure 6. Variation of the major components of the zonal flow correction, $\psi_{0,1}$ and $\theta_{0,1}$ with $V$ corresponding to the wave solution shown in Fig. 5.
distinguishable, although their corresponding values of $|\theta_{0.1}|$ and $\psi_{0.1}$ are not very different. The distinction is clearly manifested in their zonal baroclinic component $\theta_{0.1}$. Those near the stable upper branch equilibria have a significant positive value of $\theta_{0.1}$ implying a westerly zonal shear with a maximum at the centre of the domain $(-\partial[\theta]/\partial y \approx \theta_{0.1}(\sqrt{\gamma})\pi^{-1}\sin y)$. Likewise, those near the stable lower branch equilibria induce an easterly zonal shear associated with a negative value of $\theta_{0.1}$.

(e) Effect of asymmetric friction

The viscous effect of the subgrid-scale motions near the surface and that near the tropopause in the atmosphere are expected to be not equal in general. A vertical asymmetry in the friction should be the norm rather than an exception. Traditionally, the surface friction is assumed to be much larger than that near the tropopause. However, the wind condition could lead to gravity wave breaking near the tropopause level. An opposite sense of asymmetry might prevail when that happens. The dynamical influence of asymmetric friction can be pronounced in the context of nonlinear equilibration of baroclinic instability (see the references cited in Mak (1987)). It is therefore of interest to ascertain the dynamical impact of the vertical asymmetry in the friction on the topographically forced circulation.

For the purpose of making comparison, let us consider a case of strong asymmetry, $r_1 = 0.01$ and $r_2 = 0.19$, so that the average of $r_1$ and $r_2$ is the same as before. The results are shown in Figs. 7 and 8 which are the counterparts of Figs. 5 and 6. Again we find a stable upper branch connected to a stable lower branch by a multi-valued branch of unstable equilibria along a continuous path. The structure of the dominant wave $\theta_{1.1}$ and the barotropic zonal flow correction of the corresponding stable equilibria are very similar for both cases of symmetric and asymmetric friction. But unlike the case of symmetric friction there is no substantial range of $V$ at which the equilibrated state has a weak periodic fluctuation about the mean. This result corroborates the earlier finding concerning the effect of asymmetric friction on vacillation in a low-order system (Mak 1987). Furthermore, a comparison of Figs. 7 and 8 with 5 and 6 reveals that the asymmetric friction does strongly affect the branch of unstable equilibria with a weak wave field (Fig. 7(a)). It should be stressed that the growth rates of those equilibria are quite small. For example, the state labelled A in Fig. 7(a) has a growth rate of 0.0067. Thus, once a system is near such an equilibrium state, it would take a long time for the system to evolve towards an equilibrium of the lower branch. This is particularly true in the extreme case of asymmetry, $r_1 = 0.0$ and $r_2 = 0.2$. Such a state would appear as a pseudo stable state during a time integration calculation. Figure 9 illustrates the slow transition from an upper branch equilibrium to a lower branch equilibrium through a transient stage that has a large value of $\theta_{0.1}$.

The values of $\theta_{0.1}$ and $\psi_{0.1}$ of those unstable equilibria are very close to one another. It follows that while the zonal flow in the layer with virtually no Ekman layer is greatly modified ($(\psi + \theta)_{ZONAL} \equiv 2\psi_{0.1}$), the zonal flow in the layer with a much more viscous Ekman layer is only slightly modified ($(\psi - \theta)_{ZONAL} \equiv 0$). The system evidently responds to the forcing with a flow configuration that incurs a minimal frictional loss of energy.

It is also noted in Fig. 8 that there is a bias in favour of having a negative value of $\theta_{0.1}$ in the equilbria for a model with a traditional asymmetric friction. This amounts to inducing a westerly zonal baroclinic shear. Such a result may be interpreted as a consequence of a vertical differential frictional effect on the meridional circulation. The latter induces a temperature contrast in support of a vertical shear as required by the thermal wind relation. An opposite sense of asymmetry in the friction ($r_1 \gg r_2$) would
induce equilibria that have a large positive value of $\theta_{0,1}$ (i.e. an easterly zonal baroclinic shear). This dynamical effect of asymmetric friction on the zonal baroclinicity in a system driven by a barotropic forcing is suggested to be a process of 'baroclinization'.

