On the incorporation of atmospheric boundary layer effects into a balanced model

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SUMMARY

A method of coupling the thermal and frictional forcing in the atmospheric boundary layer to a semi-geostrophic model of the internal dynamics is developed. It is illustrated by two-dimensional sea-breeze simulations, which are compared with those of a two-dimensional primitive equation model. It is demonstrated that the main features of the circulation on a horizontal scale larger than 20 km can be simulated with the semi-geostrophic model, but the detailed local circulations cannot.

1. INTRODUCTION

As the resolution of numerical models of the atmosphere and the sophistication of the representation of physical processes in them increase, understanding of the interactions between explicitly represented dynamics and parametrized subgrid-scale processes becomes more important. In particular, it is necessary to ensure a correct coupling of parametrized processes with the balanced dynamics most important for weather forecasting. In this paper we study some aspects of this interaction.

The definition of balanced dynamics used here is described in a recent paper, Cullen et al. (1987), henceforth referred to as C. This is based on an energy principle. It is equivalent to Lagrangian semi-geostrophic theory away from the equator, and reduces to an evolving static solution at the equator. It can describe a variety of flows driven by diabatic effects. Shutts (1987) has shown that several important aspects of penetrative slantwise convection are described by the model. In this paper we examine the interaction of the atmospheric boundary layer with the geostrophic dynamics in a simple sea-breeze model. Such a solution was included in C, neglecting frictional effects. It gives a reasonable estimate of the inland penetration of the sea breeze, including its dependence on latitude. Chynoweth (1987) has extended this solution to much higher resolution and included the effects of imposed gradient winds.

The interaction between semi-geostrophic theory and boundary layer structure has been studied in some detail by Wu and Blumen (1982). They assume that the geostrophic flow is independent of height within the boundary layer, and derive an Ekman solution that matches the total semi-geostrophic flow at the boundary layer top. They considered that the main limitation of their work in explaining observed behaviour was the need to assume a constant eddy viscosity in order to obtain analytic solutions. Young (1973) also developed an approximation to the boundary layer equations which matches semi-geostrophic theory in the free atmosphere. He defines a low-order solution for the velocity which satisfies geostrophic balance outside the boundary layer and Ekman balance within it, and then approximates the momentum in the full equations by its lowest-order value. Neither of these solutions includes the thermal forcing.

In this paper both thermal and frictional effects are considered and the assumption of constant eddy viscosity is not made. Numerical rather than analytic solutions are therefore obtained. Observed sea-breeze behaviour suggests that the thermal and frictional effects on the wind are often of the same order of magnitude. Under these conditions the Ekman solution may not be a good first approximation. The approximate
equations are therefore derived directly from the two-dimensional primitive equations by neglecting the acceleration component normal to the coast. This produces a quasi-balanced sea-breeze response and gives the normal semi-geostrophic equations above the boundary layer. Yuen and Young (1986) carry out a detailed scale analysis which suggests that this approximation is reasonable if the boundary layer is deep, the capping inversion strong, and the sea breeze itself light.

Three numerical models are used in this paper. All use the same formulation for the lower boundary drag and surface fluxes, and the same turbulence parametrization to spread the effects in the vertical. These schemes are taken from the Meteorological Office operational forecast model, as described by Bell and Dickinson (1987). The first solves the approximate equations by a Lagrangian method which allows the treatment of discontinuous solutions. This allows the dynamical equations to be solved exactly in the absence of friction for piecewise constant data. However, against this advantage is the disadvantage that only the dependence of the friction on the geostrophic wind can be included in the solution.

The second model solves the approximate equations by a finite difference method. This allows all the friction to be included, but cannot treat the rest of the dynamics as accurately as the Lagrangian method. The third model solves the full two-dimensional primitive equations, using the same vertical and horizontal resolution. This allows the effect of neglecting the acceleration normal to the coast to be assessed. It also allows the results to be compared with other published sea-breeze simulations, such as the recent study by Savijärvi and Alestalo (1988). It should be noted that most simulations directed specifically at sea breezes use more sophisticated primitive equation models, usually with vertical extent limited to the lowest few kilometres of the atmosphere.

The resulting models are applied to a simple sea-breeze simulation. Under the conditions described by Yuen and Young the semi-geostrophic models should be able to describe the inland propagation of the sea breeze. They should be able to describe the dependence on the large-scale geostrophic wind which is known to determine the occurrence, strength and penetration of the front (Atkinson 1981; Houghton 1984). This requires a description of the balance between thermal and frictional effects at a coastline, which determines whether it is a zone of low-level convergence or divergence. The semi-geostrophic models cannot, however, be expected to describe the dynamics of the sea-breeze front itself, which appears to behave as a gravity current (Simpson et al. 1977). Comparisons with the primitive equation model should show the practical effect of this deficiency.

2. THEORETICAL DEVELOPMENT

The primitive Boussinesq hydrostatic equations for flow in an \((x, z)\) cross-section are written using the modified pressure coordinate of Hoskins and Bretherton (1972) with boundary layer forcing terms added:

\[
\begin{align*}
Du/Dt + \partial \phi/\partial x - g v &= F_u \\
Dv/Dt + f(u - u_o) &= F_v \\
\partial \phi/\partial z - g \theta/\theta_o &= 0 \\
D\theta/Dt &= H \\
\partial u/\partial x + \partial w/\partial z &= 0
\end{align*}
\]
\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \]

\[ u = u_0 \quad \text{at} \quad x = \pm L \]

\[ w = 0 \quad \text{at} \quad z = 0, H. \]

\( u \) and \( v \) are velocity components, \( \theta \) is the potential temperature with reference value \( \theta_0 \) and \( \phi \) the geopotential, \( f \) is the Coriolis parameter and \( g \) the acceleration due to gravity. The only \( y \) variation is an assumed basic state pressure gradient in geostrophic balance with a wind component \( u_0 \) which is independent of \( x \) and \( z \). \( F_u, F_v \) and \( H \) represent the frictional and thermal forcing terms. The boundary conditions are \( w = 0 \) at \( z = 0, H \); \( u = v = 0 \) at \( z = 0 \); and \( (u, v) \) fixed at \( x = \pm L \).

