A general method for the linear stability analysis of stratified shear flows

By S. D. MOBBS and M. S. DARBY

Department of Applied Mathematical Studies, University of Leeds

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SUMMARY

A general method for testing the stability of a stratified, parallel shear flow is presented. The technique allows the routine determination of unstable modes from any high resolution radiosonde data and is a development of the procedure described by Lalas and Einaudi. Stability analyses of several idealized velocity profiles have been performed in order to validate the method. A particularly important feature is the ability to predict marginally unstable modes with more than one critical level; several examples are presented. An example using real radiosonde data is given.

1. INTRODUCTION

In recent years, internal gravity waves have been recognized as having an important influence on the dynamics of the lower atmosphere (see, for example, Gossard and Hooke 1975 or Einaudi 1980). Topographic effects undoubtedly play a major part in wave generation, particularly for longer wavelengths, but in addition, there have been numerous observations suggesting that Kelvin–Helmholtz-type shear instabilities are often important (see e.g. Gossard et al. 1970; Hines 1970; Emmanuel 1973; Hooke et al. 1973; Kelicher 1975; Pellacani and Lupini 1975; Davis and Pelletier 1976; Merrill 1977; King et al. 1987). The prediction of these unstable modes requires, in the first instance, a linear stability analysis of the basic flow profiles in which the waves propagate. A full understanding of their generation, propagation and influence is likely to require a nonlinear treatment but such investigations often require, as a first step, a determination of the linearly unstable modes. We require, therefore, a general method for analysing the stability of stratified flow profiles. For most realistic atmospheric applications, such basic flows may be considered, as a first approximation, to be horizontal shear flows.

Lalas and Einaudi (1976, LE) developed a numerical technique for obtaining the stability characteristics of a stratified, parallel shear flow and presented results for a hyperbolic tangent wind profile with constant Brunt–Väisälä frequency. The method allowed a rigid, flat lower boundary and a radiation boundary condition at some upper level. Later, Lalas and Einaudi (1980) applied the method to the analysis of a rawinsonde profile which corresponded to ground-based observations of gravity waves obtained using a microbarograph array. Satisfactory agreement between the observed waves and those predicted by the stability model was obtained. The same model was also used by Fua et al. (1976) for idealized nocturnal boundary layer profiles and by Mastrantonio et al. (1976) for idealized jet stream profiles. Chimonas and Grant (1984) used the technique to study a realistic model profile containing two vertical length scales giving rise to thin regions of sub-critical Richardson number (Richardson number less than 0.25). Chimonas and Grant showed that such thin layers can cause instabilities with a wider range of horizontal wavelengths than those associated with a smooth, one-scale atmosphere.

The method developed by LE is applicable to both neutral and unstable wave modes. It is well known that the former can possess ‘critical levels’ at which the horizontal phase speed is equal to the mean flow speed in the direction of propagation. (Hereafter, we shall refer to levels where the mean flow and phase speeds are equal as critical levels, irrespective of whether the waves are neutral or growing.) Such critical levels appear
mathematically as regular singularities in the governing differential equations and LE obtained solutions by matching across the singularity, numerical solutions obtained by integrating inwards from the upper and lower boundaries. Such a procedure is clearly applicable only to profiles in which the neutral modes have no more than one critical level. More seriously, however, numerical difficulties arise for weakly growing unstable modes for which the solutions are almost singular at the levels where the phase speed and mean flow speed are equal. Such modes cannot be obtained by the method of LE. One solution to this problem was presented by Mastrantonio et al. (1976): they prescribed a minimum value below which the growth rate was not allowed to fall. If a growth rate smaller than the critical value was suggested by the iterative procedure, it was immediately set equal to the critical value. An alternative solution, allowing arbitrarily small growth rate, is presented in this paper.

Attempts to analyse the stability of atmospheric profiles obtained from radiosondes have shown that unstable modes frequently have several critical levels. The need for a more general method allowing an arbitrary number of critical levels and arbitrarily small growth rate prompted the present study. Other extensions to the technique described by LE also proved to be necessary in order to make the stability analysis fully automatic:

(i) There are usually several levels of sub-critical Richardson number. A general stability analysis technique should be able to distinguish between unstable modes resulting from different levels of low Richardson number. The method described in this paper uses as a first estimate of the phase speed of an unstable mode the mean flow velocity at a specified minimum of the Richardson number. Convergence to solutions strongly peaked around that level is usually obtained.

(ii) The search for unstable modes involves the iterative solution of an eigenvalue problem. Convergence of any practical numerical scheme can be slow or even non-existent, particularly when the vertical wavelength is small compared with the depth of the domain or if the initial estimates for the growth rate and phase speed are not good. The method presented here first obtains solutions with upper and lower 'boundaries' close to the level of minimum Richardson number; convergence is then usually reliable. The boundaries are subsequently moved stepwise out to the correct positions. It has been found that such movement of the boundaries results in only slight adjustments to the eigenvalues and that the required solutions can be easily followed as the boundaries are moved outwards.

(iii) Real atmospheric profiles frequently have many sub-critical minima in the Richardson number which have associated with them unstable modes with very small growth rates. Such modes are of little practical interest but their determination can involve considerable amounts of computer time. It is desirable, therefore, to recognize such levels prior to numerical solution of the eigenvalue problem. Numerical solution need then be performed only for the significant unstable modes. The present technique, due to Howard (1961), uses an upper bound on the growth rate to eliminate weakly growing modes. Such an approach does not remove the need to cope with small growth rates, which, as discussed above, require special treatment near critical levels. This is because modes which for some wavelengths are rapidly growing will, at other wavelengths, generally grow only slowly.