(f) Effect of baroclinic forcing

An obviously relevant issue to be addressed is concerned with the effect of an additional baroclinic forcing on the topographically induced equilibria under a condition of asymmetric friction such as $r_1 = 0.01$ and $r_2 = 0.19$. Another set of computations is therefore made with a fixed barotropic part of the forcing, $V$, in conjunction with a wide range of values for the baroclinic forcing, $U$. Without intending to make very extensive computations, we focus on one particular value of $V$ that is of special interest. It is the value that gives rise to two distinctly different stable equilibria in the absence of a baroclinic forcing, e.g. $V = 1.7$ (see Fig. 7). We seek to determine how each of the five equilibria found earlier for $V = 1.7$ and $U = 0$ would vary with $U$ from $-1.0$ to $1.7$. 

Figure 7. Effect of asymmetric friction on the variation of the equilibria with $V$ in terms of $|\theta_{1,1}|$ and its angle for $U = 0.0$, $r_1 = 0.01$ and $r_2 = 0.19$. 
It should be noted that \((U, V) = (-1.0, 1.7)\) and \((1.7, 1.7)\) corresponds to \((U_1, U_2) = (0.7, 2.7)\) and \((3.4, 0.10)\) respectively. The results are first examined in terms of the amplitude of the dominant wave, \(|\theta_{1,2}|\) (Fig. 10). The stable states are indicated by the solid curve segments and the unstable states by the dashed curve segments. The character of the instability is again interpreted as topographic instability if the largest eigenvalue \(\sigma\) has a positive real value, and as baroclinic instability if the largest eigenvalues are a complex conjugate pair. They are labelled at some sample points by the letters ‘T’ and ‘B’ respectively in Fig. 10. It is seen that the stable weak-wave state exists over a broad range of values of \(U\), \((-0.9 < U < 1.1)\), and that the stable strong-wave state exists
Figure 9. Transition from the upper branch equilibrium at $V = 1.8$ to the lower branch equilibrium at $V = 2.0$ for $U = 0.0$, $r_1 = 0.01$ and $r_2 = 0.19$. (a) Evolution of $|\theta_{1,j}|$, $j = 1$ to 6 and (b) $\psi_{0,j}$, $j = 1$ to 6.
over a narrower but well-defined range of $U$, $(-0.3 < U < 0.25)$. It turns out that the variations with the baroclinic shear, $U$, of the stable weak-wave state and a branch of unstable strong-wave state form a closed curve. One continuous curve depicts the variations with $U$ of the other three equilibria found at $U = 0$ (see Fig. 10). There is an additional stable weak-wave state obtained under conditions of a large surface flow and a strong easterly shear (e.g. $U = -1.0$, $V = 1.7$). It, however, becomes unstable when the easterly shear is weaker ($U > -0.85$). It is of the type of topographic instability. This branch of equilibria changes to a form of stable strong-wave state when the forcing is quasi-resonant, i.e. $V \sim 1.7$ with a weak shear (small $|U|$). Upon further increase of the westerly shear ($U > 0.25$), the stable strong-wave equilibrium state becomes baroclinically unstable. Finally, all equilibria are unstable for a sufficiently large westerly shear with a weak surface flow ($U > 1.1$, $V = 1.7$).

Figure 11(a) shows the corresponding results for the barotropic zonal flow correction, $\psi_{0,1}$. It confirms that when the wave state is strong, the zonal flow correction is large leading to a relatively weak total zonal flow and vice versa. For completeness, we also show the corresponding results of the baroclinic zonal flow correction, $\theta_{0,1}$ in Fig. 11(b). We see that the induced baroclinic zonal flow correction associated with the strong-wave states changes from negative values to positive values as $U$ is changed from negative to positive values. The wave field therefore tends to reduce the net baroclinic shear in the flow.

