Warm and occluded fronts in two-dimensional moist baroclinic instability

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SUMMARY

The two-dimensional Eady mode of baroclinic instability exhibits a strong cold front structure after a period of development using semi-geostrophic equations excluding moist processes. Indeed the surface wave has a single centre of cyclonic vorticity on the cold side of the warm air anomaly. Here results are presented from a semi-geostrophic numerical model including moist effects which predicts an Eady wave with a cold and a warm front associated with two centres of surface vorticity located on either side of the warm anomaly. The ascent regions are taken to be saturated and close to moist neutral which in this baroclinic environment corresponds to a condition of zero equivalent potential vorticity; the descent is taken to be unsaturated.

This development of a structure resembling a warm front occurs only when the moist effects become dominant within a pre-existing, weak, dry Eady mode. Using the same condensation representation starting from an extremely small, and random, amplitude disturbance a ‘pure’ moist Eady mode develops with a single vorticity maximum in the form of a cold front. In nature it is likely that often the moisture becomes dominant after a period, albeit sometimes a short period, of essentially unsaturated growth, although observational evidence is limited. Therefore the process of warm frontogenesis described here may be a contributory factor to those minority of mid-latitude cyclones which exhibit a strong warm front. The model indicates that the warm front moves more slowly than the cold front, resulting in a cold occlusion process.

These investigations require a detailed examination of the initial growth of small amplitude disturbances near the limit of neutral moist ascent, i.e. pure moist modes. Results of previous such calculations using an analytical 2-level model are confirmed here using a numerical multi-level model. The more accurate estimates obtained here show that the finite growth rate in the limit of zero equivalent potential vorticity has a value of 1.9 times the dry value compared with the 2.4-fold increase given by the 2-level model.

1. INTRODUCTION

Semi-geostrophic theory, as originally proposed by Hoskins and Bretherton (1972), provides a clear understanding of the dry adiabatic mechanisms leading to the formation of atmospheric fronts. One of the archetypal models studied within the theory is the nonlinear development of an Eady baroclinic wave. The ultimate collapse in that model into a shallow narrow frontal zone can be interpreted, when restricted to two dimensions, as a model of a synoptic ‘cold’ front. The corresponding structure of the flow observed in mid-latitude systems was identified as early as 1841 by Loomis (1841). Only when extended to fully three-dimensional baroclinic waves growing on jet-like basic flows does semi-geostrophic theory give some indications of a ‘warm’ front (Hoskins and West 1979). The warm frontal structure was introduced into synoptic meteorology together with the polar-front model by the Norwegian school (Bjerknes 1919), but in practice it is sometimes difficult to identify from surface observation, or by satellite imagery. However, Hoskins and Heckley (1981) proposed that the different structures leading to ‘cold’ and ‘warm’ fronts were consequences of the eastward tilt with height of the temperature anomaly in a dry baroclinic wave. When this tilt is large the upper air temperature anomaly is displaced toward the warm air, a characteristic feature of cold fronts. As the slope of the front becomes larger, the wave tilts less, the maximum upper air temperature anomaly moves towards the cold air, and this reinforces the lower-level gradients on the warm air side. Two-dimensional results illustrating this process were

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proposed within the framework of the Bergeron deformation model. However, a quasi-two-dimensional structure with cold and warm fronts such as portrayed in a vertical cross-section in the Norwegian mature cyclone model has yet to be produced within the framework of a simple dynamical model.

Recent theoretical work in baroclinic instability and frontogenesis, on the other hand, has turned to the study of the modification of dry adiabatic results by latent heat release. This is an attempt to explain some aspects of the frontal structure not well described by the ‘dry’ semi-geostrophic theory. For example, observations show that the vertical velocity is more intense but on a narrower scale than predicted (see e.g. Blumen 1980). It is also important to link together frontogenesis and the observed banded precipitation structure of fronts. This has been attributed to several processes, amongst which are the possibility of moist symmetric instability (Bennetts and Hoskins 1979), and of multiple frontogenesis (Hoskins et al. 1984). More recent observations (e.g. Emanuel 1985) suggest that, in mature cyclones, the atmosphere is very near or at neutrality with respect to moist symmetric instability in the frontal ascent. In other words, the moist potential vorticity is very near zero in the ascent. Based on this idea, a simple parametrization of latent heat release in such ascent was proposed by Emanuel (1985) and Thorpe and Emanuel (1985). The pictures of frontal circulation they produced indeed show the reduction in the scale of the ascent and the increase of vertical velocity. They also showed how potential vorticity was affected during the process of frontogenesis, with a small region of high values building up at low levels, and with widespread low values aloft.

In a recent paper, Emanuel, Fantini and Thorpe (1987, hereafter referred to as EFT), applied their idea of condensation heating expressed in terms of moist and dry potential vorticity to the study of the stability of a two-dimensional, vertically sheared flow. They first restricted themselves to a two-layer model, which allows a (nearly) analytical solution. They were able to find the asymptotic solution in the limit of moist neutrality to slantwise ascent: the growth rate remains finite even when the ascent becomes infinitely narrow, the maximum growth rate being 2-4 times larger than in the dry model. The wavelength of maximum growth is displaced, by a factor 0-6, towards the smaller scales, but the scale selectivity is weak. These results were broadly confirmed by multi-level time integrations of the semi-geostrophic equations starting from a small (but finite) amplitude dry normal mode. In these simulations, the effective stability encountered by ascending parcels, the moist potential vorticity, was a tenth of its original uniform value. In some cases, the adjustment between the initial dry normal mode and the moist unstable normal mode leads to transient features such as a double maximum in the ascent.

It will be shown here, in section 3, that when moist potential vorticity becomes even smaller, the richness in structure gives rise to complex solutions, involving broad scale ‘bands’ of ascent, and multiple maxima in the potential vorticity and vorticity fields. We postulate in this paper that such a combination of initially ‘dry’ growth followed by the competitive growth of a ‘moist’ mode, the latter growing at twice the rate of the former, may be a simple explanation of the distinctive structure described by the Bergen school. Our solution involves two narrow regions of intense ascent, one on the cold air side and the other on the warm air side, separated by the region where the temperature anomaly has its maximum. Stretching below the region of ascent generates two zones of large vorticity, which can be identified as a warm front and a cold front. Also, in section 4, the occlusion process is described; a phenomenon which occurs in this model for a range of values of the moist stability parameter.

A crucial point here is that the presence of several different modes in the initial
conditions is necessary for these complex structures to develop; this is because the structure of the 'pure' moist mode is very simple, and remains so even at the time of frontal collapse. In an attempt to show this, we will study in section 5, the stability of the Eady problem in a moist atmosphere applying an initial value technique to the semi-geostrophic equations. This opportunity will be taken to compare the results from the multi-level model with those from the 2-level model given by EFT, and to show the structure of the moist modes. It is important to extend the two-level model results to multi-levels so that the variety of structures presented in the nonlinear simulation can be understood in terms of the linear normal modes. It is by no means obvious, a priori, that two-level model results have a rich enough vertical structure to properly describe the continuous moist modes. The paper begins with a summary of the relevant equations, in section 2.