The method is described in sections 2 and 3. Some results for simple, idealized profiles are given in section 4. These compare the present method with previous results and demonstrate some of the capabilities of the technique. In section 5, results are given for a real profile obtained from a radiosonde ascent and some conclusions are presented in section 6.
The equations to be solved are the linearized versions of the full equations of motion, thermodynamics and continuity for adiabatic, stratified flow. LE restricted attention to a 'shallow convection' version of the Boussinesq approximation, although they subsequently point out that no additional numerical complexity is introduced by using the full equations. (Mastrantonio et al., in a follow-up to LE, used a version of the Boussinesq approximation less severe than that used by LE.) The Coriolis force is neglected, since the length and time scales of the unstable modes are expected to be small. (In fact, the present method could be extended to include the Coriolis force. However, the extension is not trivial, since the reduction to a two-dimensional problem, as described below, does not carry through.) The equations are

\[
\begin{align*}
\frac{\partial u}{\partial t} + U_o \frac{\partial u}{\partial x} + V_o \frac{\partial u}{\partial y} + w \frac{dU_o}{dz} + \frac{1}{\rho_o} \frac{\partial p}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + U_o \frac{\partial v}{\partial x} + V_o \frac{\partial v}{\partial y} + w \frac{dV_o}{dz} + \frac{1}{\rho_o} \frac{\partial p}{\partial y} &= 0 \\
\frac{\partial w}{\partial t} + U_o \frac{\partial w}{\partial x} + V_o \frac{\partial w}{\partial y} + \frac{1}{\rho_o} \frac{\partial p}{\partial z} + \frac{\partial g}{\partial z} &= 0 \\
\frac{\partial \rho}{\partial t} + U_o \frac{\partial \rho}{\partial x} + V_o \frac{\partial \rho}{\partial y} + w \frac{\partial \rho_o}{\partial z} + \rho_o \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) &= 0 \\
\frac{\partial p}{\partial t} + U_o \frac{\partial p}{\partial x} + V_o \frac{\partial p}{\partial y} + w \frac{\partial p_o}{\partial z} &= -\gamma \rho_o \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right). 
\end{align*}
\]

\(x\) and \(y\) are the horizontal coordinates, \(z\) is height and \(t\) is time. \(u(x, y, z, t)\) and \(v(x, y, z, t)\) are the horizontal perturbation velocity components in the \(x\) and \(y\) directions respectively. \(w(x, y, z, t)\) is the velocity component in the \(z\) direction, \(\rho(x, y, z, t)\) is the perturbation density and \(p(x, y, z, t)\) is the perturbation pressure. \(U_o(z)\) and \(V_o(z)\) are basic state velocity components in the \(x\) and \(y\) directions, \(\rho_o(z)\) is the basic state density and \(p_o(z)\) is the basic state pressure. \(N\) is the Brunt–Väisälä frequency given by

\[
N^2 = -\frac{g}{\rho_o} \frac{d \rho_o}{dz} - \frac{g^2}{c_s^2}
\]

where \(g\) is the gravitational acceleration and \(c_s\) is the speed of sound (assumed constant).

In Eqs. (1)–(5), a solution is sought in the form

\[
(u, v, w, p, \rho) = \rho_o^{-1/2} \text{Re}[[\hat{u}(z), \hat{v}(z), \hat{\Psi}(z), \hat{\rho}(z), \hat{\rho}(z)] \exp[i(kx + ly - \omega t)]]
\]

where \(\text{Re}\) denotes the real part. \(\Psi\) is an unknown function of height (the eigenfunction), \(k\) and \(l\) are the horizontal wavenumbers in the \(x\) and \(y\) directions and \(\omega = \omega_r + i \omega_i\) is the complex angular frequency. Clearly the choice of direction for the \(x\) axis is arbitrary and therefore we can choose, without loss of generality, to take the \(x\) axis to be the direction of wave propagation, with the result that \(l = 0\). It then turns out that \(V_o\) no longer enters the equations and the equation for \(\delta\) decouples from the others. This does not mean, however, that only one horizontal component of the basic flow velocity is relevant to the stability problem, since it is necessary to test for growing disturbances travelling in all possible horizontal directions. In practice, this involves testing a range of velocity profiles \(U_o(z)\) obtained by resolving the total horizontal wind vector in a number of different
directions. Substitution of (7) into Eq. (1) and Eqs. (3)–(5) and subsequent elimination of \( \dot{\alpha}, \dot{\beta}, \dot{\rho} \) results, after much algebra, in a single equation for \( \Psi \), which is

\[
d^2\Psi/dz^2 - \lambda(z)\Psi = 0 \tag{8}
\]

where

\[
\lambda(z) = k^2 - N^2 \frac{d^2 c}{dz^2} - \frac{1}{4\rho_o} \frac{d\rho_o}{dz} \left( \frac{d\rho_o}{dz} \right)^2 + \frac{1}{2\rho_o} \frac{d\rho_o}{dz} \left( \frac{d^2 \rho_o}{dz^2} \right) + \frac{1}{c_s^2} \frac{d\rho_o}{dz} \left( \frac{1}{\rho_o} \frac{d\rho_o}{dz} + \frac{2g}{c_s^2} \right) - \frac{N^2}{c_s^2} \tag{9}
\]

where \( c = \omega/k \) is the complex phase speed and \( \dot{c} = c - U_o \).

A rigid lower boundary condition is imposed on \( \Psi \):

\[
\Psi = 0 \quad \text{at} \quad z = z_l \tag{10}
\]

where \( z_l \) is the height of the lower boundary. (\( z_l \) need not be zero, since, as described below, the iterative procedure used to obtain the eigenvalues involves having the initial position of the lower boundary close to the level of lowest Richardson number.) Following LE, an upper level \( z_u \) is assumed to exist, such that

\[
\begin{align*}
N^2(z) &= N^2(z_u) \quad \text{for} \quad z > z_u, \\
U_o(z) &= U_o(z_u)
\end{align*} \tag{11}
\]

In LE the profiles of \( N \) and \( U_o \) actually satisfy (11), whereas real radiosonde profiles rarely will. Nevertheless, an assumption concerning the atmosphere above the highest level available from the radiosonde is needed in order to apply a radiation boundary condition and (11) is chosen for its simplicity. In practice, any prediction of an unstable mode whose structure is significantly dependent on the basic flow data at the top of the radiosonde ascent must be considered suspect. By enforcing condition (11), \( \lambda \) is made constant for \( z > z_u \) and the solution to Eq. (8) in this region takes the form

\[
\Psi \propto \exp(\text{im}z)
\]

where \( m \) is the vertical wavenumber, given by

\[
m = \pm [-\lambda(z_u)]^{1/2} = m_r + \text{im}_i \tag{12}
\]