Finally, the vertical structure of the wave states is examined in terms of the phase angles of the dominant baroclinic wave mode and the corresponding barotropic wave mode. Such results for the former are shown by the solid curve in Fig. 12, and those for the latter by the dashed curve. The results of the phase angle reveal that the quantity $\text{Im}[\theta_{1,1} \psi_{1,1}^*]$ of the stable strong-wave state has a positive value. For example, [angle $(\theta_{1,1}) - \text{angle} (\psi_{1,1})$] is equal to $-184^\circ$ and $-190^\circ$ for $U = -0.2$ and 0.1 respectively. A weak easterly shear would then have a stabilizing influence, whereas a weak westerly shear would have a destabilizing influence. The wavy equilibrium state becomes baroclinically unstable for $U > 0.25$. 
Figure 11. Variation of the major components of the zonal flow correction, $\psi_{0,1}$ and $\theta_{0,1}$, with $U$ for $V = 1.7$ and $\delta = 1.0$. Solid (dash) line segments denote stable (unstable) equilibria.

6. RESULTS FOR A MULTI-WAVE TOPOGRAPHY (TYPE (B))

(a) Equilibria and variability

In order to induce transitions between the two otherwise stable equilibria, Egger (1981) introduced an additional stochastic forcing in a low-order system with a single-wave topography. Be that as it may, we explore the possibility of finding spontaneous fluctuations in the neighbourhood of the equilibria in a system of a multi-wave topography. That would be the case if none of the equilibria turns out to be stable. We now demonstrate this with the use of an idealized three-wave topography as defined by (11). These three waves are deliberately chosen to be a triad so that when they have finite amplitudes they would readily exchange energy among themselves. If the model parameters have such values that the resonance conditions for these three waves are close together, these waves would be simultaneously excited by a certain background flow $V$. It follows that the strong interaction among them as well as their neighbouring spectral modes could prevent any equilibria from being stable.

The three wavy topographic components are prescribed to have equal height, $\delta = 0.5$, and their wavenumbers are $(k, l) = (1, 1), (1, 2)$, and $(2, 1)$. The resonance conditions for the corresponding waves are then $V = 0.83, 0.67$, and $0.67$ respectively for $\beta = 10, F = 5$. Our specific task is to determine the equilibria in such response and their stability properties for different values of $V$. The strongest wave components are naturally
the $\theta_{1,1}$, $\theta_{1,2}$, and $\theta_{2,1}$ waves. Figure 13 shows the variation of their amplitudes as a function of $V$. Figure 14 shows the counterpart results of the zonal flow corrections, $\psi_{0,1}$ and $\theta_{0,1}$. It is seen that there is a single stable equilibrium state for $V \leq 0.79$ and $V \geq 1.2$. Three equilibria exist in the ranges $0.79 \leq V \leq 0.93$ and $1.01 \leq V \leq 1.2$. Five equilibria exist in the central range of values of $V$ ($0.93 \leq V \leq 1.01$). Two features of the results are especially noteworthy. First, there are no topographically induced multiple stable equilibria in this system for any value of $V$. Secondly, there is a range of parameter conditions under which there is not even a single stable equilibrium state ($0.85 \leq V \leq 0.99$). In the event of not having even one stable equilibrium state, the equilibrated state is necessarily a time-dependent one. It turns out to be a vacillation with a high degree of regularity and a period in the order of tens of days. Such equilibrated state is verified by a calculation with the time-integration method for $V = 0.9$. Figure 15 shows the evolution of a few key spectral components in this case. Every wave component fluctuates periodically in a concerted but complex manner. For this parameter setting, the period is about 60 time units (equivalent to about 60 days). Hence, well defined low-frequency variability in a geophysical system can be excited by a steady flow over a topography. This process may be relevant to the study of low-frequency variability in the atmosphere.

A vacillating response has also been observed in a differentially heated annulus apparatus with a wavenumber-two topography (Li et al. 1986). But our vacillation has a
Figure 13. Variations of the amplitude of the three dominant waves of the equilibria with $V$ for $U = 0.0$ in the case of multi-wave topography with $\delta = 0.5$, $r_1 = r_2 = 0.1$. Arrows denote the linear resonant value of $V$ for the waves.

different dynamical origin in that it is primarily sustained by the resonant interaction of the three topographically forced waves. It may be more related to the regular low-frequency oscillations found by Simmons et al. (1983, see Fig. 20) in their nonlinear barotropic model using the climatological-mean 300 mb flow as the background forcing.