2. SUMMARY OF MATHEMATICAL FORMULATION

We consider a uniform 'zonal' flow $\bar{u}$, with constant vertical shear $\bar{u}_z$ and further assume that the surface and tropopause can be approximated by two rigid plates where vertical velocity vanishes. This flow is maintained by the existence of a cross-plane uniform horizontal gradient of potential temperature: an infinite (in a two-dimensional approximation) reservoir of potential energy. Along the plane, the motions are periodic, with period $L$. This flow has an initially uniform potential vorticity. We assume that the motion perturbing this flow occurs on such a scale that the change in the velocity vector of parcels is small compared with the acceleration imposed by the uniformly rotating frame of reference. This assumption leads to the geostrophic momentum approximation originally proposed by Eliassen (1948).

This approximation breaks down either because of the onset of turbulence where the Richardson number becomes too small, or because potential vorticity also becomes too small as a result of redistribution by condensation. Then, ageostrophic acceleration becomes large and the flow becomes unstable to two-dimensional symmetric rolls. This bound on the smallness of potential vorticity is studied by Emanuel (1985).

Hoskins and Bretherton (1972) further showed that by introducing the geostrophic coordinates

$$X = x + v_\phi f, \quad Y = y, \quad Z = z^* = (c_p H_s/R)[1 - (p/p_0)^{R/c_p}], \quad T = t$$

the nonlinearity of the original set of equations was greatly reduced. Here, $x$ is the horizontal coordinate in the direction of the basic flow, $y$ across it, $z^*$ the vertical coordinate of a pressure type equal to physical height in an adiabatic atmosphere, $t$ time, $v_\phi$ is the geostrophic wind normal to the plane of symmetry, $H_s$ is the scale height, $p_0$ a reference pressure, $c_p$ the specific heat, $R$ the gas constant, and $f$ the Coriolis parameter.

The geostrophic part of the flow is linked to potential vorticity $P$ through the inversion equation

$$P \frac{\partial^2 \Phi}{\partial X^2} + \frac{\theta_0 f^2}{g \rho} \frac{\partial^2 \Phi}{\partial Z^2} = f^2 P$$

where $\Phi$ is a potential such that $f v_\phi = \partial \Phi / \partial X$ and $g \theta / \theta_0 = \partial \Phi / \partial Z$; $\theta_0$ is a reference potential temperature, $g$ the acceleration due to gravity and $\rho$ is the density, normally a function of $z^*$ only. In the results presented here, we assume $\rho$ to be a constant $\rho_0$, a Boussinesq-type approximation made for the sake of simplicity, although it can be relaxed easily. Equation (2) embodies the invertibility principle advocated by Kleinschmidt (Eliassen and Kleinschmidt 1957), whereby the essential flow characteristics can be
deduced from a knowledge of potential vorticity in a given domain, and potential
temperature on the domain boundaries (Hoskins et al. 1985).

Note that we have used \( P \) both as a coefficient in the right-hand side and as a weight
on the horizontal Laplacian of \( \Phi \) as this form is more suitable for numerical solution
when \( P \) has large localized maxima.

The vertical velocity may be recovered by solving a form of the ‘omega equation’,
or the Sawyer–Eliassen cross-frontal circulation equation:

\[
\frac{\partial^2}{\partial X^2} \left( \frac{g}{\theta_s} \frac{P}{\zeta_a} v^* \right) + f^2 \frac{\partial}{\partial Z} \left( \frac{f}{\rho} \frac{\partial}{\partial Z} \left( \frac{P}{\zeta_a} \right) \right) = 2f\bar{u}_Z \frac{\partial^2 v_s}{\partial X^2} + \frac{g}{\theta_o} \frac{\partial^2 S}{\partial X^2}.
\]  

In (3), \( \zeta_a \) is the absolute vorticity given by

\[
\zeta_a = f \left( \frac{1}{1 + \frac{f^2}{\theta_s \partial X^2}} \right)
\]

\( w^* \) is the vertical velocity (\( dz^*/dt \)) and \( S \) is the diabatic source term from the thermo-
dynamic equation whose form will be described later.

Thus, the along-front wind \( v_s \), potential temperature \( \theta \) and vertical velocity \( w^* \) are
recovered diagnostically. The time dependence occurs in the potential vorticity equation

\[
(\partial/\partial T + \bar{u} \partial/\partial X + w^* \partial/\partial Z)P = (\zeta_s/\rho) \partial S/\partial Z
\]

replaced on horizontal boundaries by the thermodynamic equation

\[
(\partial/\partial T + \bar{u} \partial/\partial X)\theta + \bar{u} \partial/\partial X)v_s = 0 \quad \text{at} \quad Z = 0 \text{ and } Z = H
\]

where we have used the boundary conditions

\[
w^* = 0 \quad \text{and} \quad S = 0 \quad \text{at} \quad Z = 0 \text{ and } Z = H.
\]

\( \bar{u} \partial/\partial X \) is the constant baroclinicity, related to \( U_Z \) by the thermal wind equation. Condition
(6) also provides boundary values for solving (3). Knowledge of \( \theta \) provides boundary
conditions in order to solve (2) for \( \Phi \). As mentioned above, we assume periodic lateral
boundary conditions. A detailed derivation of this set of equation can be found in Hoskins
(1975) and Hoskins and Draghici (1977).

EFT provide an expression for \( S \) in terms of \( P \) and \( P_e \), the equivalent (moist)
potential vorticity. Assuming latent heat is released as soon as there is ascending motion,
then the diabatic term can be written as

\[
S = \frac{\rho}{f} \left( P - \frac{\Gamma_m}{\Gamma_e} P_e \right) \max(0, w^*)
\]

where \( \Gamma_m/\Gamma_e \) is the ratio of the moist adiabatic to the dry adiabatic lapse rates. We make
the further assumption that \( (\Gamma_m/\Gamma_e) P_e \) can be considered as a constant, set to \( rP_{oo} \), where
\( P_{oo} \) is a reference value of potential vorticity. According to observations presented by
Emanuel (1985), \( r \) is in fact very near zero in frontal ascent.

The numerical procedure used to solve the set comprising Eqs. (2) to (4), with
conditions (5), (6) and the simplified source term (7) is given in the appendix.

We set the constants to the following values:

\[
\begin{align*}
\rho_o P_{oo} & = 3 \times 10^{-7} \text{Km}^{-1} \text{S}^{-1} \\
\bar{u}_Z & = 3.6 \times 10^{-3} \text{S}^{-1} \\
H & = 8 \text{ km} \\
H_s = \frac{R \theta_o}{g} & = 8 \text{ km} \quad \text{or} \quad \theta_o = 273 \text{ K}.
\end{align*}
\]
FRONTS IN MOIST BAROCLINIC INSTABILITY

It is convenient to express potential vorticity in a more practical unit. Following Hoskins et al. (1985), we define a PVU (potential vorticity unit) as $10^{-6} \text{m}^2 \text{s}^{-1} \text{K kg}^{-1}$. Then, $P_{oo}$ is $0.3 \text{PVU}$ if $\rho_o$ is $1 \text{ kg m}^{-3}$.