If \( m_i \neq 0 \) then the sign in (12) is chosen so that \( m_i > 0 \), implying exponential decay of the wave amplitude above \( z = z_u \). If \( m_i = 0 \) then the sign in (12) is chosen so that the group velocity has a positive vertical component. The upper boundary condition is

\[
d\Psi/dz = \text{im}\Psi \quad \text{at} \quad z = z_u. \tag{13}
\]

3. The Numerical Model

The height-dependent coefficient \( \lambda \) is first evaluated from the radiosonde data. The density is easily deduced from the measurements of pressure and temperature: \( \rho = p/RT \), where \( R \) is the gas constant for air (=287 J kg\(^{-1}\)K\(^{-1}\)). The choice of direction in which to resolve the flow velocity in order to obtain the basic state component \( U_o \) is arbitrary. Usually, however, the direction is chosen as one which is likely to yield a dynamically unstable profile. Guidance for this comes from Howard’s (1961) bound on the growth rate, \( \omega_l \). Howard showed that

\[
\omega_l \leq \max \left\{ \frac{1}{4} \left( \frac{dU_o}{dz} \right)^2 - N^2 \right\}^{1/2} = \max(\Omega) = (\omega_l)_{\max} \tag{14}
\]
where the maximum is taken over all heights. By calculating $\Omega$ as a function of $z$ and of the direction of resolution, suitable directions for resolving can be deduced; an example is given below (case (g)).

In general the flow speed and density data will not be known at the heights required for the numerical integration of Eq. (8). Cubic splines, with knots at every radiosonde data point, are therefore fitted. $U_0$ and $\rho_0$ (and their derivatives) can then be evaluated from the splines at any desired height.

Equation (8) with boundary conditions (10) and (13) is an eigenvalue problem for the complex phase speed $c$. It is solved iteratively; the procedure used for each estimate of $c$ is to integrate Eq. (8) from $z_1$ to $z_a$, using the lower boundary condition (10) and hence obtain an error estimate for the upper boundary condition (13). Newton iteration then gives the next, improved, estimate for $c$. A fourth-order Runge-Kutta method is used to integrate Eq. (8). The integration is started with $\Psi(z) = 0$ and with an arbitrary choice of $1+i$ for $d\Psi/dz(z_l)$ (a different choice is merely equivalent to multiplying the eigenfunction by a constant, which is irrelevant since the system is linear).

The step size is reduced as critical levels are approached in order to maintain accuracy. Specifically, a reduction in step size is applied only when the modulus of the intrinsic phase speed, $|c|$, falls below a critical value, the step size $\Delta z$ then being made proportional to $|c|$. The detailed justification is given in appendix A.

The step size can become unreasonably small near critical levels when $c_l$ is small and the method would break down completely for neutral waves with $c_l = 0$. We therefore follow LE and use a Frobenius expansion for the solution in the neighbourhood of critical levels. However, LE only used a Frobenius solution when $c_l = 0$; here a Frobenius solution is used, if possible (see below), when $c_l$ falls below a critical value, thus avoiding the use of very small steps in the numerical integration. The Frobenius solution is (Booker and Bretherton 1967)

$$\Psi = A(z - z_c - i c_l/U_o')^{\alpha_1} + B(z - z_c - i c_l/U_o')^{\alpha_2}$$

where $A$ and $B$ are constants, $z_c$ is the height of the critical level and $U'_o$ is $dU_0/dz$ evaluated at $z = z_c$. $\alpha_1$ and $\alpha_2$ are given by

$$\alpha_1 = 1/2 + \sqrt{1/4 - Ri} \quad \text{and} \quad \alpha_2 = 1/2 - \sqrt{1/4 - Ri}$$

where $Ri$ is the Richardson number, $N^2/U_o'^2$, evaluated at $z = z_c$. The procedure for using this solution is to integrate Eq. (8) using the Runge-Kutta method, up to a small distance $D$ below the level $z = z_c$. The constants $A$ and $B$ can then be evaluated by requiring that $\Psi$ and $d\Psi/dz$ match continuously onto the Frobenius solution. The Frobenius solution is used to obtain $\Psi$ and $d\Psi/dz$ at a distance $D$ above $z = z_c$. Numerical integration can then proceed to the top of the domain or up to the next critical level.

The Frobenius solution (15) is invalid if $Ri = 0$ or $Ri = 1/4$: close to these values it is necessary to restrict $c_l$ to values above a very small constant (as did Mastrantonio et al. for all values of $Ri$) and use the Runge-Kutta scheme near the critical level. Although alternative Frobenius solutions can be found for $Ri = 0$ and $Ri = 1/4$, these are not valid for values of $Ri$ very close to, but not equal to, 0 or 1/4. They are therefore of little practical value and are not used. In practice, the restriction on $c_l$ rarely comes into effect for real atmospheric profiles. The full details concerning the application of the Frobenius expansion are given in appendix B.

An alternative solution to the problem of multiple critical levels described in the previous paragraph was used by Mastrantonio et al. (1976). Whenever $c_l$ fell below $5 \times 10^{-4}$ m s$^{-1}$ they set $c_l = 5 \times 10^{-4}$ m s$^{-1}$. This procedure, coupled with an adequate
reduction in the integration step $\Delta z$ in the region of a critical level, was found to be sufficient to allow integration through all critical levels without use of a Frobenius expansion. The method of Mastrantonio et al. is simpler than the one used here and gives virtually identical eigenfunction structure but it has two drawbacks. These are, firstly, that the method requires greater computer CPU time than the present one and secondly, that when a number of close eigenvalues exist, satisfactory convergence requires that the complex phase speed $c$ should be determined to a greater accuracy than $5 \times 10^{-4}$ m s$^{-1}$ (using a smaller minimum value of $c_r$ results in even larger computer requirements).