It is instructive to examine the vacillation in the physical space from the evolution
Figure 14. Variation of the zonal flow correction $\psi_{0,1}$ and $\theta_{0,1}$ of the equilibria with $V$ for $U = 0.0$ in the case of multi-wave topography with $\delta = 0.5$ and $r_1 = r_2 = 0.1$.

of the total departure field of the flow in the lower layer ($\psi_2$). Any instant may be chosen as a reference time $t = 0$ without loss of generality. The flows at 10 time-unit intervals are shown in panels (a) to (f) in Fig. 16. The topographic configuration consists of a southern mountain and a northern mountain as shown in panel (g) for comparison. The flow at $t = 10$ may be interpreted as the mature stage of a slow cyclone development on the lee side of the southern mountain. There is a corresponding high pressure centre to its north-eastern location. In the next 20 time units (~20 days), the low centre weakens and moves north-westward. It shifts to the lee side of the northern mountain, whereas the high pressure also weakens and moves eastward from its position at $t = 30$. The flow at $t = 40$ is then indicative of a secondary lee-cyclone development on the lee side of the northern mountain. In the next 20 time units, the high centre continues to gradually expand in size and weakens. There is no trace of a low centre at the lee side of the northern mountain by now. A new cycle of topographically induced cyclogenesis begins to take place at $t = 50$ associated with the southern mountain. By $t = 60$, the flow pattern essentially evolves back to that found at $t = 0$. 
Figure 15. Evolution of the major spectral components for $V = 0.9$ in the case of multi-wave topography with $U = 0.0$, $\delta = 0.5$, $r_1 = r_2 = 0.1$. 
(b) Multiple vacillations

In this section, we explore the possibility that the vacillation reported in the last subsection need not be unique for a given set of external conditions. A set of computations is first made to determine how each of the three equilibria for $\delta = 0.5$ for the condition considered in subsection 6(a) would vary with progressively larger or smaller values of the topographic height $\delta$. Figure 17 shows the result in terms of $|\theta_{1,1}|$. It shows that there only exists one steady and stable state for sufficiently low topography ($0.0 < \delta < 0.2$). This is to be expected in light of the earlier discussion in subsection 5(b). The intensity of the response varies almost linearly with $\delta$ in that range, as it should. Figure 17 also shows that three equilibria exist in two intervals of values of $\delta$. The first interval is $0.205 < \delta < 0.315$. These end points are labelled as II and I in Fig. 17. The stable branch
of equilibria extends to $\delta = 0.315$ (point I) at which the stability changes. Those unstable equilibria are indicated by the symbol $\times$. Such instability is of the wave-drag type. It is noteworthy that the values of $|\theta_{1,1}|$ are relatively large, indicating that the response for this range of $\delta$ is largely associated with the near-resonant response of the $(m,n) = (1,1)$ topographic wave. Shortly beyond point II in Fig. 17, we see that $|\theta_{1,1}|$ actually decreases with increasing values of $\delta$. This seemingly counterintuitive feature reflects the fact that the wave components with $(m,n) = (1,2)$ and $(2,1)$, rather than the $(1,1)$ wave, are preferentially excited for larger $\delta$ in the context of this topography configuration. These equilibria are also unstable but the corresponding eigenvalues are pairs of complex conjugate values. Those equilibria are denoted by dots in Fig. 17. Such instability would therefore be expected to give rise to propagating waves.