The equivalent of the static stability in terms of a frequency in semi-geostrophic theory is $N^2 = (g/\theta_o) \rho P_{oo}/f$ and here $N = 1.04 \times 10^{-2} \text{s}^{-1}$ giving the corresponding Rossby radius $NH/f$ of about 830 km.

3. WARM FRONTOGENESIS AS A MODIFICATION OF A DRY NORMAL MODE BY CONDENSATION

In nature, air has to be lifted a certain distance before saturated ascent can occur so the initial growth of the baroclinic wave may be close to that given by the dry modes. The following experiments take this fact into account. As an initial condition, we have taken an unstable dry normal mode with a small but finite amplitude: namely the initial amplitude of the temperature wave is 2 K on the boundaries, and we integrate in time with $r$ equal to $0.01$ and $10^{-5}$. Only the results for $r = 0.01$ will be shown, as the pictures for $r = 10^{-5}$ are very similar: the coincidence of solutions indicating the sense in which we are close to the moist neutral limit discovered by EFT. The constants have been set to the values given above. The initial growth rate of the dry mode implies an e-folding time of 1.08 days. It collapses to a discontinuity in 56 hours. From section 5 below, it is seen that the corresponding growth rates of a single ascent moist mode are $0.69$ day for $r = 0.01$ and 0.66 day for $r = 10^{-5}$ with a wavelength of 3247 km. For these runs, we use 40 levels in the vertical and 79 Fourier modes, the smallest fully resolved wavelength being 41.6 km. (The grid length is 13.5 km in geostrophic space.)

Initially there is a broad single ascent, although narrowed, with $\lambda$, the ratio of the horizontal widths of ascent to descent, taking the value of 0.667 (instead of 1 for a dry mode). However, this region is actually split into several narrow multiple ascents. Figure 1 shows the structure of the solution after 18 hours. In this and subsequent figures, only the ascent region will be shown as the descent region has structure similar to a dry mode. Consequently it is only in the ascent that the simulations shown here differ from those in EFT.

At this stage, there are three separate ascents, with weak descent between them. At mid-levels the distance between two ascents is about 310 km, and 205 km at the surface. The structure with $r = 10^{-5}$ looks the same (3 maxima in vorticity, for example), although the 'central' ascent tends to split in two, and all ascents are narrower. Notice that this solution does not depend on the horizontal resolution of the model; with 33, 56 or 79 modes the positions are identical, with the ascent zones slightly narrowed as resolution increases.

Moreover, each of these ascent zones acts on the vorticity field, with three associated maxima resulting from stretching. It has a small effect on the potential temperature field, but seems to be accompanied by a broadening of the separation between the low-level jets.

The reason for the splitting of the ascent zone into three multiple zones is central to our understanding of this simulation. As is well known, the vertical velocity maximum in a dry Eady mode tilts with height, in the upshear direction. The absolute momentum surfaces ($m = fX$) have a smaller tilt with height. The imposed diabatic forcing of potential vorticity is proportional to the gradient of the diabatic heating along the $m$ surface. Consequently, it has a structure which is everywhere as broad as the region of ascent except in the upper troposphere on the cold side and in the lower troposphere on the warm side, where it becomes rather narrow. The diabatic potential vorticity (PV)
(a) Solid lines: vertical velocity with contours 0 (bold solid), 3, 9, 15 cm s⁻¹ (light solid). Negative values have moduli smaller than 1·1 cm s⁻¹, and so no contours appear. Both dashed lines depict constant absolute momentum surface \( m = f x + u_x \), drawn every 20 m s⁻¹, with arbitrary origin. Arrows show the cross-front ageostrophic flow in the system-relative frame of reference. Vertical velocity is multiplied by a constant ratio \( N/f \) equal to 100. The reference arrow on the right-hand side has a length equivalent to 55 m s⁻¹.

(b) Continuous lines: along-front geostrophic wind \( v_x \), contour interval 3 m s⁻¹, zero contour bold, negative contours dotted. Dashed lines: potential temperature, contour interval 3 K.

(c) Potential vorticity, contour interval 0·05 PVU (see text for this unit). Bold contour: 0·3 PVU, the initial uniform value. Dash–dotted lines are values below the reference, corresponding to a negative anomaly.

(d) Potential temperature departure from the hydrostatic rest state of reference. Contour interval 0·5 K, bold contour 2 K. This shows the warm anomaly in the wave.

(e) Absolute vorticity scaled by \( f \), the Jacobian of the change to geostrophic coordinates. Contour interval, 0·1, bold contour 1, dash–dotted contours correspond to negative relative vorticity.

Figure 1. Vertical cross-section showing the structure of two-dimensional Eady wave with moist slantwise stability parameter \( r = 0·01 \). The initial state is a dry mode with thermal amplitude 2 K. The wavelength is 3247 km, but here only 1/3 of the domain is shown, with ticks every 200 km horizontally and 2 km vertically (total height 8 km). The remainder of the wave is occupied by weak descent and structure similar to the equivalent dry mode. The figure shows the solution after 18 hours.

This figure features the splitting of the original broad ascent into three wide bands separated by weak descent. Each band has generated, through condensation, its own maxima in potential and absolute vorticity. Comparison between (a) and (b) reveals the two most intense ascent zones to be in the regions of largest horizontal gradient in potential temperature. The positive jet below the warm side ascent reaches 17·3 m s⁻¹, the negative jet away from the cold front -12·0 m s⁻¹.
anomaly thus produced has a much shorter horizontal scale and consequently a larger horizontal gradient in these two regions than in the bulk of the region of ascent. This can be seen in Figs. 2(a) and (b) where fields are shown at 3 hours. It can be shown that the direct consequence of this PV-scale contraction and large gradients is a local bimodal forcing of vertical velocity. In Fig. 2(c) the vertical velocity is shown at 6 hours at which point the triple structure in the vertical motion field is evident. No further splitting occurs as the diabatic forcing is soon aligned more closely with the $m$ surface. Thus the splitting is due to the introduction of latent heating and the consequent PV anomalies into a pre-existing dry mode. As will be shown in section 5 below, the pure moist modes do not exhibit this behaviour as the changes of PV are always on the same horizontal scale as the vertical motion. That mode evolves with vertical motion nearly aligned with the $m$ surfaces at all times. In nature we might expect that the state of zero moist potential vorticity used here will only occur after a period of essentially dry growth.