There is an additional complication for the case of critical levels with Richardson number greater than 1/4. The amplitude of a disturbance propagating through such a level is attenuated by a factor $m = \exp \left[ - 2\pi \sqrt{(Ri - 1/4)} \right]$ (Booker and Bretherton 1967). For $Ri > 1$ greater than one, only a very small fraction of an incident disturbance is transmitted through the critical level. If this small fraction is used in the iteration procedure at the upper boundary, convergence is poor or even nonexistent. This is reasonable on physical grounds, since it would not be expected that the small transmitted fraction should have a decisive controlling effect on the flow below the critical level. Therefore, if $Ri > 1$ at any critical level above the one controlling an instability, the upper boundary is moved down to a level just above the critical level where $Ri > 1$ and a rigid boundary condition ($\Psi = 0$) is applied there. Similarly, if a second critical level with $Ri > 1$ occurs below the controlling critical level, the lower boundary is moved up to a position just below the critical level where $Ri > 1$. Since the theory of Booker and Bretherton is strictly valid only for small growth rates, false boundary conditions are used only if the imaginary part of the phase speed, $c_r$, is less than a critical value. A value of 0.1 m s$^{-1}$ has been found to be adequate; no problems with poor convergence due to absorption at critical levels have been encountered for larger values of $c_r$. The effect of using false boundary conditions is discussed more fully in the next section (case (d)).

The details of the iterative procedure are as follows. For a given estimate for the complex phase speed $c$, Eq. (8) is integrated from $z = z_1$ to $z = z_u$ using the method described above. A normalized measure of the error in the upper boundary condition (14) is

$$
\varepsilon_1 = \frac{|d\Psi/dz - im\Psi|_{z = z_u}}{(d\Psi/dz)_{max}}
$$

where $(d\Psi/dz)_{max}$ is the maximum value of $d\Psi/dz$ between $z_1$ and $z_u$. Convergence is achieved when $\varepsilon_1$ becomes less than a small number; 10$^{-4}$ is typically found to be satisfactory. However, if $\varepsilon_1$ is used as the error estimate in a Newton iteration scheme to obtain the next estimate of $c$, convergence is found to be very slow. A much superior method is to use a second, non-normalized estimate $\varepsilon_2$, given by

$$
|d\Psi/dz - im\Psi|_{z = z_u}
$$

within the Newton iteration scheme.

For successful iteration it is important to have a good initial guess for $c$. This is chosen by setting $c_r$ equal to the value of $U_o$ at a minimum of the Richardson number $N^2/(dU_o/dz)^2$. Initially, the range of horizontal wavenumbers, $k$, over which a disturbance may be unstable is unknown. A suitable wavenumber at which instability is likely to occur is found using the formula

$$
k = 2\pi/n\delta \quad \text{for} \quad 1 \leq n \leq 14
$$

where $\delta$ is the depth of the subcritical Richardson number region at which $c_r$ has been
set equal to $U_o$. Values of $n$ starting from 1 are tried successively until convergence is obtained or $n = 14$. This value for the upper limit on $n$ is chosen because experience shows that if convergence is to occur, it will occur for a value of $n$ in the range 5 to 10. As a first estimate of $c$, the value $(\omega_i)_{\text{max}}/k$ is used (see Eq. (14)). Once a solution is known for one value of $k$, the eigenvalue $c$ can be used as the initial estimate for the next value of $k$, which will not in general need to satisfy Eq. (17).

At each wavenumber, convergence may be difficult to obtain, even if the initial estimate for $c$ is quite good, if the thickness of the region of sub-critical Richardson number is much less than the depth of the domain considered. This is frequently the case for real atmospheric profiles. It has been found, however, that rapid convergence can be obtained if the upper and lower boundaries are placed close to the minimum in the Richardson number. This has the additional advantage that for profiles with several sub-critical minima of the Richardson number, an initial calculation can be made including only one minimum; thus the resulting unstable mode can be positively associated with that particular minimum. Having obtained a solution with the boundaries close to the minimum in $Ri$, the process can be repeated successively as the boundaries are moved out towards their ‘true’ positions. For each calculation, the value of $c$ obtained for the immediately previous boundary position is used as the initial estimate. The boundaries are gradually moved outwards either to the top and bottom of the known profile, or to the most separated positions for which convergence can be obtained. A full discussion of this technique is given in the next section (cases (e) and (f)).

In order to avoid expending an excessive amount of computer time calculating weakly unstable modes which are of little physical significance, the maximum possible growth rate, $(\omega_i)_{\text{max}}$, is first calculated for each sub-critical minimum in the Richardson number. Minima for which the maximum possible growth rate is less than some predetermined value are not considered as possible sources of physically relevant instability.

4. Examples of the use of the stability model

Six examples illustrating the use of the stability model described in section 3 are now given. The first five have constant Brunt–Väisälä frequency and uni-directional velocity profiles of the general form

$$U_o = A_0 + A_1 \tanh \left( \frac{z - a_1}{h_1} \right) + A_2 \tanh \left( \frac{z - a_2}{h_2} \right).$$

(a) Case (a): single minimum in $Ri$

In this case, $A_0 = A_2 = 0$, $A_1 = 30\cdot22 \text{ m s}^{-1}$, $a_1 = 1000 \text{ m}$, $h_1 = 100 \text{ m}$ and the Brunt–Väisälä frequency, $N$, is 0.0956 Hz. The upper boundary is at $z = 3000 \text{ m}$. Figure 1(a) shows the velocity profile and Fig. 1(b) shows the corresponding Richardson number. The minimum Richardson number is 0.1 and the profile corresponds closely to one of those studied by LE. Figures 2(a) and (b) show the resulting phase speed and growth rate (i.e. $kc_0$) plotted against wavenumber. Profiles of this type have been studied on numerous occasions and here we restrict attention to the comparison with the method of LE. Figures 2(a) and (b) also show the phase speed and growth rate as calculated with the current model, but using LE’s shallow convection expression for $\lambda$ rather than the full version given in Eq. (9). Comparison of these results with those presented by LE

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1 In fact LE took an isothermal basic state which is slightly different from the present one. However, the difference between the LE and present profiles changes only those terms in $\lambda$ (Eq. (9)) proportional to $1/H^2$, where $H$ is the scale height. This difference is very small.
Figure 1. (a) The velocity profile for case (a). The profile is defined by Eq. (18) (see text for the values of the constants used). The dashed line indicates the level of minimum Richardson number. (b) The Richardson number profile for case (a).