The second interval of values of $\delta$ where there are also three equilibria is $0.365 < \delta < 0.815$. They are all unstable equilibria. There are two bifurcation points labelled as III and IV at $\delta = 0.815$ and 0.36 respectively. The branch of equilibria between III and IV is analogous to that between points I and II in that the dominant instability in both cases is of the wave-drag instability type. Those equilibria are thus also indicated by the symbol $\times$. The values of $|\theta_{2,1}|$ increase with decreasing values of $\delta$ in the branch between III and IV, whereas those of $|\theta_{1,2}|$ and $|\theta_{1,1}|$ decrease with decreasing values of $\delta$ (not shown for brevity). The equilibria in this branch may then be interpreted as being mainly associated with the near-resonant response of the $(m,n) = (2,1)$ topographic wave. The bottom branch of equilibria in this interval of $\delta$ exists for $\delta > 0.36$. The largest

![Figure 17. Variation of the amplitude of $\theta_{1,1}$ wave of the equilibria as a function of the height of the three-wave topography $\delta$ for $V = 0.9$, $U = 0.0$, $r_1 = r_2 = 0.1$. Solid line segment denotes stable equilibria; $\times$, wave-drag-type unstable equilibria; dots, propagating-wave-type unstable equilibria.](image-url)
eigenvalues of the equilibria in this branch are a pair of complex conjugate values, implying that the unstable waves would be propagating. The equilibria are characterized by having their $|\theta_{1,1}|$ increase strongly with $\delta$. We may then interpret this branch of equilibria as stemming mainly from the resonant response in the $(m,n) = (1,2)$ topographic wave. Beyond the point $V$ ($\delta > 0.815$), there is only one equilibrium solution which is naturally unstable. The nonlinear effect due to such large topographic height is that the response is no longer mainly associated with any single topographic wave.

A time integration analysis is next performed to determine the nature of the equilibrated state at large time for different topographic heights. The integration confirms the existence of a steady state at large time for small $\delta$, $0 < \delta < 0.315$. The result of the amplitude of $\theta_{1,1}$ for $\delta = 0.25$ is indicated by a circle in Fig. 18. The family of equilibria is also shown here as a dotted curve for easy reference. If we use a somewhat larger value of $\delta$ in the integration, say $\delta = 0.35$, the system is found to quickly evolve from a weak arbitrary initial flow field to a vacillatory state. This vacillation has a low frequency with a period of about 60 units, corresponding to $\sim 60$ days, similar to the one discussed in subsection 6(a). Now we use this equilibrated state at some arbitrary instant as the initial condition to determine the evolution of the system after resetting $\delta$ to 0.25. The integration reveals that the newly equilibrated state is still a vacillation although its intensity is much weaker. In other words, we have established that the equilibrated state for $\delta = 0.25$ can be either a steady state or a vacillation. Which attractor the system would evolve to depends upon the initial state. This is a mixed type of hysteresis. Both

![Figure 18. Portrait of the multiple vacillations. Variation of the range of fluctuations in the amplitude of $\theta_{1,1}$ of the equilibrated states with the height of the three-wave topography $\delta$ for $V = 0.9$, $U = 0.0$, $r_1 = r_2 = 0.1$. The circle indicates a stable steady state. The dotted curve denotes the equilibria.](image-url)
forms of equilibrated state are structurally stable in this system since they exist over a finite range of the parameters.

The branch of vacillatory states described above, however, ceases to exist beyond $\delta = 0.525$. For example, when $\delta$ is set to 0.65, the integration asymptotically evolves to a distinctly different vacillatory state. The differences are manifested both in its structure and period. The period of this new vacillation is found to be only $\sim 24$ days. Additional computations with different values of $\delta$ confirm the existence of this second branch of vacillations. It exists for $\delta > 0.41$. It is of particular interest to note that two different branches of vacillation coexist in an overlapping range of values of $\delta$, namely $0.41 \leq \delta \leq 0.525$. The existence of multiple vacillations in this geophysical system is therefore confirmed.

![Figure 19](image)

Figure 19. As Fig. 18 except for the result of the zonal component $\psi_{0,1}$.