The presence of three regions of ascent appears to be an intermediate step in an adjustment process which ends after 21 hours. The final structure is depicted in Fig. 3. There are only two ascent zones, which correspond to the two regions of maximum baroclinicity, one on the warm air side, where the gradient reaches 6 degC in 100 km, the other on the cold air side, with a value of nearly 3 degC in 100 km. Vertical velocities as large as 50 cm s$^{-1}$ are reached. A remarkable property of this behaviour is that it is apparent in all the other fields: two maxima in potential vorticity and vorticity, the strongest one being on the warm air side. At this stage close to frontal collapse, the distance between the ascent regions is 350 km at mid-level, but less than 125 km near the surface. On the other hand, the overall effect on baroclinicity as shown by $\theta - \bar{\theta}(z)$ is rather small, although the equivalent potential temperature, as in nature, would show marked frontal gradients. Here, by definition, they are nearly parallel to the $m$ surfaces in the areas of ascent. The ‘warm side’ of the structure is strongly reminiscent of the warm front type B described by Hoskins and West (1979) with, in particular, the strengthening of the ‘northward’ low-level jet and the sharpening of vorticity at the rear of the jet, together with little signature in $\theta$.

Simultaneously, we observe all the characteristics of a very strong ‘cold’ front behind this first structure. A summary of this adjustment process towards a double structure is given in Table 1. The ratio of ascent to descent width, $\lambda$ (within geostrophic space), the number and spacing of ascent zones, as well as the several maximum values of vorticity at the surface, are given. Also shown in Table 1 is the distance between the extreme values of absolute vorticity. This length is normalized by the width of the descent.

As the number of ascent regions increases, $\lambda$ falls, and it decreases further when the third, ‘middle’, ascent vanishes. But when the double structure is formed, after 24 hours, $\lambda$ remains constant in geostrophic space. However, the two ascent regions and the associated features move nearer to one another as the wave grows, as shown by the
distance between the high values of vorticity at the surface. Even after 24 hours, it is decreasing by about 0.02 units every hour, much more with $r = 10^{-5}$. Relative to the system, the 'warm front' moves upwind towards the resting 'cold front'. We may assume that, if frontal collapse was prevented by, for example, turbulent mixing at the front, the two ascents would sooner or later merge into one: the so-called 'occlusion' concept. Note that the total updraught width is respectively 5 times and 3.6 times the smallest resolved wavelength: the reduction from 0.097 to 0.048 when $r$ decreases from $10^{-2}$ to $10^{-5}$ shows that the final structure is not constrained by horizontal resolution.

The cold front slopes at 1 in 25 westward, and the warm front at 1 in 50 eastward. As expected from synoptic experience, the warm front has a more shallow slope than the cold front, although both are somewhat steeper than in nature.
Figure 3. As Fig. 1 but after 30 hours. Changes in contour intervals are: (a) 0 (bold), 10 and 30 cm s\(^{-1}\) for vertical velocity. Arrows are directly comparable to those of Fig. 1(d), as are the \(m\) surfaces. (b) 6 m s\(^{-1}\) for the along-front wind \(u_x\). (c) 0-4 PVU for potential vorticity, bold line 0-4 PVU. The dash-dotted contour is 0-2 PVU, the dashed line 0-6 PVU. (d) 1 K for the potential temperature anomaly. (e) \(f\) for absolute vorticity contour, bold line 1f.

This shows a detailed picture of the cold and the warm fronts. Each of the two ascent regions is accompanied by large values of vorticity and potential vorticity, indicating the presence of fronts. The ascent follows the local \(m\) surface. Among realistic features are the reduced slope of the warm front compared with that of the cold front.

A more detailed analysis of the process of frontogenesis shows that the increase in the horizontal gradient of \(\theta\) in geostrophic space has three possible sources. The primary one is the geostrophic or \(Q\) vector forcing, here given by \(-\partial \theta / \partial y \partial v_x / \partial X\). It is opposed by the vertical motion which maintains the thermal balance. This is given by \(-\left( \rho_0 / \rho \right) \partial w^* / \partial X\). Condensation is the last source, as given by \(\partial S / \partial X\). At the base of the warm frontal ascent, near the time of frontal collapse, the magnitudes of these terms are respectively 6-3 K (100 km)\(^{-1}\) d\(^{-1}\), 4-4 K (100 km)\(^{-1}\) d\(^{-1}\) and 5-3 K (100 km)\(^{-1}\) d\(^{-1}\). Thus, the effect of the diabatic source term is ten times larger than the geostrophic forcing. But it is mostly nullified by the compensating ageostrophic
TABLE 1. Frontal parameters as a function of time for \( r = 0.01 \) and \( r = 10^{-5} \)

<table>
<thead>
<tr>
<th>Time (h)</th>
<th>Number of ascents</th>
<th>( \lambda )</th>
<th>( J ) at 'cold' front</th>
<th>Third maximum in ( J )</th>
<th>( J ) at 'warm' front</th>
<th>( d_{j,j_2} )</th>
<th>( (d_{j,j_3}) )</th>
<th>( d_{j,j_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = 0.01 )</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.667</td>
<td>1.13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1 with 3 maxima</td>
<td>0.490</td>
<td>1.19</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>0.290</td>
<td>1.35</td>
<td>1.57</td>
<td>1.46</td>
<td>0.269</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>3</td>
<td>0.116</td>
<td>1.60</td>
<td>1.57</td>
<td>1.68</td>
<td>0.075</td>
<td>(0.211)</td>
<td>0.136</td>
</tr>
<tr>
<td>24</td>
<td>2</td>
<td>0.07</td>
<td>2.42</td>
<td></td>
<td>3.11</td>
<td>0.113</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
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<td>2.83</td>
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<tr>
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<td>12.03</td>
<td>0.107</td>
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<tr>
<td>( r = 10^{-5} )</td>
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<td>0.667</td>
<td>1.13</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
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<td>6</td>
<td>1 with 3 maxima</td>
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<td>1.20</td>
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<td>0.082</td>
<td>(0.221)</td>
<td>0.139</td>
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<td>1.68</td>
<td>0.076</td>
<td>(0.176)</td>
<td>0.100</td>
</tr>
<tr>
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<td>2.71</td>
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</tr>
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<td>3.85</td>
<td>0.118</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( J \) is the absolute vorticity divided by \( f \).
\( d_{j,j_2} \): Distance between the first two maxima in \( J \) at the surface divided by the width of the descent region in the wave.
\( d_{j,j_3} \) and \( d_{j,j_3} \): non-dimensional distances between the next two maxima in \( J \) and the extremes respectively.
\( \lambda \) is the ratio of the widths of the ascent to descent region.

motion. Hence, the net rate of increase is roughly of the order of twice the geostrophic forcing. Also, the increase in vertical velocity, due to condensation, bears directly on the generation of vorticity. In geostrophic space, the source of vorticity is stretching, that is \( f^2 \partial (w^* / \zeta) / \partial Z \). To compare the two aspects of the frontogenesis, we estimate the time scale it would take to double the vorticity or the horizontal gradient of \( \theta \). It takes two hours for the gradient of \( \theta \), one hour only for the vorticity. We conclude from these figures that the primary forcing of frontogenesis is stretching due to the large gradients of vertical velocity. These gradients have their origin in the necessity of maintaining thermal wind balance against the destabilizing effects of condensation. However, though small, the geostrophic forcing is essential in producing the tendency for vertical motion. Note that in dry frontogenesis the two time scales quoted above are in the ratio 1:3 rather than 2:1 for the moist case.

