Response to small-scale forcing on two staggered grids used in finite-difference models of the atmosphere

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SUMMARY
‘Source–sink’ shallow water integrations are performed, using two limited area models with the Coriolis terms included. One of the models was defined on the staggered C grid, and the other on the semi-staggered E/B grid. For each grid, experiments were performed with grid sizes of 250, 125 and 62.5 km. A source and a sink, of constant intensity, were placed symmetrically in the central part of the domain. They covered, for the three resolutions considered, areas of one, four and sixteen grid points, respectively.

At the sink ‘point’, after 24 hours, substantial differences in the depth of the water existed between the lowest resolution experiments on the two grids. The differences were very much reduced in the two 125 km experiments, and were virtually absent in the highest resolution, 62.5 km, experiments. With increasing resolution, the depths at the sink on the two grids converged with about equal rapidity towards a value in between the two lowest resolution values. As to be expected on the basis of the geostrophic adjustment theory when applied to the spatially discretized systems, the depth on the low resolution C grid solution was found to be an underestimate, and that on the E grid an overestimate, of the true value. It is demonstrated that the rate of convergence on the E grid can be improved by increasing the weight of the modification introduced to cope with the grid separation problem.

An implication of the results obtained in the simple experiments reported here is that forcing schemes should be considered which would avoid forcing at single grid points. Where appropriate, simultaneous forcing at several neighbouring points may be an attractive alternative approach.

1. INTRODUCTION

Following the early works of Winninghoff (1968) and Arakawa (1970) and also Arakawa and Lamb (1977) on the simulation of geostrophic adjustment, a number of studies have been made in order to examine the second-order finite-difference shallow water dynamics on rectangular grids. The accumulated evidence (e.g. Mesinger and Arakawa 1976; Mesinger 1981; Janjić and Mesinger 1984; Arakawa and Lamb 1981; Janjić 1984; Dragosavac and Janjić 1987) suggests that with presently available second-order finite-difference schemes, the staggered C grid and the semi-staggered B/E grid in the Arakawa notation (Mesinger and Arakawa 1976) are better than the other grids considered. Grids C and E are shown in Fig. 1 together with the grid distance d and the coordinate systems used later to define differencing and averaging operators. The B grid can be obtained by rotating the E grid through 45°.

In simulation of the geostrophic adjustment process, the B/E grid has a grid separation problem with the short waves, particularly in the case of the external and fast-moving internal modes (Mesinger 1973; Mesinger and Arakawa 1976; Janjić 1974, 1979). The effects of the grid separation can be clearly seen, e.g. in the four- to two-grid-interval wave ranges along the E grid x and y axes. In these ranges the frequencies of the gravity–inertia waves decrease with increasing wavenumber, resulting in a group velocity of the wrong sign (Mesinger and Arakawa 1976). However, a technique has been developed (Mesinger 1973; Janjić 1974, 1979) which to a large extent overcomes the problem (Vasiljević 1982; Cullen 1983; Janjić and Mesinger 1984). The C grid, on the other hand, has a difficulty with slow internal modes, but for all wave lengths (Mesinger and Arakawa 1976): in the case of small equivalent depths, owing to averaging of the Coriolis force.

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2. THE FINITE-Difference SCHEMES

The finite-difference shallow water equations used in the experiments have the form

\[ u^{r+1} = u^r - \Delta t g \delta_x h^{r+1} + \Delta t (v^r \delta_y \bar{u}^r - \Delta t (u \delta_x \bar{u}^r + \bar{v} \delta_y \bar{u}^r) \]  
\[ v^{r+1} = v^r - \Delta t g \delta_y h^{r+1} - \Delta t (v^r \delta_x \bar{v}^r - \Delta t (u \delta_x \bar{v}^r + v \delta_y \bar{v}^r) \]  
\[ h^{r+1} = h^r - \Delta t (\delta_x (\bar{h}^r \bar{u}^r) + \delta_y (\bar{h}^r \bar{v}^r)) \]  

on the C grid, and

\[ u^{r+1} = u^r - \Delta t g \delta_x h^{r+1} + \Delta t f v^r - \Delta t (u \delta_x \bar{u}^r + \bar{v} \delta_y \bar{u}^r) \]  
\[ v^{r+1} = v^r - \Delta t g \delta_y h^{r+1} - \Delta t f v^r - \Delta t (u \delta_x \bar{v}^r + \bar{v} \delta_y \bar{v}^r) \]  
\[ h^{r+1} = h^r - \Delta t (\delta_x (\bar{h}^{r-\frac{1}{2}} \bar{u}^{r-\frac{1}{2}}) + \delta_y (\bar{h}^{r-\frac{1}{2}} \bar{v}^{r-\frac{1}{2}})) + \frac{(\Delta t)^2}{2} \nabla \cdot (\nabla^2 \bar{h} + \nabla^2 \bar{h}) \]