A supplementary result is presented in Fig. 19. It shows the variation of the correction of the barotropic component of the zonal flow for the equilibria, $\psi_{0,1}$, as well as the range of its fluctuations in the equilibrated state. It reveals more clearly the different origin of the two vacillatory modes mentioned above. So far, we have not examined the results of the phase angle of the equilibrated waves. Panel (a) of Fig. 20 shows the projection of the solution vectors of the different equilibrated states on the plane defined by $(\Re(\theta_{1,1}), \Im(\theta_{1,1}))$ for the case of $\delta = 0.45$. The three equilibria are also plotted as three points with labels $\times$, A and B. Their instability characteristics are described in the figure legend. The two vacillatory solutions portray two closed curves. It is clear that the Hopf bifurcation associated with the equilibrium state A gives birth to the vacillatory state that encircles the state A. The other vacillation is clearly associated with the equilibrium B. Let us simply refer to them as 'vacillation A' and 'vacillation B' for short. Vacillation A has a period of $\sim 60$ days, whereas vacillation B has a period of $\sim 30$ days. The fact that vacillation B encircles the origin point in an anticlockwise direction also means that the phase angle of the $\theta_{1,1}$ wave continually increases in time. In other words, the $\theta_{1,1}$ wave
propagates steadily eastward in spite of the presence of a topography. Vacillation A does not encircle the origin point, signifying that this wave zonally oscillates about a mean position. Panel (b) of Fig. 20 depicts the counterpart results in terms of $\theta_{1,2}$. The $\theta_{1,2}$ wave of vacillation A has a net westward phase propagation, whereas the $\theta_{1,2}$ wave of vacillation B only oscillates in the zonal direction. Panel (c) of Fig. 20 depicts the counterpart results of the remaining dominant wave component in the response, $\theta_{2,1}$. Here we see that the $\theta_{2,1}$ waves of both vacillations do not propagate zonally but oscillate slightly about their mean values in the zonal direction.

7. Interpretation of a Time-Mean Wave Flow

We now wish to make use of the model-generated time-mean flow to draw attention to a fairly obvious but often overlooked issue in the study of planetary stationary waves. The most common approach to studying such a wave field has been to calculate separately each of the constituent waves as a forced linearizable perturbation on a zonally symmetric basic state. It is fair to say that Lindzen's (1986) statement, "a (linear) model might in fact tell us . . . how stationary waves in the atmosphere work", echoes a widely accepted view. In my opinion, there is a key caveat in this type of analysis associated with the a priori prescription of a 'realistic' zonally symmetric basic state. If the zonally symmetric
flow prescribed in a linear model is, in reality, strongly influenced by the actual time-mean wave field, one cannot logically establish that the wave field is 'caused' by the given zonal flow, in conjunction with the topography, by simply showing a reasonably good reproduction of the former in such a model. Ignoring the strong mutual influences is not only unjustifiable but also can be misleading. This controversy is analogous to that concerning an instability analysis of an observed time-mean flow. Pedlosky (1979, section 7.1) has already succinctly pointed out the inherent conceptual limitation in trying to interpret such unstable modes in terms of any particular observed growing disturbances.

Let me offer an example to substantiate the criticism above. Recall that Fig. 15 is an example of an equilibrated state of our nonlinear model that is time dependent with a pronounced low-frequency variability. We can, nevertheless, define a time-mean flow field by simply averaging the spectral components of the model output over a long time. The time-mean flow field is presented in Fig. 21(a) showing a pronounced trough to the east of the southern mountain with a corresponding ridge to its north-easter location.

Figure 21. (a) The time-mean streamfunction of the lower layer in the nonlinear model computed from the evolution shown in Fig. 15. (b) The steady state response of the lower layer streamfunction in the linearized model. Friction coefficients in (a) and (b) are $r_1 = r_2 = 0.1$ and $r_1 = r_2 = 0.3$ respectively.