Figure 2. (a) The phase speed $c_p$ and (b) the growth rate $k_\sigma$, plotted against wavenumber for the profiles in Fig. 1. The solid lines are the solutions obtained using the present method and the dashed lines are the solutions obtained using the shallow convection equations of LE (observed by the full line in (b)).
shows that when the same expression for \( \lambda \) is used, the current model reproduces almost exactly the results of LE. However, the results using the full expression for \( \lambda \) given in Eq. (9) have a significantly different phase speed, although the growth rate curves are very similar. Clearly the phase speed is sensitive to the particular form of the Boussinesq approximation. All the subsequent cases use the full version of \( \lambda \).

(b) Case (b): unstable modes with a second critical level where Ri is low

For this case, \( A_1 = 3.16 \text{ m s}^{-1}, A_2 = -3.16 \text{ m s}^{-1}, h_1 = h_2 = 100 \text{ m}, a_1 = 1000 \text{ m}, a_2 = 1500 \text{ m} \) and \( N = 0.01 \text{ Hz} \). The velocity profile is given in Fig. 3(a) and the corresponding Richardson number profile is shown in Fig. 3(b). The Richardson number has two minima, each having a value of 0.1. It might be expected that there would be unstable modes associated with each of the two minima in the Richardson number and indeed this is found to be the case. Figures 4(a) and (b) show the phase speeds and growth rates for the two modes plotted against wavenumber. Figure 5 shows the moduli of the eigenfunctions, \( \Psi \), plotted against height for a wavenumber of 0.00612 m\(^{-1}\). Clearly, one mode has its maximum amplitude at the minimum of \( Ri \) at 1000 m and the other mode has its maximum amplitude at the minimum of \( Ri \) at 1500 m. Consideration of the phase speeds, \( c_r \), of each mode indicates that both modes have two critical levels (levels where \( U_o = c_r \)) where the Richardson number is considerably less than one. The attenuation by a factor of \( \exp(-2\pi\sqrt{(Ri - 1/4)}) \), predicted by Booker and Bretherton, has little effect for the small values of the Richardson number found here.

It is worth remarking that the ability to move the boundary positions is crucial to finding the two separate modes. For each mode, the iteration was begun with the upper and lower boundaries sufficiently close to the appropriate minimum in \( Ri \) that the other minimum was excluded; thus the mode could be positively identified with a particular minimum in \( Ri \). The solution was then refined by moving the boundaries out to their true positions in steps. If the iteration is started with the boundaries in their true positions, there is a tendency to always converge to one particular mode and the other mode is never found.

![Figure 3.](image)

(a) The velocity profile and (b) the Richardson number profile for case (b). The dashed lines in (a) indicate the levels of minimum Richardson number.
Figure 4. (a) The phase speed $c$, and (b) the growth rate $k_0$, plotted against wavenumber for the profiles in Fig. 3. The solid lines show the solution for the mode generated by the low-Richardson-number layer near 1000 m and the dashed lines show the solution for the mode generated by the low-Richardson-number layer near 1500 m.

Figure 5. The modulus of the eigenfunction, $|\Psi|$, for case (b) plotted against height for the mode generated by the low-Richardson-number layer near 1000 m (full line) and the mode generated by the low-Richardson-number layer near 1500 m (dashed line). The wavenumber is 0.00612 m$^{-1}$. 
(c) Case (c): unstable mode with a second critical level where Ri is high

The profile in this example, shown in Fig. 6(a), has a jet-like structure as in case (b). However, there is now only one sub-critical minimum in the Richardson number (Fig. 6(b)) and hence only one unstable Kelvin–Helmholtz mode. The profile is achieved using \( A_0 = 0 \), \( A_1 = 3.16 \text{ m s}^{-1} \), \( A_2 = -3.16 \text{ m s}^{-1} \), \( a_1 = 1000 \text{ m} \), \( a_2 = 1500 \text{ m} \), \( h_1 = 100 \text{ m} \), \( h_2 = 300 \text{ m} \) and \( N = 0.01 \text{ Hz} \). The values of the Richardson number at the two minima are about 0.1 and 1.0. The phase speed and growth rate are plotted against wavenumber in Figs. 7(a) and (b). The results are similar to those found in case (b) for the mode with its critical level close to 1000 m. The modulus of the eigenfunction is shown in Fig. 8. Note that the amplitude goes rapidly to a small value near 1500 m, which corresponds to a second critical level of the unstable mode. The Richardson number is greater than one here and so the strong attenuation is to be expected from the theory of Booker and Bretherton. This contrasts with case (b), where the Richardson number was low at the second critical level and little attenuation was evident.

The results presented for this case were calculated with the final boundary positions in their ‘true’ positions (i.e. 0 m and 3000 m). However, as described in section 3, convergence can be slow or non-existent if the amplitude of a disturbance is strongly attenuated at a second critical level. The reason is quite clear from the present results: if strong attenuation occurs, then there is negligible amplitude at the upper boundary for any value of \( c \) and hence the upper boundary condition does not satisfactorily determine \( c \). In normal use the model selects a new upper boundary just above a second critical level (if \( Ri > 1 \) at that level) and a rigid boundary condition is applied there. The application of this method to the present profile is described in case (d).

Figure 6. (a) The velocity profile and (b) the Richardson number profile for case (c). The dashed line in (a) indicates the level of minimum Richardson number.
Figure 7. (a) The phase speed $c$, and (b) the growth rate $kc$, plotted against wavenumber for the profiles in Fig. 6. The boundaries were placed at their true positions (i.e. $z = 0 \text{ m}$ and $z = 3000 \text{ m}$).

Figure 8. The modulus of the eigenfunction, $|\Psi|$, plotted against height for the profiles shown in Fig. 6. Note the strong attenuation above the second critical level at 1500 m.
\[ (d) \text{ Case (d): unstable mode with a second critical level where } Ri \text{ is high; solution with rigid upper boundary} \]

The profile is the same as for case (c) but solution is performed by placing a rigid upper boundary just above the second critical level of the unstable mode at 1500 m. Figures 9(a) and (b) show the phase speed and growth rate plotted against wavenumber. Comparison with Figs. 7(a) and (b) reveals that the two methods give virtually identical results. (In fact the results are not plotted on the same axes since they would be indistinguishable.) Note, however, that convergence was obtained much more quickly using the rigid upper boundary condition. Figure 10 shows the eigenfunction for a wavenumber of 0.0055 m\(^{-1}\). At levels below the critical level, the eigenfunction is virtually identical to that obtained using the radiating upper boundary condition in case (c).