Hoskins and Bretherton show, by considering a coordinate frame moving with a front, that the two-dimensionality allows \( Du/Dt \) to be neglected compared with the other terms in (1). This scale analysis is unlikely to apply to strong sea-breeze flows because the values of \( u \) relative to the front may be large and \( v \) may initially be small or zero. However, the neglect of \( Du/Dt \) may also be valid if there is a balance between the pressure gradient and friction terms. It is also valid if the Ekman balance is valid. Yuen and Young’s ‘quasi-geostrophic’ response to a cold air outflow is obtained if \( Du/Dt \) is neglected. Since we wish to consider both thermal and frictional boundary layer effects, the Ekman solution, whose vertical structure is dynamically unstable, may not be accurate. We therefore do not use the system of equations obtained by Young (1973), which use the Ekman solution as a first approximation.

In this paper we therefore neglect the term \( Du/Dt \) in (1) and make no other approximations. In particular the term \( Du/Dt \) in (2) is not approximated. The resulting system is

\[ \frac{\partial \phi}{\partial x} - f v = F_u \]

(9)

together with (2) to (8).

We now derive the energy equation for this system. To be consistent with the neglect of \( Du/Dt \), the kinetic energy must be approximated by that of the \( v \) component. Thus we define the energy of a fluid parcel \( E \) as

\[ E = \frac{1}{2} v^2 - g \theta z / \theta_0. \]

(10)

This obeys the equation

\[ \frac{D E}{Dt} = u F_u + v F_v - g z H / \theta_0 - u \frac{\partial \phi}{\partial x} + f v u_0 - w \frac{\partial \phi}{\partial z}. \]

(11)

The last three terms represent the rate of working by the pressure field. The other terms represent energy changes due to thermal and frictional effects. Friction acts as an energy sink if \( F_u \) and \( F_v \) are specified as drag terms, \( -C_D u \) and \( -C_D v \).

The behaviour of the system can be understood by first considering a linearized ‘shallow water’ form of the equations:

\[ \frac{\partial \phi}{\partial x} - f v = -C_D u \]

(12)

\[ \frac{\partial v}{\partial t} + f (u - u_0) = -C_D v \]

(13)

\[ \frac{\partial \phi}{\partial t} + \phi \frac{\partial u}{\partial x} = H. \]

(14)

Assuming variations proportional to \( \exp(\lambda t + i k x) \), the homogeneous part of this system has two eigenvalues:

\[ \lambda = (-A \pm \sqrt{B})/2C_D \]

(15)
with \( A = C_D^2 + f^2 + k^2 \phi_o \) and \( B = A^2 - 4k^2 \phi_o C_D^2 \). Both have negative real parts and zero imaginary parts representing damped non-oscillatory solutions. The eigenvectors can be written
\[
(u, v, \phi) = (-\lambda, f\lambda/(C_D + \lambda), \phi, ik).
\]

Now consider the solution of these equations in a simple sea-breeze configuration. The basic state is a uniform geostrophic flow, and perturbations are forced by the differing frictional effects over land and sea and by heating of the lower levels over the land. The sea breeze is an ageostrophic response to the thermal forcing. The associated vertical gradient of thermal wind and the ageostrophic flow are almost entirely confined to the boundary layer.

Consider first a frictionless flow with only thermal forcing. One of the eigensolutions disappears in this case. Equation (9) shows that \( v \) is geostrophic and Eq. (2) shows that \( (v + fx) \) is conserved following the motion. A simple solution is shown in C. It is represented in terms of a small number of finite elements on each of which \( (v + fx) \) and \( \theta \) are constant. Elements in contact with the lower boundary over land are heated, and the heat spread evenly through the element in question. These solutions were developed by Chynoweth (1987) to include a much larger number of elements. These show, for instance, the evolution of a well-mixed convective boundary layer over the land, the formation of a sea-breeze front and its propagation inland, and the formation of a much stronger front if there is a basic state offshore wind.

Now consider the converse case with no thermal forcing. The equations are slightly more general than those treated by Young (1973), in that \( v \) is not replaced by the \( v \) implied by Ekman balance in the \( Du/Dt \) term. Though his simple analytic solution is no longer possible, the frictional isallobaric wind that he describes should still be represented. The behaviour of the solution can be deduced from the eigenvectors and eigenvalues (15).

In the case \( f \ll C_D \), which is the normal situation, the eigenvalues are approximately \(-C_D\) and \(-k^2 \phi_o/C_D\). In the limit \( f \to 0 \) the eigenfunctions are respectively \((0, 1, 0)\) and \((k^2 \phi_o, 0, ik \phi_c C_D)\). The first represents spin-down of the \( v \) component of the wind, the second represents spin-down of the pressure gradient. The spin-down of the pressure gradient is slower if the drag coefficient is large because less wind is needed for friction to balance the pressure gradient. Thus, though the cross-isobar angle of the flow is increased, the cross-isobar mass transfer is decreased because the wind speed is less. If \( f \) is small but \( f \ll C_D \), the different spin-down rates for pressure gradient and wind lead to a geostrophic departure which is balanced by the friction term.

In the case \( k^2 \phi_o \ll f^2 \), which requires very large horizontal scales or very small vertical scales, the eigenvalues are approximately \(-C_D^2/(C_D^2 + f^2)\) and \(-k^2 \phi_c C_D/(C_D^2 + f^2)\). The first represents the establishment of the Ekman layer and the second the independence of the basic pressure gradient from the boundary layer wind field. Finally, if \( C_D^2 \ll f^2 + k^2 \phi_o \), there is one very large and one very small eigenvalue. The first eigenvalue tends to infinity as \( C_D \) tends to zero, indicating that the frictionless limit is singular. Physically it represents rapid adjustment to semi-geostrophic balance. The second eigenvalue represents the evolution of the balanced solution. The presence of the large eigenvalue indicates that implicit numerical methods will be required to solve the system. These will ensure that numerical solutions for small \( C_D \) will be close to those for zero \( C_D \). A similar difficulty in the nonlinear balance equations was pointed out by Moura (1976).
3. METHODS OF NUMERICAL SOLUTION