In this moist Eady mode the warm front collapses to a discontinuity before the cold front. Also the collapse occurs before the system is occluded, i.e. there is still a finite width warm sector at the time of formation of the discontinuity. The relative timing of these three events (cold front, warm front, and occlusion formation) is considered in the next section.

4. The Occlusion Process

The timing of the occlusion process is dependent on the size of the stability parameter \( r \). In summary, decreasing \( r \) from 0.1 gradually speeds up the process of frontogenesis
and reduces the horizontal scale of the ascent. The latter aspect explains why the two maxima embedded in a wide single ascent given in the \( r = 0.1 \) solution by EFT are gradually separated into individual ascents. The air between the two maxima is forced to zero vertical velocity as \( r \) reduces to 0.07 and then to a very slow descent as smaller values are taken. The rapidity of the frontogenesis for small \( r \) prevents the two maxima merging into a single ascent before the discontinuity forms. There is, however, a range of values of \( r \) which allow the semi-geostrophic equations to describe the occlusion process, or at least, its two-dimensional counterpart. For \( r \) between 0.075 and 0.05 there is a selection of scale allowing a period when individual ascents can exist, and an increase in surface vorticity slow enough for these two ascent zones to merge. This merging of the ascent zones is depicted on Fig. 4, with half-hourly pictures of some of the most characteristic fields.

Contrary to the Norwegian conceptual models of an occlusion, the warm air does not seem to be moved upward: the maximum in potential temperature anomaly still lies at the surface, very much like the dry analytical solution of Hoskins and Bretherton (1972). Horizontal shear, the primary cause for the potential temperature anomaly resulting from the along-front differential displacement of isentropes, remains a maximum at the surface.

As the region of small vorticity between the two fronts shrinks and disappears, the winds along the front turn very abruptly. Table 2 shows how the two fronts move toward one another, together with the usual other parameters. The cold front moves nearly at the same speed as the baroclinic wave, the warm front is slower, another well-known fact from observation.

The mechanism of occlusion derives from the same process that leads to frontal collapse. In the system-relative frame of reference, the warm front moves towards the cold front. We now note that the cold front ascent roughly follows the \( v_\theta = 0 \) surface, while the warm front ascent is right above the accelerating 'northward' low-level jet. This implies that the parcels in the warm front region will be moved substantially by ageostrophic displacement 'westward', toward the cold front. On the other hand, parcels in the cold frontal ascent are not accelerated along the front, and have small displacements in the across-front direction. Thus, we understand the relative motion of the fronts. A

<table>
<thead>
<tr>
<th>Time (h)</th>
<th>Number of ascents</th>
<th>( \lambda )</th>
<th>( J ) at 'cold' front</th>
<th>Position in SG space</th>
<th>( J ) at 'warm' front</th>
<th>Position in SG space</th>
<th>( d_{1/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.633</td>
<td>1.16</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.558</td>
<td>1.19</td>
<td>0.058</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>1 with 2 max</td>
<td>0.420</td>
<td>1.29</td>
<td>0.089</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>1 with 2 max</td>
<td>0.311</td>
<td>1.48</td>
<td>0.12</td>
<td>1.51</td>
<td>0.32</td>
<td>0.191</td>
</tr>
<tr>
<td>24</td>
<td>2</td>
<td>0.224</td>
<td>2.03</td>
<td>0.13</td>
<td>2.46</td>
<td>0.27</td>
<td>0.127</td>
</tr>
<tr>
<td>27</td>
<td>1 with 2 max</td>
<td>0.212</td>
<td>1.91</td>
<td>0.14</td>
<td>4.18</td>
<td>0.25</td>
<td>0.106</td>
</tr>
<tr>
<td>30</td>
<td>1</td>
<td>0.176</td>
<td>1.23</td>
<td>0.20</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The position in SG space is the distance of the front from the location of the maximum surface vorticity at 00 h normalized by the width of the descent zone. Other parameters are as defined in Table 1.
Figure 4. Vertical cross-sections showing the evolution of vertical velocity in a two-dimensional Eady wave with moist stability parameter $r = 0.07$. Initial conditions are otherwise as in Fig. 1. The bold solid line is the zero contour; light solid lines are drawn every 3 cm s$^{-1}$. The light dashed line in (a) is the $+1.5$ cm s$^{-1}$ contour. The only significant negative vertical velocity contours ($-1.5$ cm s$^{-1}$) are drawn as dotted lines in (e) and (f). The heavy dashed lines are $m$ surfaces, contour interval 20 m s$^{-1}$. The arrows represent the system-relative cross-front ageostrophic flow ($u_v(N/f)w$), the unit arrow on the top right corner of (f) is 15 m s$^{-1}$.

The cross-sections are made after 15 hours (a), 21 hours (b), 27 hours (c), 29 hours (d), 29.5 hours (e), 30 hours (f). In (e), maximum vertical velocity is 10.6 cm s$^{-1}$ and minimum $-1.5$ cm s$^{-1}$.

The reduction in scale of the regions of ascent is not as drastic as with $r = 0.01$. However, as shown by (a) and (b), two independent ascent zones temporarily form. The frame of reference is moving with the system. Note that the cold frontal ascent moves 200 km 'eastward', whereas the warm frontal ascent is displaced by more than 400 km 'westward'. In a fixed frame of reference, this means that the cold front is moving at twice the speed of the warm front. This merger is interpretable as the occlusion process.
less obvious aspect appears when the evolution is studied in geostrophic space. The two fronts move evenly toward one another owing to the long term modal behaviour of the system. The solution we are examining depicts an adjustment from a dry to the most unstable moist normal mode. The flow is weakly nonlinear owing to the absence of feedback onto the baroclinicity. A nonlinearity is the vertical advection of PV. Therefore, the long term behaviour converges toward the dominant eigenmode. In the next section, we show the similarity of structure of the 'cold occlusion', as shown by Fig. 5, and a moist normal mode.

It is important to consider the influence of the initial amplitude. As the nonlinearity is weak we do not find any dramatic change in the solution as a result of a decrease in the initial amplitude of the dry mode. The time taken for collapse to a discontinuity is increased, so that the original ascent can split, and the resulting ascent zones can merge again before the discontinuity occurs.

5. MOIST NORMAL MODES IN A MULTI-LEVEL MODEL

It is important, so as to gain an understanding of the dynamics of these simulations, to determine the moist normal modes of the Eady problem. This is possible analytically for a two-layer model, as given in EFT, but the changes due to a multi-layer description must be found numerically.