on the E grid. Here, the symbol \( \delta \) and the overbar represent, respectively, the simplest second-order differencing and averaging operators applied in the directions of the coordinate axes, indicated by the suffix (more than one suffix indicates repeated application of the averaging operator). The orientations of the coordinate axes \( x' \) and \( y' \) used in (6) are shown in Fig. 1, and the auxiliary velocity components along these axes are defined by

\[ u' = \frac{1}{2} \sqrt{2} \cdot (u + v) \]  
\[ v' = \frac{1}{2} \sqrt{2} \cdot (-u + v) \]

The operators \( \nabla^2_x \) and \( \nabla^2_y \) are the finite-difference analogues of the \( \nabla^2 \) operator which are calculated, respectively, on the

The group velocity of the gravity-inertia waves may have the wrong sign, or vanish in a special case, in the entire admissible wavenumber range (Mesinger and Arakawa 1976). This problem has received little attention so far, and it is not known how serious its impact can be and whether anything can be done about it.

The present paper is devoted to a further study of the geostrophic adjustment process on the staggered C grid and the semi-staggered E grid. Specifically, the convergence of the simplest second-order finite-difference schemes is examined in a series of numerical experiments with strong small-scale forcing. On the E grid, a modification (hereafter referred to as 'the modification') (Mesinger 1973, Janjić 1974, 1979), is applied which prevents the separation of the gravity waves on the subgrids of the E grid. It is shown that the rate of convergence on the E grid can be improved by increasing the weight of the modification term in the E grid continuity equation.

The results of these experiments are believed to be of interest in view of parametrization schemes which to a large degree result in small-scale forcing.
RESPONSE TO SMALL-SCALE FORCING

Figure 2. Arrangement and notation of the grid points used in the discussion of the modification term introduced to prevent the grid separation on the E grid.

'x' and '+' shaped arrays of grid points shown schematically in Fig. 2. The time level is denoted by $\tau$, and $\Delta t$ is the time step. The mean height of the free surface of the fluid, $H$, was 1000 m, the Coriolis parameter, $f$, was assumed to be 0.0001 s$^{-1}$, and gravity, $g$, was taken as 9.80 m s$^{-2}$.

A forward–backward time-integration scheme (Gadd 1974, 1978; Mesinger 1974, 1977) is used for the gravity-wave terms, and the simplest forward scheme is applied for the Coriolis and advection terms in the equations of motion.

The last term on the right-hand side of the E grid continuity equation (6) represents the modification introduced in order to prevent the separation of the gravity waves on the two elementary C subgrids of the E grid (Mesinger 1973, 1974; Janjić 1974, 1979). It has been shown (Janjić 1979) that, with the maximum time step allowed by the CFL stability criterion for the scheme without modification, the presence of the modification term does not cause instability provided the weighting factor $w \leq 0.25$.

The modification as defined in (6) is of a higher order in $\Delta t$ than the mass divergence, and therefore the total effect of the modification in a given finite time interval will depend on the value of the time step chosen. In particular, the modification term vanishes in the limit as $\Delta t$ tends to zero. This unattractive dependence of the modification on the time step can be removed by increasing the weight of the modification with decreasing time step.

In view of such a possibility, it is interesting to determine the maximum value of $w$ for a given ratio $\Delta t/d$ which is smaller than the maximum value allowed by the CFL criterion. The stability criterion for the forward–backward scheme with the modification (Janjić 1979, Eq. (42)) can be rewritten in the form $4wgH(\Delta t/d)^2 \leq 1$. Hence, the maximum weight of the modification allowed by the stability criterion is

$$w_{\text{max}} = \frac{1}{4}d^2\{(\Delta t)^2 gh\}^{-1}. \quad (7)$$

Substituting (7) into (6), one obtains

$$h^{\tau+1} = h^\tau - \Delta t\{\delta_x\left(\frac{\partial}{\partial x}\vec{v}^\tau \frac{\partial}{\partial y}\vec{v}^\tau\right) + \delta_y\left(\frac{\partial}{\partial y}\vec{v}^\tau \frac{\partial}{\partial x}\vec{v}^\tau\right)\} + (d^2/4)(\nabla_x^2 - \nabla_y^2)h^\tau. \quad (8)$$