(a) Equations for numerical solution

In order to derive finite difference solutions in which the lower boundary is a coordinate surface, we use $\sigma$ coordinates in the vertical, rather than the $z$ coordinate that is most convenient for theoretical studies. The model includes the full depth of the atmosphere with an upper boundary at zero pressure. The equations for the basic state geostrophic flow, defined by quantities $(u_o, v_o, \phi_o)$, are

$$\frac{\partial \phi_o}{\partial x} + c_p \sigma^k \theta_o \frac{\partial \pi_o}{\partial x} - f v_o = 0 \tag{16}$$
$$\frac{\partial \phi_o}{\partial y} + c_p \sigma^k \theta_o \frac{\partial \pi_o}{\partial y} + f u_o = 0 \tag{17}$$

where $\pi_o$ is the basic state Exner function $(p_*^*/p_o)^{\kappa}$, $p_o$ is the surface pressure, $p_*$ is a reference pressure and $\kappa = R/c_p$. In the equation for the perturbed flow we assume that the surface pressure variation implied by the basic state is sufficiently small for $\partial \pi/\partial x$ to be expanded as $\partial \pi_o/\partial x + \partial \pi^*/\partial x$. The equations for the perturbation from the basic state are then, using unprimed variables to represent perturbation quantities,

$$\frac{\partial \phi}{\partial x} + c_p \sigma^k \theta \frac{\partial \pi}{\partial x} - f v = F_u \tag{18}$$
$$\frac{\partial \phi}{\partial \sigma} = -RT/\sigma \tag{19}$$
$$T = \theta(\sigma^k \pi) \tag{20}$$
$$Dv/Dt - fu = F_v \tag{21}$$
$$D\theta/ Dt = H \tag{22}$$
$$\frac{\partial p_*}{\partial t} + \frac{\partial}{\partial x}[p_* + p_0] - \frac{\partial}{\partial x}[p_* + p_0] = 0 \tag{23}$$
$$\frac{\partial}{\partial x}(p_* + p_0)(u - \bar{u}) + (u_o - \bar{u}_o)\frac{\partial p_*}{\partial x} + (v - \bar{v})\frac{\partial p_*}{\partial y} + \frac{\partial}{\partial \sigma}(p_* + p_0)\bar{\delta} = 0. \tag{24}$$

The tilde in these equations represents a vertical mean.

The finite difference models solve the equations in an $(x, \sigma)$ cross-section at right-angles to the coastline. The basic domain used for output extends 100 km on either side of the coast. In the primitive equation integrations the computational domain was extended to 200 km either side of the coast to remove lateral boundary effects, and in the semi-geostrophic integrations to 160 km either side. The horizontal gridlength is 4 km, with no temperature gridpoint actually on the coast. The geometrical model uses an $(x, z)$ cross-section with surface pressure calculated diagnostically.

(b) Finite difference grid and boundary layer parametrization

In the finite difference models the temperatures and horizontal velocities are held at different levels, the 'Charney-Phillips' grid. This is necessary for satisfactory integration of the semi-geostrophic equations (Cullen 1989), and is used for both models to ensure comparability of layer depths. These models are written using $\sigma (= p/p_*)$ as vertical coordinate and 11 layers, with boundaries at $\sigma = 1, 0.975, 0.9, 0.79, 0.65, 0.51, 0.37, 0.27, 0.195, 0.125, 0.06, 0$. Temperatures and vertical velocities are stored at layer boundaries and winds at layer midpoints. In the geometric model the initial data are defined to give four isentropic layers of depth 1600 m above two lower layers of thickness 800 m. The depth of the model is fixed at 8000 m.

This distribution of levels is typical of that used in operational forecast models, though many can now use more levels than this. It is thus appropriate for a study of the
interaction of boundary layer dynamics with the rest of the atmosphere. Most sea-breeze studies use very much higher vertical resolution and models which are confined to the lowest 2–4 km of the atmosphere.

The surface exchanges are parametrized in all the models, including the geometrical model, by the method used in the Meteorological Office operational forecast model (Bell and Dickinson 1987). This defines fluxes:

\[ F_u = -C_D |v| u \]  
\[ F_v = -C_D |v| v \]  
\[ F_\theta = -C_H |v| (\theta - \theta_w) \]

where \( u, v, v \) and \( \theta \) refer to bottom layer values, and \( \theta_w \) is the surface potential temperature. The drag and exchange coefficients are determined from the Richardson number in the bottom layer. In the geometrical model the layer thickness used in the calculation is half the depth of the bottom element. In the finite difference models the estimates of vertical gradients allow for the fact that velocities are stored half a level above the surface and temperatures a full level.

In the geometric model the fluxes are assumed to spread only through elements in contact with the ground. In the remainder of the boundary layer in the finite difference models, the fluxes take the form \( F_x = -K_x \partial X/\partial z \) where \( K_x \) depends on the diagnosed boundary layer depth, the Richardson number, and the velocity gradient. The Richardson number is calculated at layer midpoints and then interpolated as necessary. The definitions of the coefficients are given by Bell and Dickinson. However, to allow the scheme to be used with a long time step, the fluxes are added on using an implicit method.

The models are driven by incrementing the surface temperature with a heating function which varies sinusoidally in time. The surface potential temperature then obeys the equation

\[ \partial \theta_w/\partial t = A \sin(2\pi t/86400) + C_H |v| (\theta - \theta_w) \]

where \( t \) is the time in seconds, \( \theta \) and \( v \) refer again to bottom layer values, and \( A \) is a scaling factor. Values of \( A \) of 1 and 2.5 K h\(^{-1}\) were used to give weak and strong heating rates.

(c) Geometric model

In this method the atmospheric cross-section is represented by finite elements on each of which \( \theta \) and \( (v_g + f x) \) are constant. The method is completely Lagrangian, and at each time step the elements are arranged to satisfy the geostrophic and hydrostatic equations. The method is described in detail by Chynoweth (1987) and is summarized in C.