Suppose a researcher only knows about the topography, has measurements of the flow field and wishes to understand the dynamics of the mean flow. He would regard the time-mean wave field and the time-mean zonal flow as two parts of an 'observed' circulation. If he attempts to interpret the mean wave field in terms of linear dynamics, he would use the 'observed' zonal flow as a given basic flow in the linearized version of this model. He would soon find out that by using different values of the damping parameters ($r_1$ and $r_2$), he could get a steady wave field with different degrees of 'realism' compared with the 'observed' wavy flow. Just as in other studies of this kind, the desirable damping for his linear model calculation needs to be somewhat larger than the actual values used in
the nonlinear model (e.g. Kang and Held 1986). Specifically, our hypothetical researcher would generate a steady state wave field with his linear model, fairly resembling the observed wave field if he uses $r_1 = r_2 = 0.3$ (Fig. 21(b)). But it would be very misleading for him to conclude, on this basis, that these steady waves are generated in the actual system by the linear dynamical processes. We happen to know the truth in this case. It is that the dynamics of this flow is decidedly nonlinear in character and the flow fluctuates greatly about the mean. His success in tuning the linear model primarily stems from the use of the ‘observed’ zonal flow. However, such success is a somewhat empty one because the linear model, in essence, builds upon the knowledge that can only come out of a nonlinear model. All it proves is that a realistic looking linear steady wave field may be made compatible with the given steady zonal flow because a judiciously chosen dissipation has been used as a proxy for the missing nonlinear dynamics. Moreover, a different dissipation coefficient would have to be chosen for a different level of forcing and different mean zonal flow. The price for doing so is to lose all information concerning the time dependency of the equilibrated flow. The root of this logical fallacy is that our researcher has ignored the fact that the mean zonal flow and the mean wave field are inseparably coupled together through the transient eddies. The above example suggests that it is prudent not to make too literal an interpretation of the results of a linear model for an atmospheric flow which is in all probability nonlinear in character.

8. Concluding remarks

We have attempted in this analysis to delineate the nature of nonlinear response to a topographic forcing in a sufficiently high resolution generic model. Multiple stable equilibria have been established for the case of a single-wave topography under a condition of weak baroclinicity. Their existence is attributable to the detuning effect of the nonlinearity on a quasi-resonant response. They have the familiar strong wave/weak zonal flow combination and vice versa. In this sense, our high resolution baroclinic model result is in support of the original findings of Charney and De Vore (1979) and Hart (1979). The unstable equilibria themselves have multivalued dependence upon the background flow when the topography is sufficiently high. Linearly unstable but nonlinearly stable equilibria have also been found to give rise to a weak vacillation about a large mean response.

No multiple stable equilibria exist for the case of a three-wave topography under consideration. There is a range of background flow values for which there is not even one stable equilibrium state. In such a case, the equilibrated state is characterized by regular, pronounced, but complex fluctuations with a well-defined low frequency. Furthermore, such a vacillatory state is found to have multiplicity at a given set of external conditions. The existence of multiple vacillations and their distinctively different properties, has been determined as robust characteristics. The two vacillatory states are associated with the two unstable equilibria of which the eigenfunction has the characteristic of a propagating wave field. One has a period about twice as long as the other, both being in the order of tens of days. The pronounced fluctuations arise from the strong resonant interaction among the three different families of waves associated with the directly forced topographic modes. In physical space, the fluctuation is manifested as sequential formation of lee-cyclones on the lee side of the two topographic ridges. This is a possible mechanism that generates low-frequency variability in the atmospheric circulation. The multiplicity of vacillation may underlie the relatively broad frequency band in the observed spectra.

Finally, we have used the known response in the nonlinear model to illustrate that it could be misleading to interpret a time-mean flow with a linear model. Better per-
spective of the time-mean response in our atmosphere should be generally sought in the context of nonlinear, rather than linear, dynamics.

There are a number of obvious limitations in this model, such as the channel geometry of the domain, the presence of a top in the model and idealized topography. Nevertheless, as long as there exists a complete set of vertical normal modes that can be used to depict the vertical structure of the flow in the atmosphere, the nonlinear resonant response investigated in this paper seems to be relevant. This model is intended to serve as a point of departure for investigating the essence of low-frequency variability with less restrictive models.

ACKNOWLEDGMENTS

This research was supported by the National Science Foundation under grant ATM-8610329. The programming assistance from Ming Cai is much appreciated. Ted G. Shepherd brought my attention to Benjamin’s paper that proves the odd multiplicity of equilibria. The thoughtful criticisms of the reviewers and the comments from Ted Shepherd, Brian Reinhold, Greg Holloway and Erland Källen have been particularly helpful. The results were presented partly at the Sixth Conference on Atmospheric and Oceanic Waves and Stability of the AMS, August 25–28, 1987, Seattle, U.S.A., and partly at the European Geophysical Society Conference, March 21–25, 1988, Bologna, Italy. The computations were made with the facility at the National Center for Supercomputing Applications at the University of Illinois.

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