Figure 5. The structure of the cold occlusion after 29.5 hours. The contour intervals are as in Fig. 1 except for: (a) Along-front wind $v_1$ contours: 4 m s$^{-1}$. (b) Potential vorticity contours: 0.4 PVU, bold line 0.4 PVU. (c) Potential temperature anomaly contours: 0.5 K, bold line 2 K. (d) Absolute vorticity contours: $f$, bold line $1f$.

A single region of very large potential vorticity and vorticity is left. The transition from the ‘southerly’ low-level jet (of $-21.4$ m s$^{-1}$) behind the front to the northerly jet ahead of it (reaching 30.3 m s$^{-1}$) does not show the transition zone of Fig. 1(b) of nearly constant wind.
In order to determine the growth rate, phase speed and structure of the normal modes of the problem we have just described, we applied an initial value technique similar to the one introduced by Simmons and Hoskins (1976). This technique was originally applied to a quasi-geostrophic model by Brown (1969). We have applied it here to the fully nonlinear model, starting from small random values perturbing $\theta$ on both boundaries. Details of the procedure are to be found in the appendix.

Figure 6 shows the resulting growth rates, obtained to an accuracy of at least three significant figures for several values of $r$. The dry case $r = 1$ offers an opportunity to compare the numerical results to the analytical Eady problem: the agreement clearly is excellent. We present these curves using the same non-dimensionalization as in EFT. If $\sim$ denotes a dimensional value, we have

$$\tilde{X} = (H \lambda / 2f) X = 415 \text{ km} \times X$$

$$\tilde{T} = (2N / fu_x) T = (0.67 \text{ d}) \times T.$$ 

These curves are profoundly different from those that can be inferred from a reduction of static stability in the original Eady problem: in the latter model although the short-wave cut-off is also displaced towards shorter wavelengths and the growth rate is increased, the interpretation is different. Here each wavelength is composed of a narrow ascent and a broad descent whereas in the reduced static stability model it is symmetrical. Also here there is a finite growth rate in the limit of neutral ascent whereas the other model has infinite growth in this limit. For small horizontal wavenumbers, $M$, it is not easy to find the growth rate using the current technique for small values of $r$. This is because sub-harmonics of the gravest mode grow at a faster rate and dominate the solution. This behaviour itself gives an indication that the growth rate curve has a maximum at a non-zero value of $M$. Note that for all the cases stated here, the phase speed is found to be zero, as predicted by EFT.

For the sake of comparing these numerical results with the analytical calculations of EFT, a different presentation must be chosen. Since the technique provides the eigenvector structure as well as the eigenvalue $\sigma$, we also monitor the ratio of the width of the ascent to the width of the descent, $\lambda$, in geostrophic space. The accuracy of $\lambda$ is limited, in as much as the widths can take on only integer multiples of the grid length.

![Figure 6. The non-dimensional growth rate as a function of the non-dimensional horizontal wavenumber $M$, as determined by the initial value technique. The solid lines are for 40 vertical levels. Various values of the stability parameter $r$ are used, including that for the analytical solution at $r = 1$, shown with the dashed curve. The dash-dotted curve was obtained with $r = 10^{-20}$ and 5 levels. The two triangles are values obtained with 4 levels.](image-url)
From the knowledge of the total width $L$ and of $\lambda$ of the most unstable mode, we can deduce the descent half-width $L_2 = L/(2(1 + \lambda))$. Figure 7 is constructed following this procedure, and it can be directly compared to Fig. 2 of EFT. The problem of multiple ascents does not arise in the part of the spectrum shown on this figure and in EFT.

The overall agreement with the 2-level model is very good. It confirms EFT's main finding, that is the shift towards smaller scales of the maximum growth rate and, above all, the finite value of the maximum as $r$ goes towards zero. The stability calculations for small $r$ were performed with $r = 10^{-4}$, $10^{-5}$, $10^{-6}$ and $10^{-20}$. Only the third significant figure was changed in the resulting growth rates, and that by not more than 1 unit, in the vicinity of the maximum growth rate: this remarkable property suggests that we have reached the neutral limit. Calculations are also insensitive to changes in the horizontal resolution.

However, there are some differences between the two-layer calculation and our 40-level curves. The whole diagram is shifted towards larger values of $L_2$, with a short-wave cut-off moving from 1·11 to 1·32. This is a well-known difference between the two-layer and the continuous Eady problem. In the latter case, the short-wave cut-off occurs when the non-dimensional wave number $2\pi/L$ is such that

$$\tanh(2\pi/L) - 2\pi/L = 0 \quad \text{or} \quad 2\pi/L = 1.2.$$ 

Taking $L_2$ to be $L/4$ because $\lambda = 1$ in that case, we have $L_2 = 1.31$. The two-layer solution on the other hand reads $2\pi/L = \sqrt{2}$ and so $L_2 = 1.11$. This is exactly the difference between the two curves in Fig. 7.

The increase in the maximum growth rate predicted by EFT between the dry and the neutral $r = 0$ limit is 2·4, whereas we predict a factor of 1·9. If the vertical resolution in the multi-level model is reduced then the growth rate maximum is increased. It is likely, therefore, that these differences can be explained by vertical resolution. Note that EFT use some slight linearizations, such as the use of a constant $P_0$ in the descending region, when computing the diabatic source term. Also, the potential vorticity does change with time in the 2-layer model contrary to the impression given in EFT.

Running these stability calculations with intermediate values of $r$ allows us to construct Fig. 8 where maximum growth rate and the ratio of widths are plotted as
functions of $r$. The growth rate curves are very similar to one another, except for the above-mentioned differences. Small values of $\lambda$ can be evaluated in these numerical solutions as relatively large horizontal resolution has been used.

Thus, the two-layer analytical approach and the multi-level numerical technique support one another as far as their main conclusions are concerned: the finite limit of the growth rate is not an artefact of the very severe vertical truncation of the two-layer model, and it is also not an effect of the numerical scheme. However, it is not possible to say whether the limit of $\lambda$ when $r = 0$ is zero or merely a small value.

In EFT it is shown that very large wavelengths are destabilized; i.e. the limit when $r = 0$ gives an asymptotic value of the growth rate of $\sqrt{2}$ as $L_2$ becomes large. This limit cannot be approached in these calculations, because at small wavenumbers, the non-separability of (2) and (3) selects sub-harmonics. This indicates that the gravest is not the most unstable mode.

These results show that the moist processes modelled with $r = 0.01$ are representative of those in the limit of neutrality to moist slantwise ascent.