The thermal forcing can easily be incorporated in this model. The length of each element in contact with land or sea surface is calculated, and the appropriate heat input is converted into a temperature increment. The momentum flux is calculated in the same way. The component in the \( y \) direction is added onto \( v_g \). However, it is not possible to include the friction in the \( x \) direction in this method. This is because the construction of the element configuration uses values of \( \theta \) and \( (v_g + f x) \) which are assumed to satisfy static and inertial stability. In the presence of friction, \( v_g \) may be large but the friction rather than the term \( f v \) may balance the pressure gradient. There is then no need for \( v_g \) to satisfy an inertial stability condition, and the construction cannot be uniquely carried out.
The results from this model are essentially exact for piecewise constant data, given the neglect of the $F_u$ term. They therefore act as a valuable check on the finite difference results, given the absence of analytic solutions to the complete equations. The value of $F_u$ can be diagnosed and checked against the actual differences between the finite difference and geometric results. It is not at present practicable to run the model with as large a number of elements as the number of gridpoints in the finite difference models. Given this, the different method of representing the data leads to difficulties in obtaining output comparable to that of the finite difference models. Wind components can be inferred from the movement of element centroids and interpolated to a grid for plotting. The results, however, then appear excessively smooth. Only plots of element positions, from which the cross-coastal wind components can be inferred, and the $\theta$ field, which is smoother than the wind fields, are therefore included in the results section.

(d) Finite difference model with semi-geostrophic dynamics in interior

Equations (16) to (24) are approximated by second-order centred finite differences using a staggered grid in which $\theta$ and $\tilde{\theta}$ are held at points on layer boundaries, and $u$ and $v$ at layer midpoints displaced a half grid length in the horizontal. The horizontal staggering corresponds to the Arakawa B grid. The system of equations is implicit and solved by a predictor-corrector algorithm described in more detail by Cullen (1989). Equations (23), (21) and (22) are first advanced in time through a time step $\delta t$ to give provisional values of $p^*, v$ and $\theta$ at time $t + \delta t$, denoted by a superscript $\#$. The calculation uses values of $u$ and $\tilde{\theta}$ from the previous timestep:

\[
\begin{align*}
    p^* - p^\# &= -\delta_x \tilde{U} - \tilde{u}_a \delta_y \tilde{p} - \tilde{v} \partial p/\partial y \\
    \tilde{p}^\#(v^* - v\#) &= -fU - (U + U_0)\delta_x \tilde{\theta} - \tilde{S} \delta_y \tilde{\theta} \\
    p^\#(\theta^* - \theta\#) &= -(U + U_0)\delta_x \tilde{\theta} - \tilde{S} \delta_y \tilde{\theta}
\end{align*}
\]

where

\[
U = p\tilde{u}, \quad S = p\tilde{\theta}.
\]

Standard finite difference averaging notation has been used. The boundary layer increments are then added to $\nu$ and $\theta$ as described in section (b) above. In addition, the matrix coefficients

\[
C_{ij} = \partial F_{ui}/\partial u_j
\]

where the subscripts refer to vertical levels, are calculated. The next stage is to update $U$ and $S$ to time $(t + \Delta t)$ by enforcing (18) and (19). The equation that has to be solved for these variables is elliptic, provided that the potential vorticity at time $t$ is positive. This condition is frequently difficult to satisfy in the test problem because of the creation of a well-mixed boundary layer. It was shown in Cullen (1989) that a stable numerical solution could be obtained by:

(a) Modifying the $\theta$ field to ensure static stability.

(b) Modifying the $\nu$ field to ensure inertial stability in the absence of friction. In the presence of friction the modification need only enforce the weaker condition $\partial/\partial x((\nu + f)x) - C_{ij} > 0$ at level $i$, where $C_{ij}$ is as defined in (33).

(c) Calculating $U$ and $S$ from $U$ using a reduced version of the cross-front circulation equation which is elliptic provided only that the data are statically and inertially stable.

(d) Updating $p^*$, $\theta$ and $\nu$, and iterating to convergence.
This procedure enhances numerical stability by ensuring that the corrections to \( U \) and \( S \) are usually underestimated at each iteration. Each calculation of \( U \) and \( S \) is elliptic if the data are statically and inertially stable, though the iteration may diverge if the potential vorticity is negative. No algorithm for enforcing positive potential vorticity on the fields was found which did not have unacceptable side effects. Such an algorithm could be used instead of (a) and (b) and would probably improve the performance of the scheme.

In this model the procedure set out above is implemented by using vertical and horizontal convective adjustment schemes successively, and iterating them to ensure that both (a) and (b) are satisfied. Corrections \( \Delta U \) and \( \Delta S \) to \( U \) and \( S \) are written in terms of the vertical mean of \( \Delta U \) and a streamfunction:

\[
\Delta U = \Delta \bar{U} - \partial \psi / \partial \sigma \tag{34}
\]

\[
\Delta S = \partial \psi / \partial x. \tag{35}
\]

Equations (18) and (19) are then combined by eliminating \( \phi \) and the residuals calculated using the values of \( p, \theta \) and \( \nu \) denoted with a \# in (29) to (31):

\[
R_\# = \delta_x \phi_\# + c_p \delta_x \pi - f \delta_\sigma \psi_\# - F_\nu \tag{36}
\]

\[
R = k c_p \sigma^{x-1} \pi \delta_x \theta - c_p \sigma^{x-1} \delta_\sigma \delta_x \pi + f \delta_\sigma \nu = \delta_\sigma F_\nu. \tag{37}
\]

The subscript \# denotes surface values throughout. These residuals must be eliminated by correcting \( \pi, \theta \) and \( \nu \). The corrections are derived in terms of corrections to \( U \) and \( S \) by using (29) to (31):

\[
k c_p \delta_x \pi \delta_x \bar{U} - f^2 \delta_\sigma (\Delta \bar{U} - \delta_\sigma \psi)_\# + C_\# (\Delta \bar{U} - \delta_\sigma \psi)_\# = R_\#
\]

\[-k c_p \sigma^{x-1} \pi \delta_x (\delta_\sigma \delta_x \psi) + f^2 \delta_\sigma (p_\# \delta_\sigma \psi) + \delta_\sigma C_\# (\Delta \bar{U} - \delta_\sigma \psi)_\# = R. \tag{38}
\]

Note that after deriving these equations from (30) and (31), the cross-derivative terms \( \delta_x \delta_\sigma \psi \) have been omitted to enhance the numerical stability as described above. The boundary conditions were \( \psi \) constant on upper and lower boundaries and \( \partial \psi / \partial x \) and \( \partial \bar{U} / \partial x \) zero on lateral boundaries.