The present results illustrate the effectiveness of replacing the radiating upper boundary condition by a rigid boundary when an unstable mode has a critical level of high Richardson number above the minimum of \( Ri \) responsible for the instability. Exactly the same procedure can be applied if a second, high-\( Ri \) critical level occurs below the level of minimum \( Ri \). Then a rigid boundary condition is applied just below the second critical level.

![Graphs showing phase speed and growth rate](image)

Figure 9. (a) The phase speed \( c_0 \), and (b) the growth rate \( \kappa_0 \), plotted against wavenumber for the profiles in Fig. 6. A rigid upper boundary condition was applied just above the second critical level near 1500 m.

\[ (e) \text{ Case (e): overreflection modes of instability} \]

The aim here is not to study any detailed property of overreflection modes of instability; this has been done by Lindzen and Rosenthal (1976), Fritts (1980, 1982) and many others. The intention is to demonstrate that the present method can predict such modes and to point out a possible problem associated with moving the lower boundary...
in order to obtain convergence. The profile used has $A_0 = A_1 = 3.16 \text{ m s}^{-1}$, $A_2 = 0$, $a_1 = 500 \text{ m}$, $h_1 = 100 \text{ m}$ and $N = 0.01 \text{ Hz}$ (see Eq. (18)). The velocity profile is shown in Fig. 11(a) and the corresponding Richardson number profile is given in Fig. 11(b). In fact the velocity profile is similar to that for case (a) but the minimum in the Richardson number is at 500 m instead of 100 m and the upper boundary is at 1500 m instead of 3000 m (the latter change has no effect on the unstable modes). The value of $N$ is much more realistic than in case (a). The phase speed and growth rate are shown in Figs. 12(a) and (b). For high wavenumbers they are very similar to those obtained for case (a) but at low wavenumbers the instability clearly has a different character. This is the overreflection-type instability discussed by Lindzen and Rosenthal and by Fritts; it relies for its existence on the proximity of the ground. As the phase variations with height of the eigenfunctions are of particular interest in this case, they are represented by contours of equal $\text{Re} \{ \Psi(z) \exp(ikx) \}$ plotted against $x$ and $z$. Figure 13(a) shows the eigenfunction for a wavenumber of 0.0055 m$^{-1}$; this corresponds to a Kelvin–Helmholtz-type instability. The eigenfunction has its maximum amplitude near the level of minimum Richardson number and has considerable phase variation with height. Figure 13(b) shows the eigenfunction for a wavenumber of 0.0026 m$^{-1}$; this is the overreflection mode of instability and has no phase variation with height below the level of sub-critical Richardson number. (The reader is referred to the paper by Fritts (1980) for a comparison of Kelvin–Helmholtz and overreflection modes of instability.)

The primary reason for presenting results for an overreflection mode is to point out that such modes are likely to occur when a level of sub-critical Richardson number occurs close to the rigid lower boundary. If, for a real atmospheric profile, such modes are found with the lower boundary in its true position, then they can be interpreted as having physical significance. However, they may also appear when false, raised boundaries
Figure 11. (a) The velocity profile and (b) the Richardson number profile for case (e). The dashed line in (a) indicates the level of minimum Richardson number.

Figure 12. (a) The phase speed $c_p$ and (b) the growth rate $k_0$ plotted against wavenumber for the profiles in Fig. 11.
are used to obtain accelerated convergence, as described in section 3. Under those circumstances, the overreflection modes should be rejected, unless they persist as the lower boundary is moved downwards to its true position. The overreflection modes can be identified by the closed contours of vertical velocity which exist between the ground and the critical level. Another distinguishing feature of overreflection modes is that the phase speed tends to vary strongly with horizontal wavenumber (see Fig. 12(a)).

(f) Case (f): the thin shear layer profile of Chimonas and Grant

This profile has been studied previously by Chimonas and Grant (1984). The velocity profile is shown in Fig. 14(a) and the corresponding temperature profile is shown in Fig. 14(b). Note the ‘kink’ in the velocity profile at 13200 m; without this kink the profile would be stable but the kink induces a thin layer of sub-critical Richardson number with a minimum value of 0.072. The phase speed and growth rate of the resulting unstable mode are shown in Figs. 15(a) and (b), along with those calculated by Chimonas and Grant. There is clearly good agreement, particularly when it is noted that Chimonas and Grant used the shallow convection equations of LE. The eigenfunction for a wavenumber of 0.01 m⁻¹ is given in Fig. 16. It is clearly highly trapped near the shear layer and its amplitude tends very rapidly to zero away from the shear layer. In fact the final positions of the upper and lower boundaries in this case were about 12000 m and 14000 m.
Figure 14. (a) The velocity profile and (b) the temperature profile for case (f) (from Chimonas and Grant 1984). Note the 'kink' in the velocity profile near 13200 m, which gives rise to a sub-critical minimum in the Richardson number at the level indicated by the dashed line.

Figure 15. (a) The phase speed $c$, and (b) the growth rate $\gamma c$, plotted against wavenumber for the profiles in Fig. 14. The solid lines are the solutions obtained using the present method and the dashed lines are the solutions obtained by Chimonas and Grant (1984).
Figure 16. The modulus of the eigenfunction, $|\Psi|$, plotted against height for the profiles in Fig. 14, for a wavenumber of 0.01 m$^{-1}$.

Convergence became very slow with larger boundary separations but the results were negligibly different from those shown here, indicating that it is unnecessary to consider wider separations. The model used by Chimonas and Grant did not have the provision for moving the boundaries and so they used an asymptotic solution for large wavenumbers. The agreement between the present results and those of Chimonas and Grant provides a check on the two methods. The advantage of the present method, which involves moving the boundary positions, is that it can easily be applied to any profile.