The most rapidly growing mode when $r = 0.01$ occurs at wavenumber $2\pi/L = 1.20$. Its structure is shown on Fig. 9. In this case, $\lambda = 0.047$. The updraught is vertical. The temperature wave tilts eastward only at middle levels. This moist normal mode can be integrated forward into the nonlinear regime. The structure in geostrophic space remains constant but increases in amplitude. In physical space, there is collapse to a frontal discontinuity. The structure of the mode just before a front is formed on the lower boundary, is shown in Fig. 10. The ascent region has a characteristic wedge geometry, as it tends to follow surfaces of constant absolute momentum $m$. As $\nu_g$ in the lower layers decreases with height, $m$ surfaces are inclined westward, and the reverse is true above mid-levels. This shape is also noticeable in vorticity and potential vorticity. The distribution of wind and temperature is similar in the 'occluded front' and in the present normal mode.

We find a change of tilt in the potential temperature wave, in the lowest layers. In the dry mode, the tilt is uniformly eastward. A very similar picture is offered by the most unstable mode with $r = 10^{-5}$ (not shown here), except that frontal collapse occurs even earlier and the front region is even narrower: the bow-like vertical profile is also clearly visible.
These simulations were obtained from very small random initial perturbations of $\theta$ on the boundaries, and they give rise to structures with one ascent, and finally one vorticity maximum, that is, one front. Compared with the dry front, the transition between ascending cloudy air and descending clear air behind the front is made sharper, and the frontal slope is more vertical, even reversed in the low layers. As in the dry mode, however, the ascent is roughly in phase with the maximum positive temperature anomaly.

6. DISCUSSION

In this paper, using a realistic representation of condensation heating, a range of frontal structures has been simulated. Using a multi-level numerical model the results of EFT have been extended to allow descriptions of warm frontogenesis and the occlusion process. These structures have not hitherto been simulated in a two-dimensional context and it is shown, for the first time, that the natural phenomena may not necessitate three-dimensional dynamics. A summary of the important conclusions made here is as follows:

* Moist normal modes of a sheared flow are similar to their dry counterparts in that there is a single cold frontal structure. Their growth rate is at most twice that of the dry Eady mode with a horizontal wavelength of about 2000 km for typical atmospheric values.

* The moist baroclinic development from a small amplitude Eady wave can produce a cold front and a warm front separated by the warm sector depending on the magnitude of the stability of the air to moist slantwise ascent. The more stable the air is, the less distinct are the two fronts with an evolution which quickly merges the ascent to produce a single feature. The closer to moist slantwise neutrality the stronger is the warm front and the larger is the separation from the cold front. Close to the neutral limit, the warm front collapses to a discontinuity before its motion toward the cold front is complete. For an intermediate value of stability, the relative motion of the distinct fronts is such as to allow an occlusion to be produced. Observations are only just becoming available to allow an accurate determination of this slantwise stability parameter. The values used here have been chosen to explore the likely range found in the atmosphere.
(a) Solid lines, vertical velocity. Contours are 0 (bold line), 20 cm s⁻¹ and 40 cm s⁻¹. Arrows show the ageostrophic flow, reference arrow 60 m s⁻¹. Dashed lines are m surfaces, contoured every 18 m s⁻¹.

(b) Continuous lines, along-front flow \( u_z \); contour interval: 3 m s⁻¹; bold line is 0, negative contours dashed. Dashed lines, potential temperature; contour interval: 3 K.

(c) Potential vorticity; contour interval: 0.2 PVU, bold line 0.4 PVU. The 0.2 PVU contour is dash-dotted, indicating a negative anomaly.

(d) Potential temperature departure from the mean; contour interval: 0.5 K, bold line, on the bottom right corner, 0.

(e) Absolute vorticity normalized by \( f \); contour interval: 1, bold line: 1.

The structure can be compared with the 'cold occlusion' of Fig. 5; the transformation of the latter into a moist normal mode is almost complete.

Figure 10. The structure of the moist mode after nonlinear evolution for 54 hours from the moist normal mode of Fig. 9. The initial amplitude was set consistent with a thermal anomaly of maximum value 0.08 K.
* The dynamics of the splitting of the ascent zone, prior to the formation of the warm front, can be understood in terms of the potential vorticity structure. Owing to the weak static stability along absolute momentum surfaces, the condensation heating produces a PV anomaly which has zones of large horizontal gradients. These regions, being of smaller horizontal extent than the broad sloping ascent in the Eady wave, imply, because of the PV gradients, a localized forcing of enhanced ascent and descent. This ultimately produces a splitting of this broad region of ascent. The dynamics of this process is probably applicable to many such situations where condensation is enhanced locally in a baroclinic zone.

* The small horizontal scale of the potential vorticity maximum, generated by condensation heating, dictates that this anomaly of potential vorticity is expressed almost totally in the vorticity field. This explains the relatively minor impact of the moist frontogenesis on the dry thermal field.

* As the nonlinearity of the moist baroclinic development is weak when viewed in geostrophic coordinates, the long-term behaviour is to tend towards a moist normal mode. This final state can be regarded as the occlusion in these simulations. The warm front thus appears as part of an adjustment from the initial flow, which is non-modal for moist development, to the final flow, which is close to a moist normal mode. It should be noted that these frontal structures also develop if the simulation is initialized with a neutral dry mode. The range of horizontal scales for which the neutral mode is moist unstable is approximately 1000 to 2000 km.

These studies can be regarded as related to several other approaches in which simplifications of the Eady model are refined. In particular, in a model of multiple frontogenesis, Hoskins et al. (1984) show that the introduction of small local anomalies in the surface distribution of potential temperature or potential vorticity can lead to distinct multiple vorticity maxima. Despite this, in their two-dimensional model, the vertical velocity field did not exhibit multiple ascent/descent regions. The model described here represents the dynamical effects of condensation heating, and leads to a splitting of the vertical velocity to produce two distinct fronts. Further, these fronts move relative to one another and so can clearly be identified as cold and warm fronts. These differences may be related to the dynamical feed-back inherent in condensation heating and to the deep nature of the consequent potential vorticity anomalies. Also we consider the shear mechanism of frontogenesis whereas the two-dimensional model in Hoskins et al. (1984) describes the deformation mechanism. Undoubtedly the natural processes of frontogenesis rely on all these mechanisms to different degrees. These models serve to further demonstrate the dynamical significance of tropospheric potential vorticity and/or surface thermal anomalies.