The implicit equations were solved directly by a block tridiagonal algorithm. The corrections to \( U \) and \( S \) are substituted back into (29) to (31) and iterated. It was found best to use the longest time step that allowed the advection to be treated accurately and to iterate the corrections. Satisfactory results could then be obtained for the test problems. However, the need to satisfy the ellipticity conditions prevented a satisfactory solution being obtained if the friction term in the \( x \) direction was omitted and the heating included. This meant that a direct test against the geometric model was not possible. Tests of this algorithm against the geometric method in other problems have indicated that it gives satisfactory results (Cullen 1989).

(e) **Primitive equation model**

In this model Eq. (18) is replaced by

\[
Du/Dt + \partial \phi / \partial x + c_p \sigma^x \theta \partial \pi / \partial x - f \nu = F_\nu. \tag{40}
\]

Equations (19) to (24) are solved as before. The vertical finite differencing is the same as in the semi-geostrophic model. There are differences in the horizontal averaging to ensure numerical stability. Explicit leapfrog time integration was used, with a 6-second time step and a weak time filter. A splitting approach was used to represent the physical processes. The boundary layer parametrization was called every 15 minutes together with
a convective adjustment scheme to remove instability in the vertical. Second-order horizontal and vertical diffusion was added to the equations, with a coefficient increased towards the lateral and upper boundaries to reduce wave reflections. At the lateral boundaries \( u \) was specified and \( \partial u / \partial x \) and \( \partial \theta / \partial x \) were set to zero. \( \partial \rho_c / \partial x \) was specified as the basic state value derived from (16).

4. NUMERICAL RESULTS

(a) Design of experiments

Two types of experiment are illustrated. In the first, the surface temperature is initially the same, 290 K, over land and sea. A heating coefficient \( A \) in equation (28) of 2.5 K/h is used. Figure 1 shows time traces of surface potential temperature at land gridpoints 10 and 50 km from the coast. In the second series of experiments gradient winds of 5 m s\(^{-1}\) are imposed in various directions.

![Figure 1](image_url) Time evolution of surface and 300 m potential temperature for land gridpoints at (A) 50 km and (B) 10 km from coast.

(b) Integrations with no basic state wind

The typical observed behaviour is discussed in Atkinson (1981), and that over the water immediately offshore by Houghton (1984). It is for a sea-breeze circulation to develop which increases in horizontal scale. The fully developed extent at middle latitudes is of the order 30 km offshore and 80 km inland, though the inland penetration is very variable. The sea breeze propagates as a near-discontinuity in wind speed and direction; the air in front of it is calm. The vertical extent of the sea breeze is about 500 m to 1 km, with a return circulation about twice as deep and half as strong. The sea breeze dies away around sunset at the coast but can accelerate inland in the evening. The nocturnal
land breeze is usually much shallower and weaker. There is a very large scatter of results in the different studies reviewed by Atkinson, emphasizing that the real behaviour is strongly influenced by the details of the coastal topography and the airmass.

The initial distribution of elements in the geometric model is shown in Fig. 2. The elements are numbered so that trajectories can be followed. The solutions after 12 hours, when the heating of the surface stops, are shown in Fig. 3(a). At this time, however, there is still significant upward heat transfer into the atmosphere. The results from the geometric model show the development of a ‘well-mixed’ boundary layer over the land, and the deepening of the layer as illustrated by the distortion of the boundary above the second layer of elements. The air originally at the coastline has penetrated 31 km inland, but the strongest frontogenesis is about 15 km inland where elements originally 30 km apart have come together. The neglect of friction in the $x$ direction means that thermal wind balance in the $y$ direction is enforced throughout the boundary layer. This means that the cold air is pushed well across the coast and the temperature never reaches 292 K near the surface within the domain, as can be seen in Fig. 3(b). The average maximum implied sea breeze over the 12 hours for any element interface is 1.3 m s$^{-1}$. With friction coefficients over land for the bottom element of a typical value of $5 \times 10^{-4}$s$^{-1}$ calculated from the parametrization scheme, the friction term could balance a wind along the coast of about 6 m s$^{-1}$, which would be generated by inland penetration of air of 60 km, about the maximum that occurs in the simulation. A simple calculation suggests that the degree
Figure 3. Solutions after 12 hours with no basic state wind and surface heating coefficient of 2.5 K/h in Eq. (28). \( \theta \) is in kelvin with contour interval 0.5 K and \( u \) and \( v \) in m s\(^{-1}\) with contour interval 1 m s\(^{-1}\). GEOM—geometric model; SG—finite difference semi-geostrophic model; PE—finite difference primitive equation model.
of penetration will be reduced by about 30% by friction. The vertical extent of the circulation is restricted almost entirely to the bottom two layers of elements, 1.6 km deep.

The finite difference results using semi-geostrophic interior dynamics are shown in Figs. 3(c), (d) and (e). The perturbation is confined to the lowest 2.5 km. The temperature contrast is confined closer to the coast with friction balancing the horizontal temperature gradient in the lowest layers. This agrees with the solution expected by modifying the geometric solution to allow for friction, and thus confirms that the finite difference solution is reasonably accurate. The wind near the coast is about 4 m s\(^{-1}\) and has veered to about 135°. A clear sea-breeze front is visible 55 km inland, where the wind suddenly increases from 135° 1.5 m s\(^{-1}\) to 155° 5.5 m s\(^{-1}\) 500 m above the surface. The surface value is reduced by friction.