Thus, the modification as implemented in (8) will operate with its maximum weight, no matter how small the time step. Note that the modification term written in this form is of order $d^2$ since the difference of the two Laplacian approximations is of order $d^2$. 
Originally, the modification was introduced on the basis of physical arguments (Mesinger 1973, 1974; Janjić 1974, 1979). However, in the case of shallow water equations, this term can be derived quite formally. Consider the E grid continuity equation in the form

$$h^{*+1} = h^* - \Delta t(\delta_x(h \vec{u} \vec{v}) + \delta_y(h \vec{v} \vec{u}))^r.$$

Here, $h^* = \alpha h^r + (1 - \alpha)h^r$, and $h^r$ is obtained by fourth-order interpolation from the eight surrounding height points. The constant, $\alpha$, varies between 0 and 1. Using the notation of grid points introduced in Fig. 2, after some algebra, one obtains

$$h^*_0 = \alpha h^*_0 + (1 - \alpha)[h_0 + \frac{1}{2}(h_1 + h_2 + h_3 + h_4 - 4h_0) - \frac{1}{2}(h_5 + h_6 + h_7 + h_8 - 4h_0)]^r$$

or

$$h^*_0 = h^*_0 + \frac{1}{2}(1 - \alpha)[(h_1 + h_2 + h_3 + h_4 - 4h_0) - \frac{1}{2}(h_5 + h_6 + h_7 + h_8 - 4h_0)]^r.$$

The second term on the right-hand side of the last expression will have the same form as the last term of the E grid continuity equation (6), provided the factor $wgH(\Delta t)^2$ is replaced by $d^2(1 - \alpha)/2$. Thus, the alternative E grid continuity equation has the form

$$h^{*+1} = h^* - \Delta t(\delta_x(h \vec{u} \vec{v}) + \delta_y(h \vec{v} \vec{u}))^r + \frac{1}{2}d^2(1 - \alpha)(\nabla^2_x - \nabla^2_y)$$

which is identical to (8) if $\alpha = 1/2$. In other words, as in (8), with $\alpha = 1/2$ the modification will have its maximum weight permitted for stability, no matter how small the time step. With $\alpha = 1/4$, on the other hand, it has its maximum weight only if the maximum time step permitted by the CFL condition for the gravity-wave part of the equations is used.

3. Design of the Numerical Experiments

The same basic experiment design was adopted as that of the early 'source–sink' experiments of Arakawa (1972) and Mesinger (1973). The shallow water equations were integrated in a rectangular domain starting with the fluid at rest. The size of the integration domain was 5250 km by 3500 km in the case of the C grid, and 5303:30 km by 3535:53 km in the case of the E grid. The slight difference between the sizes of the integration domains is due to the different geometry of the two grids and the requirement that the dimensions of the two domains be as nearly the same as possible, with the grid distance $d$ (shown in Fig. 1) the same in both cases.

No-slip boundary conditions were used. As schematically represented in Fig. 3, a source and a sink, 1750 km apart on the C grid, and 1767:77 km apart on the E grid.

![Figure 3](image-url) Schematic representation of the integration domain. The locations of the source and sink points/areas are shown.
were placed symmetrically in the central part of the domain along a line oriented east–west. The intensities of both source and sink were 2 m min\(^{-1}\).

Three groups of experiments were performed:

(i) Experiments with varying horizontal resolution, in order to compare the solutions of (1)–(3) and (4)–(6), and to estimate the rates of convergence on grids C and E.

(ii) Experiments with fixed horizontal resolution on the E grid and varying time steps, in order to assess the impact of reduced time steps on the modification as defined in (6).

(iii) Experiments on the E grid as in group (i), but with the continuity equation (6) replaced by (8), in order to check whether the rate of convergence can be improved by increasing the weight of the modification term.

In groups (i) and (iii), the lowest resolution experiments were performed with the grid distance \( d = 250 \text{ km} \). The experiments were then repeated with 125 km and 62.5 km mesh sizes. In the experiments with the lowest resolution, the source and the sink were restricted to single grid points. In the 125 km experiments, the source and the sink consisted of four grid points each, with equal intensity of forcing at each of the four points. Analogously, in the 62.5 km experiments, there were 16 points with equal intensity of forcing at both source and sink. Since the integration domains of the higher resolution experiments were of the same sizes as those of the lowest resolution experiments, the centres of these higher resolution source and sink areas had to be shifted slightly with respect to their lowest resolution symmetrical positions. They were shifted to the north–west for the C grid experiments, and to the west for the E grid experiments; by \( \frac{1}{2} \sqrt{2.5} \) and by \( \frac{1}{4} \sqrt{2.5} \) in the medium resolution and the highest resolution experiments, respectively.