5. APPLICATION TO A REAL ATMOSPHERIC PROFILE

The profile used here was obtained from a radiosonde ascent made at Halley station (75°36'S 26°40'W) in the Antarctic, starting at 1139 GMT on 9 April 1985. The measured profiles of wind speed, wind direction and potential temperature (calculated from the pressure and temperature) are shown in Figs. 17(a) and (b). The profile was sampled at about one point every 50 m, up to about 3 km. Since there is significant variation of wind direction with height, it was necessary to make an assessment of the best direction in which to resolve the profile in order to find the fastest growing modes of instability. This was done by plotting contours of $\Omega^{-1}$ (see Eq. (14)) against direction of resolution and height. The minimum possible e-folding time can be found from Howard's (1961) bound on the growth rate (see Eq. (14)) since the e-folding time is $1/(\omega_i)_{\text{max}}$. The contour diagram of the quantity $\Omega^{-1}$ is shown in Fig. 18. It is clear that the potentially fastest growing modes are associated with levels around 1000 m and 2200 m and that resolution along 0° is likely to reveal significant modes of instability. Hence the profile used for the stability analysis was obtained by resolving along a north–south direction. Figure 19(a) shows the resulting velocity profile in the form used for the stability analysis (i.e. the profile obtained after resolving and fitting cubic splines, with knots at every data point,
to the measured data). The Richardson number $N^2/(dU_z/dz)^2$ for the resolved profile is shown in Fig. 19(b). It is clear that there are two levels of sub-critical Richardson number, near 1000 m and 2200 m, each associated with a shear layer.

Figure 17. Radiosonde ascent made at 1139 GMT on 9 April 1985 at Halley Base, Antarctica. (a) The wind speed (full line) and direction (broken line). (b) The potential temperature.

Figure 18. Contour of minimum e-folding time equal to 300 s, calculated from Howard's bound on the growth rate, for the profile in Fig. 17.
Figure 19. (a) The velocity profile obtained from the profile of Fig. 17(a) by resolving along the north-south direction and fitting cubic splines to the resolved profile. The dashed lines show the levels of the sub-critical minima in the Richardson number. (b) The Richardson number profile corresponding to the velocity profile in Fig. 19(a) and a potential temperature profile obtained by fitting cubic splines to that in Fig. 17(b).

Figure 20. (a) The phase speed $c$, and (b) the growth rate $kC$, plotted against wavenumber for the profiles in Fig. 19. The solid line corresponds to the mode generated by the low-Richardson-number layer around 1000 m and the dashed line to the mode generated by the low-Richardson-number layer around 2200 m.
It should be emphasized that this profile has not been smoothed. Sometimes, however, radiosonde profiles show clear evidence of large amplitude wave activity and such profiles should be smoothed if they are to be regarded as horizontally uniform basic states for a stability analysis. An example is given by Lalas and Einaudi (1980).

The stability analysis reveals two unstable modes; Figs. 20(a) and (b) show their phase speeds and growth rates plotted against wavenumber. One mode of instability is associated with the minimum in $\text{Ri}$ near 1000 m and the other is associated with the minimum near 2200 m. Figure 21 shows the eigenfunctions of the unstable modes plotted against height at wavenumbers corresponding to the maximum growth rate. It is interesting to note that both modes of instability have second critical levels at which the Richardson number is high. For the mode centred around 1000 m, this occurs below the minimum in $\text{Ri}$ at around 800 m. For the upper mode, centred near 2200 m, the second critical level occurs above the minimum in $\text{Ri}$ at around 2500 m. In both cases, it can be seen that the eigenfunction tends rapidly to zero at the critical level. At the wavenumbers corresponding to Fig. 21, the normal boundary conditions (i.e. a rigid boundary at $z = 0$ and a radiating condition at the top) were used, since the growth rates are quite large. However, at wavenumbers for which the growth rates were small, rigid boundary conditions were applied near the critical levels at 800 m and 2500 m, as in case (d) above. Without the provision to apply this type of boundary condition, it proves to be impossible to obtain convergence at low growth rates for this type of profile and so the stability curves cannot be traced to low growth rates as has been done in Fig. 20(b).

No further discussion of this particular profile is given here, since it is not the purpose of this paper to discuss particular profiles; rather it is the intention to demonstrate that the method described in section 5 can adequately perform stability analyses on unmodified

![Diagram](image.png)

Figure 21. The moduli of the eigenfunctions, $|\Psi|$, plotted against height for the profiles in Fig. 19. The wavenumbers are 0.026 m$^{-1}$ for the solid line and 0.0205 m$^{-1}$ for the dashed line.
radiosonde data. However, application of the model to other profiles is discussed by King et al. (1987) and by Rees and Mobbs (1988). The latter also describe the use of the model to predict neutral wave modes.

6. CONCLUSIONS

The purpose of this paper is to describe in detail the practical implementation of a scheme which allows the stability of any atmospheric profile to be tested. This has been achieved by extending the method of Lalas and Einaudi (1976). A particularly important extension involves, at the start of the iteration procedure for finding eigenvalues, placing false boundaries close to the minima in the Richardson number. Without this, it is often extremely difficult to obtain sufficiently good 'first guesses' for convergence of the iteration process. The fact that initial convergence is obtained with the boundaries close to a level of minimum Richardson number means that when several unstable Kelvin–Helmholtz modes are present, they can be positively associated with the particular minimum in the Richardson number responsible for their development. Another extension concerns the use of a Frobenius expansion for weakly unstable eigenmodes near their critical levels. The Frobenius method is computationally more efficient than approaches which artificially increase the growth rate of unstable modes for computational stability and offers advantages when several close eigenvalues exist.

The method has been successfully verified using a range of idealized and real atmospheric profiles.