The primitive equation results are shown in Figs. 3(f), (g) and (h). The \(\theta\) field is not very different, except that the air is warmer over the land ahead of the sea-breeze front. This is because the cold air can only propagate at the speed of a gravity current. The semi-geostrophic model implies higher and unrealistic propagation speeds because of the balance constraints. Waviness in the fields higher up in the atmosphere is visible. The \(v\) field also indicates reduced penetration of the cold air. The sea-breeze front is only 30 km inland and more clearly marked. There also appears to be an upper-level wind discontinuity in the return flow over the sea. The total wind contrast at the front is about 10 m s\(^{-1}\), with zero motion in front, in agreement with observed behaviour. The wind direction behind the front is about 115°. At the coast the wind is 7 m s\(^{-1}\), with direction still 115°. The \(u\) field shows marked internal wave activity with vertical phase lines and wavelength about 4 km. This type of behaviour is not seen in most sea-breeze simulations, such as those reviewed by Atkinson (1981), because the models often only represent the lowest few kilometres of the atmosphere rather than the full depth.

These results suggest that the semi-geostrophic model can describe the basic inland penetration and veering of the sea breeze, but smears out the front because the dynamics that maintain it are not represented. There is thus some wind ahead of the front, rather than calm air, and the front itself is too far inland. The semi-geostrophic model gives a greater wind veer, nearer the typical observed value. This is because it produces a smaller cross-coastal wind component than the primitive equation model and a similar component parallel to the shore. The internal waves generated in the primitive equation model are only just resolved by the vertical grid. They were only weakly affected by increased damping at the top of the model, suggesting that wave reflection is not the main reason for their presence. Expansion of the integration domain laterally also had little effect. It is likely that much higher vertical resolution is needed to model the real internal wave response, while the resolution appears sufficient to model the simpler semi-geostrophic response.

Figures 4 and 5 show time sequences at 6-hour intervals of the cross-coastal wind from the two finite difference models. After 6 hours the main difference is in the horizontal scale of the response. The semi-geostrophic model forces 'instant' balance, giving significant wind 50 km either side of the coast. There is a strong convergence zone at the coast because of the discontinuity in surface drag. In the primitive equation model the significant wind only spreads 30 km and there is no coastal convergence. After 12 hours the disparity in horizontal scale is less. The wind strength has dropped a little in the semi-geostrophic model. While \(C_D\) is large, the semi-geostrophic sea breeze will be proportional to the unbalanced part of the pressure gradient, consistent with the observation that the sea breeze is strongest when the temperature contrast is strongest (Atkinson 1981, p. 162). It has increased a great deal in the primitive equation model.
because the acceleration rather than the wind itself is proportional to the unbalanced pressure gradient. Internal wave motions have become established.

After 18 hours, the semi-geostrophic model shows frontogenesis over the land, 80 km inland. This acceleration occurs as the frictional drag decreases and the solution has to move towards thermal wind balance. This agrees with observed behaviour. A land breeze 300 m deep (the bottom model layer) has developed over the coast. The primitive equation model shows the inland front in a similar place, but stronger. The sea-breeze circulation is still blowing over the coast because it can be removed only by an adverse pressure gradient. This aspect of the simulation can be demonstrated in simple analytic models, such as those of Haurwitz (1947). The 'instant' reversal of the semi-geostrophic circulation is probably more realistic. After 24 hours, the semi-geostrophic model still has an inland front and a land breeze at the coast. The primitive equation model has set up a land breeze, of 5 m s\(^{-1}\) rather than the 1 m s\(^{-1}\) in the semi-geostrophic model. It also occupies the lowest level. Most land-breeze observations give speeds of less than 2 m s\(^{-1}\), but values of 5 m s\(^{-1}\) have been found (Atkinson 1981, pp. 137–8).

After this diurnal cycle, as shown in Fig. 1, the surface temperature is well below that of the lowest model layer because the coupling is almost removed once the lowest layer becomes stable. The initial state for the second day is thus significantly different from the first. After 30 hours, heating of the air is only just beginning. The semi-geostrophic model has essentially no wind, indicating no forcing, and the primitive equation model has maintained its land breeze. After 36 hours, both have developed a sea breeze, but almost entirely over the land. Both are weaker than on the first day.

These sequences indicate the fundamental difference between the models. One diagnoses the circulation from the forcing and the pressure gradient, and the other accelerates it. The observational study of Hsu (1970) suggests that the strength and direction of the circulation is approximately proportional to the pressure gradient. This gives a diurnal cycle closer to that produced in the semi-geostrophic integration.

(c) Results with different basic state winds

Houghton (1984) classifies the effect of a basic state wind on the sea breeze over the sea in terms of whether there is an offshore or onshore component of wind to add to the return circulation, and whether there is coastal convergence or divergence. The picture over land, as found in the various studies referred to by Atkinson (1981), is more complicated. Houghton states that an onshore wind prevents the sea breeze by impeding the return flow. However, the thermal effect may accelerate the wind close to the coast and it is difficult to distinguish the result from a sea breeze. The reduction in speed and change in direction over the land determines whether the coast is a zone of convergence or divergence. This zone may be displaced from the coast by the gradient wind. The sea-breeze convergence and divergence patterns have to fit in with this zone, resulting sometimes in a displacement of the circulation away from the coast.

The basic state gradient winds used were 5 m s\(^{-1}\) in directions 225°, 340°, 160° and 045°. These are chosen to give the four combinations of coastal convergence or divergence and offshore or onshore winds. This corresponds to the classification used in Houghton's book.

The results after 12 hours for a 225° wind and the same heating rate as used in the previous experiments are shown in Fig. 6. The layout of the results is the same as in Fig. 3. This wind direction gives a divergence zone at the coast, which will be displaced offshore. This is consistent with the sea-breeze circulation, so the frictional effect from the basic state will reinforce the thermal effect. Observations show that this is the best
situation for sea breezes, with a strong front developing near the coast. Initially the sea breeze is normal to the coast but veers to about 160° by the end of the day. There is a zone of calm or very light winds between the sea breeze and the gradient wind over the sea, which moves steadily out to sea. The sea breeze over the land is reduced or prevented. The results from the geometrical model, Fig. 6(a), are obtained by moving the coastline relative to the elements. The width of the cross-section is 200 km. The initial position of the coast is 18 km from the right-hand side of the cross-section and the final position 32 km from the left, as indicated in the diagram. The results show stronger frontogenesis than without the basic state wind. Element 60 in the centre is displaced 12 km further than in Fig. 3, though the front is now out to sea. There have also been more element interchanges, with element 78 now to the left of elements 65 and 71. The front is a wind-shift line, with little temperature contrast. The cooler sea air again
penetrates well inland; Fig. 6(b) shows that the surface air temperature is only 291.5 K 30 km inland. As in the case with no basic state wind, this results from the neglect of friction normal to the coast and the resulting need to maintain geostrophy of the wind parallel to the coast.