The model described here assumes a constant supply of saturated air to the regions of ascent and an adjustment to neutral slantwise stability. More evidence of the neutrality of real frontal ascent to slantwise convection is needed. If it is found to be the case, then improvement of mesoscale forecasting with primitive equation models demands that this feature be properly implemented in such models. On the other hand, it may be that humidity supply is crucial, something the semi-geostrophic model as used here does not consider. A primitive equation model with an explicit calculation of moist processes, may provide estimates of the spatial and temporal variations of \( r \). The mechanism described in this paper would explain those coupled fronts in which moisture supply, possibly through evaporation from the sea surface or convergence, is sufficiently large.
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APPENDIX: NUMERICAL SOLUTION PROCEDURE

(a) Nonlinear evolution

In order to make numerical solutions of the elliptic equations (2) and (3) economical and accurate even at very low values of $r$, a few improvements to the original scheme proposed by Heckley (1980) and used in EFT have been introduced. As mentioned in section 2, (2) was re-conditioned. Equations (2) and (3) are non-separable. However, by introducing a horizontal representative value of $P$, $\bar{P}(z)$, in both equations (having incorporated the diabatic source term (7) into the operator of (3)), and setting the residual as an additional forcing term, we can then start an iterative procedure, having to solve at each step horizontally separable equations. This is done by using a finite difference scheme in the vertical and a Fourier series expansion of all variables in the horizontal. At each step, for each horizontal harmonic, we obtain updated values of $\Phi$ (or $w^*$) by solving a tridiagonal system by Gaussian elimination. The grid is staggered in the vertical, $P$, $\phi$, $v_g$ and $\zeta$ being defined at $K + 1$ levels, and $\theta$ and $w^*$ at intermediate $K$ levels. All horizontal derivatives are deduced from the Fourier expansion, and nonlinear terms are computed on a 'real space' grid having at least $3M + 1$ points, where $M$ is the truncation of Fourier series, in order to eliminate aliasing on the quadratic terms. The time discrete scheme is a simple leapfrog. In order to use standard graphic software packages, the transformation from geostrophic to physical space also makes use of the Fourier expansion of $v_g$, in order to find the value of $X$ corresponding to regularly spaced values of $x$. The iterative procedures are stopped when the change in $\Phi$ (or $w^*$) is less than a given accuracy. Typically, we used $\varepsilon_\phi = 10^{-3} \text{J kg}^{-1}$ and $\varepsilon_w = 10^{-6} \text{m s}^{-1}$. Most of the results reported here were obtained with $K = 40$ levels and $M = 33$ harmonics.

(b) Normal mode growth determination

The basic set of equations with a constant vertical shear, and no feedback from the waves on the mean baroclinicity $\bar{\nabla} \cdot \theta$, admits normal modes of the form

$$\Phi = \hat{\Phi}(X, Z) e^{\sigma T} = \sum_{m=\pm \infty}^{+\infty} \phi_m(Z) e^{i m 2 \pi (X/L)} e^{\sigma T}.$$ 

The growth rate and phase speed were recovered by using the so-called initial value technique also used by Simmons and Hoskins (1976). In the present filtered model, the solution is completely determined by the potential $\Phi$.

At each time step we compute the following amplitude-weighted quantities, with $m$ the horizontal wavenumber and $k$ the vertical level:
\[ W = \sum_{k=1}^{K+1} \sum_{m=1}^{M} |\phi_{m,k}(t + \delta t)| \]

\[ G = \frac{1}{W} \sum_{k} \sum_{m} |\phi_{m,k}(t + \delta t)|^2 / |\phi_{m,k}(t)| \]

\[ \delta G = \frac{1}{W} \sum_{k} \sum_{m} |\phi_{m,k}(t + \delta t)| \cdot \left\{ \frac{|\phi_{m,k}(t + \delta t)|}{|\phi_{m,k}(t)|} - \frac{|\phi_{m,k}(t)|}{|\phi_{m,k}(t - \delta t)|} \right\} \]

\[ \tau_g = \frac{1}{W} \sum_{k} \sum_{m} |\phi_{m,k}(t + \delta t)| \left\{ \frac{(\phi_{i,m,k}(t + \delta t)(\phi_{r,m,k}(t + \delta t) - (\phi_{r,m,k}(t + \delta t)(\phi_{i,m,k}(t + \delta t)}}{(\phi_{r,m,k}(t + \delta t) + (\phi_{i,m,k}(t + \delta t)(\phi_{i,m,k}(t + \delta t)) \right\} \]

where \[ \Phi(X, Z_k) = \sum_{m=1}^{M} \phi_{m,k} e^{i2\pi m X/L} = \sum_{m} ((\phi_{r,m,k} + i(\phi_{i,m,k})) e^{i2\pi m X/L} \]

Then, assuming a normal mode form as above with \( \sigma = \sigma_r + i\sigma_i \) as \( \sigma \) is in general complex, a mean value of the growth rate \( \sigma \) is given by \( \sigma_r = \ln(G)/\delta t \) and the frequency \( \sigma_i = \tan^{-1}(\tau_g)/\delta t \). \( \delta G \) is a measure of the precision, the change in the estimate of the growth rate at two successive steps. Since in semi-geostrophic theory the flow is completely determined by the knowledge of \( \Phi \), it seemed the natural parameter to choose. Even if the flow is unstable, potential vorticity does not necessarily grow, as for example, in the case of uniform potential vorticity flows. In all cases, however, the growth of \( \Phi \) determines the growth of \( \nu_g, \theta \) and \( \nu^* \).

Taking a more mathematical point of view, this initial value technique makes use of a general property of any matrix \( A \), that \( A^n E_0 \) where \( E_0 \) is an initially random vector converges towards the dominant eigenvector of \( A \).

Defining the series

\[ E_{n+1} = A E_n \]

there exists a complex constant \( \lambda_i \) such that

\[ \lim_{n \to +\infty} (E_{n+1} - \lambda_i E_n) = 0. \]

It can be seen that \( \lambda_i \) is the dominant eigenvalue and \( E_n \) converges toward the corresponding eigenvector. It allows, via a simple iterative procedure, which may be called ‘time integration’, both the eigenvalue and the structure of the most unstable mode to be obtained without having to store \( A \) explicitly. Such storage is difficult to provide for the large values of \( K \) and \( M \) we have been using. Thus, this technique makes the most of the storage available. When the numerical scheme is properly coded for a vector computer, the cost becomes comparable to the cost of the actual calculation of the eigenvalues of \( A \), when \( A \) is known and stored, especially when one mode clearly dominates. Convergence becomes slower when several eigenvalues have comparable moduli. Called the ‘power method’ this technique appears suitable for any problem leading to non-symmetric, non-banded matrices, and, after Dautray and Lions (1983), may be the only way to recover eigenvalues in complicated cases. Strictly speaking, however, these ideas apply only to linear problems as nonlinear ones cannot be cast into matrix form.
We expect success from the 'power method' in this case because nonlinearity appears to be rather weak: nonlinearity occurs only in the vertical advection of potential vorticity, which changes slowly, and in the dependence of $S$ on the vertical velocity. Although crucial to the physics of our problem, it does not involve a quadratic time-dependent term, but changes one of the coefficients of the elliptic equation (3). The main point is that there is no feedback from the growing wave onto the mean flow $\bar{u}(z)$ (which would involve terms growing with $2\sigma$, and may obscure the normal mode structure as it saturates) because of the assumption of two-dimensionality. The initial random perturbation (or vector) must be small, in order for the procedure to converge before frontal collapse occurs. This also prevents nonlinear terms from becoming large. A typical value of the initial amplitude is $5 \times 10^{-4} K$ for the perturbation of $\theta$ on both boundaries.

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