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APPENDIX A

As a critical level is approached, the dominant term in $\lambda$ is $N^2/c^2$, unless of course $N^2$ is very small. Therefore $\lambda$ is proportional to $c^{-2}$. For the 4th-order Runge–Kutta scheme, the error at each step is proportional to

$$ (\Delta z)^5 \frac{d^5 \Psi}{dz^5} \tag{A1} $$

and hence from Eq. (8), the error must be proportional to $|\Delta z|^5 c^{-5} |\Psi|$ as the critical level is approached. Far from the critical level, if the step size $(\Delta z) = h_0$, then the error is proportional to $h_0^5 |\Psi|$. Therefore, in order to maintain accuracy as the critical level is approached, we make

$$ \Delta z \propto |c|. \tag{A2} $$
No adjustment is needed unless $|\dot{c}|$ is small and so a constant step size is maintained unless $|\dot{c}| < C_2$, where $C_2$ is a constant. We have used $C_2 = 1$ m s$^{-1}$ for radiosonde data. If $|\dot{c}| < C_2$, then $\Delta z$ is given by

$$\Delta z = h_0 |\dot{c}| / C_2.$$  \hfill (A3)

If $N$ is very small, then this estimate will be over-conservative. However, $N$ is never very much larger than average values, so that the estimate is unlikely to lead to too large a step length. In order to keep $\Delta z$ within reasonable bounds, a minimum value is chosen. If the required $\Delta z$ falls below this, no solution is attempted. A minimum value of $h_0/1000$ has been found to be adequate; in fact we have not analysed any atmospheric profiles for which this cut-off came into effect, the reason being that modes with very small $c_i$ are rejected using Howard's growth rate criterion and in any case, the Frobenius expansion (15) is usually used if $|\dot{c}|$ is very small.

**APPENDIX B**

Close to the critical level $z = z_c$, the dominant terms in Eq. (9) are those involving $\dot{c}$. Furthermore, the 4th term on the right-hand side of Eq. (9) is usually much smaller than the others involving $\dot{c}$ and this is also neglected. The resulting equation for $\Psi$ is

$$\frac{d^2\Psi}{dz^2} + \left[ \frac{N^2}{\dot{c}^2} + \frac{1}{\dot{c}} \frac{d^2 U_0}{dz^2} \right] \Psi = 0.$$  \hfill (B1)

Following Booker and Bretherton (1967), it is appropriate to transform to a new independent variable $Z$ given by

$$Z = z - z_c - i c_i / U'_0$$  \hfill (B2)

where, by definition, $U'_0 = c_r$ at $z = z_c$ and $U'_0$ is $dU_0/dz$ evaluated at $z = z_c$. Then by expanding $U_0$ in a Taylor series in $Z$, it follows that

$$-\dot{c} = U_0(z_c) - c_r + ZU'_0 + \frac{1}{2}Z^2 U''_0 + \ldots$$  \hfill (B3)

and by elementary manipulation the equation for $\Psi$ becomes

$$\frac{d^2\Psi}{dz^2} + \left[ \frac{RI}{Z^2} \left( \frac{U''_0}{U'_0} (1 + RI) \frac{1}{Z} \right) \right] \Psi = 0$$  \hfill (B4)

where, inside the square brackets, terms involving $Z^0$ and higher powers have been neglected. $RI$ is the Richardson number evaluated at $z = z_c$, i.e. $N^2/U'_0^2$. Equation (B4) can be solved using the method of Frobenius. The solutions are

$$\Psi = Z^\alpha + \frac{U''_0 (1 + RI)}{2\alpha U'_0} Z^{1+\alpha} + O(Z^{2+\alpha})$$  \hfill (B5)

where $\alpha$ is either of the roots of

$$\alpha^2 - \alpha + RI = 0.$$  \hfill (B6)

The solution given in Eq. (15) results from taking only the first term in Eq. (B5). It is invalid if the roots of Eq. (B6) are equal or differ by an integer. The two cases are:

(i) $RI = 0$ and the roots are $\alpha_1 = 1$ and $\alpha_2 = 0$. In this case, the first term in (B5) is still the first term of the solution series but later terms are different. Since only one term is used, no particular action is needed but, as indicated below, the range of values of $z$ over which the expansion is valid can become very small.
(ii) \( Ri = 1/4 \) and \( \alpha_1 = \alpha_2 = 1/2 \). An alternative solution involving \( \ln Z \) can be found in this case. However, for simplicity, this is not done since it is rare that \( Ri \) is very close to \( 1/4 \) and such profiles are not generally unstable. Therefore, if \(|Ri - 1/4| < 0.001\), the Runge–Kutta method is used to integrate through the critical level. It is possible then that no solution will be found, since the required step size may fall below a pre-determined minimum, as described in appendix A.

In order that the retention of only the leading term in Eq. (B5) should be a good approximation, it is necessary that

\[
|U_0''(1 + Ri)Z/2\alpha U_0'| < 1
\]

which upon rearrangement becomes

\[
|z - z_c| \leq d
\]

where

\[
d^2 = \frac{4U_0'^2 \left[ \frac{1}{4} - \left(\frac{1}{4} - Ri\right)^{1/2} \right]^2}{U_0'^2(1 + Ri)^2} - \frac{c_i^2}{U_0'^2}.
\]

This estimate is not valid when \( Ri \) is close to \(-1\); however, such mean profiles are unlikely to occur in practice.

The practical application of the above theory proceeds as follows. If \( c_i \) is greater than a critical value \( C_1 \), then the Runge–Kutta method is used to integrate through the critical level. We have typically used \( C_1 = 0.1 \) m s\(^{-1}\). If \( c_i \) is less than \( C_1 \), then the possibility of using the Frobenius expansion is investigated. Firstly, if \(|Ri - \frac{1}{4}| < 0.001\), then the Frobenius method is rejected and the integration proceeds using the Runge–Kutta method. Otherwise \( d \) is evaluated using Eq. (B9). If \( d > h_0/100 \) (where \( h_0 \) is the

![Flow diagram](image)

**Figure B1.** Flow diagram illustrating the practical application of critical level theory in the stability model.

\( C_1 \) = Value of \( |c| \) below which Frobenius is checked, e.g. \( C_1 = 0.1 \).

\( C_2 \) = Value of \( |c| \) at which reduction of step size starts, e.g. \( C_2 = 1.0 \).
vertical grid spacing far from the critical level, then the Frobenius solution is used from
\( z = z_c - d/10 \) to \( z_c + d/10 \). If \( d < h_c/100 \) or if \( d^2 \) in Eq. (B9) is negative (i.e. \( d \) is not
defined), the integration proceeds using the Runge–Kutta method. The method, including
the precautions taken to ensure that the grid spacing does not become too small, is
summarized as a flow diagram in Fig. B1.

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