Results for the semi-geostrophic finite difference model are shown in Figs. 6(c) to (e). The cold air has been confined to the sea, with a well-mixed boundary layer over land extending to within 10 km of the coast and surface air temperatures greater than 293 K. Friction is crucial in allowing this strong thermal gradient close to the coast to be maintained in a balanced solution. The sea-breeze front is about 5 km inland. The wind shifts from 185° 3 m s⁻¹, to 170° 8 m s⁻¹. The strong wind component parallel to the coast in offshore basic state winds was a major feature in the study of Savijärvi and Alestalo (1988). The return circulation results in a veering of the wind to 240° 1.5 km above the
Figure 6. Solutions after 12 hours with basic state wind of 5 m s$^{-1}$ at 225° and heating coefficient of 2.5 K/h. Layout of figure as Fig. 3.
surface out to sea. There is no surface calm zone out to sea, but a gradual change towards the basic wind direction. The stronger front agrees with the geometric model and the observed behaviour. The breeze is only a little stronger than without the basic state wind.

The primitive equation results are shown in Figs. 6(f) to (h). The effect of the basic state wind on the $\theta$ field is similar to the semi-geostrophic model. The sea breeze reaches 13 m s$^{-1}$ at direction 160°, about 40% stronger than with no basic state wind. The return circulation is about 3 km deep and the upward propagating wave has phase lines which tilt upstream. As in the semi-geostrophic case there is no calm zone, and the wind veers even more markedly at upper levels offshore. The region over the sea with an onshore wind component is about 60 km wide rather than the 10 km in the semi-geostrophic model. This is closer to the observed behaviour, and probably reflects the advection of the sea-breeze circulation by the basic state wind, which is neglected in the semi-geostrophic models.

Figure 7 shows the results for direction 340°. This gives a coastal convergence zone which will be displaced offshore, impeding the sea-breeze circulation. The observed behaviour is for the sea breeze to be displaced offshore with the convergence zone on the landward side of the breeze matching the convergence caused by the friction difference. With this basic state wind, the frictional contrast between land and sea causes a difference of 0.5 m s$^{-1}$ in the cross-coastal wind component giving weak convergence at the coast. If the same heating as in the previous experiment is used, the thermal effect swamps this frictional convergence. A reduced heating coefficient of 1 K/h is thus used in Eq. (28).

In the geometric model there is marked frontogenesis about 10 km offshore. The semi-geostrophic finite difference model gives a sea breeze of 2.5 m s$^{-1}$ at the coast with direction 020°. Just inland the wind is backed to 270°, further inland it settles to 315°. The change in direction out to sea is more gradual. The typical observed behaviour would require a wind of 160° over the sea, with the convergence zone displaced out from the coastline. The results in this case are, however, extremely sensitive to the heating rate. With the heating coefficient of 2.5 K/h a wind of more than 10 m s$^{-1}$ was generated at the coast, which completely swamped the effect of the frictional convergence. The primitive equation model results show much less internal wave generation than the previous cases, presumably because of the reduced forcing. A sea breeze of 6 m s$^{-1}$ in direction 080° develops with a maximum value 10 km inland. There is a strong sea-breeze front. This direction is much closer to that described by Houghton (1984) but the displacement of the sea-breeze circulation out to sea is not modelled.

Figure 8 shows the results for direction 160°. In this case the coast is a zone of divergence, which will be displaced inland. This again impedes the sea-breeze circulation. In order to demonstrate the effect, the smaller heating coefficient of 1 K/h is employed. The observed behaviour is that the true sea-breeze overturning circulation is prevented, but that the thermal effect strengthens the onshore component of the wind. This tends to cancel out the speed reduction which would otherwise occur due to coastal divergence. Most studies reviewed by Atkinson (1981) suggest that the sea-breeze front is weakened or prevented.

In the geometric results all the element rearrangement occurs over the land. There has been a considerable overturning of the elements, in particular note the final position of elements 35 and 54. This displacement fits in with the coastal divergence zone, the corresponding effect for wind direction 340° was not captured in the results shown in Fig. 7. In the finite difference semi-geostrophic model the wind over the sea is increased to 6.5 m s$^{-1}$ with little change in direction. The only evidence of a sea-breeze circulation is over the land, as in the geometric model. The perturbation winds are only 0.5 m s$^{-1}$ and
Figure 7. As Fig. 6 with basic state wind direction 340° and heating coefficient of 1 K/h.
Figure 8. As Fig. 6 with basic state wind direction $160^\circ$ and heating coefficient of 1 K/h.
are not visible in Fig. 8. The inland displacement of the circulation is clear in the primitive equation model results. This model also produces a wind out to sea increased to 7 m s\(^{-1}\) with direction veered to 170°. Since this wind is up-gradient, it must indicate an unbalanced inertial response.

Figure 9 shows the results for direction 045°. The coast is now a convergence zone, which is displaced inland, helping the sea-breeze circulation. However, the onshore component of the basic state wind impedes the return circulation and the development of the temperature contrast. The larger heating coefficient of 2.5 K/h is employed.

In the geometric model results there is more of a circulation than in Fig. 8, but little frontogenesis. The semi-geostrophic finite difference model shows an enhanced onshore wind of 6 m s\(^{-1}\) at the coast. This seems incorrect in this case, and a similar ‘bulls-eye’ is visible in the earlier cases very close to the coast. It appears to result from the imposition of exact balance between gridpoints only 4 km apart across a discontinuity in forcing. It does not happen in the geometric model where elements are continuously moved through the discontinuity. The rest of the solution shows a strong overturning circulation inland. This veers the wind to 090° up to 50 km inland, overcoming the frictional backing. The primitive equation model shows a strong front 30 km inland with calm air at the surface in front of it and a wind of 090° 9 m s\(^{-1}\) behind it. Over the sea the wind is little disturbed and gradually reaches a value of 040° 4 m s\(^{-1}\).