The time steps chosen in the experiments from group (i) on the E grid were 30, 15 and 7.5 min, depending on the resolution used. These values represented about 0.71 of the maximum time steps allowed by the CFL criterion for the forward–backward scheme for the gravity-wave part of the equations. The E grid results from this group were obtained using the modification as defined in (6) with the weighting factor \( w = 0.25 \). As can be inferred from (6) and (8), with this value of the weighting factor, and the time steps chosen, the modification operated with about 50% of its maximum weight allowed by the stability criterion.

In the case of the C grid, depending on the resolution, the time steps were 20, 10 and 5 min. With equal spatial resolution, shorter time steps are required on the C grid than on the B/E grid owing to more accurate differencing in the pressure gradient force and divergence terms.

The experiments on the E grid from group (ii) were performed with the grid distance \( d = 250 \text{ km} \), time steps of 7.5, 15 and 30 min, and the weighting factor \( w = 0.25 \) as before. As can be inferred from (6) and (8), with this choice of the weighting factor and time steps, the modification operated, respectively, with about 12.5%, 25% and 50% of its maximum weight allowed by the stability criterion.

The only difference between the E grid experiments from group (iii) and group (i) was that the continuity equation (6) was replaced by (8), and, therefore, the modification operated with the maximum weight allowed by the stability criterion.
4. DISCUSSION OF EXPERIMENTAL RESULTS

(a) Group (i): Comparison of grids C and E

The plots in Fig. 4 represent the height of the free surface at the sink point after 24 hours of forcing in the case of 250 km experiments, and the height averaged over four and sixteen forced sink points in the cases of 125 km and 62.5 km experiments. The dots on the light and the heavy lines correspond to the results obtained on the C and on the E grid, respectively. As can be inferred from the figure, the solutions on both grids have a rather large error in the experiment with the lowest resolution, and converge with about equal rapidity. However, the C grid tends to underestimate the height depression at the sink, while the reverse is true for the E grid. Thus, the solutions on the two grids converge, approaching the true solution from different sides.

This result is consistent with the linear geostrophic adjustment theory when applied to spatially discretized systems: in the case of linearized shallow water equations on an f plane, if a disturbance is introduced in the height field, the ratio of the amplitude of the wave solution corresponding to the geostrophic part, \( G_g \), and the amplitude of the wave component of the initial disturbance, \( G_o \), is given by

\[
\frac{G_g}{G_o} = \frac{f^2}{[f^2 + gH(k^2 + l^2)]}
\]  

(10)  

(see, e.g., Janjić and Wiin-Nielsen 1977; Daley 1980). Here, \( k \) and \( l \) are the wavenumber vector components, and the other symbols have already been defined.

Following the same procedure, after horizontal discretization on the C grid, expression (10) is replaced by

\[
\frac{G_{gC}}{G_o} = \frac{\cos^2(kd/2) \cos^2(ld/2)f^2}{[\cos^2(kd/2) \cos^2(ld/2)f^2 + gH(\sin^2(kd/2) + \sin^2(ld/2))]}.
\]  

(11)

**Figure 4.** Deviation from the mean height at the sink point after 24 hours of forcing in the case of 250 km experiments, and the deviations from the mean height averaged over four and sixteen forced sink points in the cases of 125 km and 62.5 km resolutions for the C grid (dots connected by light line), and the E grid (dots connected by heavy line). The height deviations are given in metres.
Similarly, on the E grid without modification
\[
\frac{G_g}{G_o} = \frac{f^2}{[f^2 + gH(\sin^2(kd/\sqrt{2}) + \sin^2(ld/\sqrt{2}))]} \tag{12}
\]
As can be seen from (11), owing to averaging of the Coriolis force term on the C grid, the numerator tends to zero as the shortest resolvable scale is approached. Therefore, the amplitudes of the geostrophic part of the solution will be underestimated in the shortwave range of the spectrum. Specifically, for the shortest resolvable scale the amplitude will be zero. For this scale, only the gravity waves can exist. On the other hand, as can be seen from (12), as the shortest resolvable wave is approached on the E grid without modification, the analogue of the term \(gH(k^2 + \ell^2)\) appearing in the denominator, tends to zero. Thus, the amplitudes of the geostrophic part of the solution are overestimated. In particular, for the shortest resolvable scale, the amplitude of the geostrophic part will be equal to the amplitude of the initial disturbance.