Overall, these experiments show that the effect of the basic state wind on the strength of the sea-breeze front, and on the sea-breeze velocity, can be qualitatively captured by the semi-geostrophic model. However, the displacement of the circulation and of the coastal convergence or divergence in the direction normal to the coast do not seem to be reliably captured. These require the inclusion of the advection of the ageostrophic circulation by the basic state wind. The primitive equation model includes this effect. In general it gives much stronger responses than the semi-geostrophic model, both in the sea-breeze circulation and in the generation of additional motions such as internal waves.

5. THREE-DIMENSIONAL MODEL

Though only two-dimensional calculations are performed in this paper, it is important to consider how the approximations used could be made in a three-dimensional model. Hoskins (1975) derived the three-dimensional semi-geostrophic equations by seeking a system which would reduce to his original two-dimensional form in appropriate axes if the pressure gradient was in one direction only. The equations are valid only if the flow is approximately two-dimensional so that the trajectories are nearly straight. Applying the same philosophy in the presence of friction gives equations

\[
\frac{\partial \phi}{\partial x} - fv_0 = F_{u_1} \tag{41}
\]
\[
\frac{\partial \phi}{\partial y} + fu_0 = F_{v_1} \tag{42}
\]
\[
\frac{\partial u_0}{\partial t} + \frac{\partial \phi}{\partial x} - fu_1 = F_{u_0} \tag{43}
\]
\[
\frac{\partial v_0}{\partial t} + \frac{\partial \phi}{\partial y} + fu_1 = F_{v_0} \tag{44}
\]
\[
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{45}
\]
\[
\frac{D}{\partial t} = \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \tag{46}
\]

together with (3) and (4). The subscript zero represents a first approximation to \( u \), playing the role of the geostrophic wind in the frictionless case, and subscript 1 represents a
Figure 9. As Fig. 6 with basic state wind direction 045° and heating coefficient of 2.5 K/h.
more accurate approximation. This system is not simply that which would be obtained by using the Ekman velocity instead of the geostrophic wind in the geostrophic momentum approximation. The reason can be seen by considering the energy equation. Representing the friction terms by drag terms \(-C_D u_i\), \(-C_D v_i\), as in (12) and (13), we have

\[
E = \frac{1}{2}(u_0^2 + v_0^2) - g\frac{\theta z}{\theta_0} \tag{47}
\]

\[
\frac{DE}{Dt} = u_0(-C_D u_0 + fu_1 - \partial\phi/\partial x) + v_0(-C_D v_0 - fu_1 - \partial\phi/\partial y) - w \partial\phi/\partial z
\]

\[
= -u_1 \cdot \nabla \phi - C_D (u_0^2 - u_0 u_1 + u_1^2 + v_0^2 - v_0 v_1 + v_1^2). \tag{49}
\]

The first term of (49) represents the rate of working by the pressure field and the second is negative definite, representing the frictional dissipation. If \(u_0\) were the Ekman velocity and the equations constructed similarly to the geostrophic momentum approximation then the subscripts 0 and 1 on the friction terms in Eqs. (41) to (44) would have to be interchanged. The energy equation (49) then reduces to

\[
\frac{DE}{Dt} = -u_1 \cdot \nabla \phi + C_D ((u_0 - u_1)^2 + (v_0 - v_1)^2 - (u_1^2 - v_1^2)). \tag{50}
\]

This could give positive energy changes due to friction if \(u_1\) is small and \(u_0\) is larger. Though the approximation is not accurate in such cases anyway, the possibility of energy generation makes it unsuitable for time-dependent calculations.

6. DISCUSSION

The main purpose of the paper was to demonstrate how to couple boundary layer processes to semi-geostrophic internal dynamics. This has been achieved and demonstrated in two dimensions, and a method described for three dimensions. The model includes dynamics which should allow the gross features of the sea breeze and its dependence on the basic state gradient wind to be represented. In particular, the fact that the wind veer is caused by conservation of \((v + f)x\) as parcels are swept inland is included; it does not require parcels to be in the circulation for any fixed length of time as in the explanations reviewed by Atkinson (1981, pp. 162–3). The results substantiate these expectations. The finite difference semi-geostrophic model which includes all the frictional effects gives more realistic results than the geometric model, showing that a full treatment of friction is worthwhile even in a balanced model. The imposition of balance between the pressure gradient, friction and Coriolis terms at all times means that the circulation reverses when the pressure gradient reverses, giving a good description of the observed diurnal cycle. If allowance is made for the different treatment of friction, there is good agreement between the finite difference and geometric results, suggesting that the resolution of the finite difference model is sufficient to model the simplified sea-breeze circulation that can be described by these equations.

The results also indicate the limitations of this theory. The balance requirement causes the initial scale of the response to be too large and the front to be smeared. The neglect of the advection of the sea-breeze circulation by the basic state wind prevents the simulation of the detail of the observed sea-breeze structure in the presence of a gradient wind. This second limitation might be removed by the use of a different definition of balance, such as the nonlinear balance equation.

The primitive equation model reproduces both these aspects, in common with other published modelling studies. However, it also has deficiencies. The diurnal variation does not seem to be well modelled, the sea breeze continues after the forcing has stopped.
This is because an adverse pressure gradient has to be set up to decelerate it, which takes time to establish. In reality, much of the velocity contrast, for instance at the gravity current head, is probably concentrated in thin shear zones, which are subject to Kelvin–Helmholz instability. This will lead to turbulent entrainment which should act as an effective brake on the circulation and result in the observed cessation of the sea breeze at sunset. These effects are missing in our simulations because of the lack of vertical resolution and the use of a two-dimensional model. The model also produces a substantial inertial wave response on a vertical scale close to the model gridlength. Such waves may well be generated by sea breezes, and can be observed, e.g. Barat and Cot (1986). However, the generating mechanism in reality is believed to be the movement of the inversion, which is not well represented in the model used here. It is likely that much higher vertical resolution is needed to treat the inertial wave response properly. The results illustrate that the extra flow features that can be simulated in a primitive equation model require higher resolution if they are to be simulated properly.

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