The following heuristic considerations may be helpful in an attempt to visualize the causes of the problems on the two grids in the extreme case of forcing applied at a single grid point. Consider a velocity point on the C grid neighbouring the source or the sink point. At that point, the effect of the Coriolis force will be reduced since, of the four velocity points entering the averaging needed to calculate the Coriolis force, only two are located on the same side of the source/sink point. For example, at the \(u\) point east of the sink point (circled point in Fig. 1), only the two \(v\) points east (south-east and north-east) of it should be expected to contribute effectively to the positive value of the \(fv\) term of the \(u\)-momentum equation. The contributions of the two remaining \(v\) points will mostly cancel, since one of them is located directly to the north and the other directly to the south of the sink point. The same will happen with the remaining three velocities surrounding the sink point. Thus, the Coriolis force will insufficiently reduce the inflow towards the sink at these four velocity points. As a result, the sink will be filled more efficiently than that of the true solution, and the depth at the sink point will be underestimated.

On the E grid, the overestimation of the depth at the sink is related to the separation of solutions. In the extreme case of total separation, a sink at an E grid height point will be felt by its C subgrid as a sink of an intensity double that of the prescribed E grid intensity, since only half of the height points belong to a single C subgrid. Thus, the depth at the sink point should be about twice that of the true solution. With the modification (and other terms) reducing such separation, but not entirely eliminating it, the depth at the sink point will be reduced compared with that of the completely separated solution, but will remain overestimated.

(b) Group (ii): The impact of the time step on the modification

As pointed out in section 2, the efficiency of the modification as defined in Eq. (6) depends on the time step. The larger the time step used, the more efficient the modification will be. This situation is illustrated in Fig. 5, showing the height deviation at the sink point after 24 hours in the 250 km resolution experiment as a function of time step. Indeed, with small time steps, as one would expect from the considerations at the end of the preceding subsection, the depth at the sink point is found to be about twice that of the two highest resolution experiments shown in Fig. 4.

(c) Group (iii): The impact of increased weight of modification

In order to check the impact of the increased weight of the modification term, the E grid experiments of group (i) were repeated using Eq. (8) instead of (6). The results
Figure 5. Deviation from the mean height at the sink point (dots) after 24 hours in the 250 km resolution experiment as a function of time step. The height deviations are given in metres.

Figure 6. Deviation from the mean height at the sink point after 24 hours of forcing in the case of 250 km experiments, and the deviations from the mean height averaged over four and sixteen forced sink points in the cases of 125 km and 62.5 km resolutions for the weight of the modification which is the maximum permitted for stability (dots connected by heavy line), and for the maximum weight which does not affect the stability condition with the parameters chosen as in the experiments shown in Fig. 4 (dots connected by light line). The height deviations are given in metres.

obtained are indicated by the dots on the heavy line in Fig. 6. For comparison, the E grid results already shown in Fig. 4 are again displayed (dots on the light line). Apparently, with the maximum weight of the modification, the situation has improved considerably, the convergence being accelerated by approximately a factor of two. The results of the low resolution E grid experiments are now clearly better than those on the C grid.
4. Conclusions

Experiments with small-scale forcing have been performed on the staggered C grid and on the semi-staggered E grid with a simple shallow water model. The resolution has been varied in order to examine the convergence of the solutions, and, if convergence were to be established, the rate of convergence on the two grids.

With forcing by a source and a sink at single grid points, a substantial difference between the solutions on the E and the C grids has been demonstrated. Considering the depth at the sink, the solutions on the two grids have been found to have large errors in the experiment with single-grid-point forcing, and to converge toward the same value as the resolution is increased.

With moderate weight of the modification (Mesinger 1973; Janjić 1979), the solution on the E grid approaches the true solution with about the same rapidity as on the C grid. However, the C grid tends to underestimate the amplitude of the disturbance, while the reverse is true on the E grid. Thus, the solutions on the two grids converge from different sides. This result is consistent with the linear geostrophic adjustment theory applied to spatially discretized systems.

It has been demonstrated, however, that in the set-up of the present experiments, the convergence properties of the E grid can be considerably improved by increasing the weight of the modification, with no penalty in terms of economy of computation.

An obvious implication of the results shown in Fig. 4 is consideration of forcing schemes which would avoid forcing at single grid points. Where appropriate, simultaneous forcing at several neighbouring points may be an attractive alternative approach. This idea was suggested a long time ago by Egger (1971), and has recently been applied in construction of the four-point mountains on the E grid (Mesinger 1985; Mesinger et al. 1988).

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