An extended version of a nonhydrostatic, pressure coordinate model

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SUMMARY

The acoustically-filtered, nonhydrostatic flow model formulated by M. J. Miller and R. P. Pearce in 1974 may be modified so as to remove dependence on a certain thermodynamic reference state. The modified model seems appropriate for mesoscale integrations. It implies an elliptic diagnostic equation for the height field, a satisfactory energy equation and an analogue of Ertel’s potential vorticity theorem. These properties are also possessed by Miller and Pearce’s original model, but in different forms (which have been obtained previously by various authors).

1. INTRODUCTION

An important and novel feature of the nonhydrostatic flow model described and used by Miller (1974) and Miller and Pearce (1974) was its use of pressure, $p$, as vertical coordinate. Previously, pressure coordinates had been used only in models which incorporated the hydrostatic approximation.

In Miller and Pearce’s model—here called the MP model—the true vertical acceleration $Dw/Dt$ is replaced in the vertical momentum equation by $D\bar{w}/Dt$, where

$$\bar{w} = -\omega/\rho \mu g = -\omega RT_s/gp.$$  \hspace{1cm} (1)

(Here $D/Dt$ is the material derivative, $w$ is the vertical velocity, $\omega = Dp/Dt$, $g$ is the gravitational acceleration, and $R$ is the gas constant for unit mass of air; $T_s = T_s(p)$ and $\rho_s = \rho_s(p)$ are reference profiles of temperature ($T$) and density ($\rho$).) In conjunction with approximations made in the continuity and horizontal momentum equations, this replacement of $Dw/Dt$ by $D\bar{w}/Dt$ ensures the absence of vertically propagating acoustic modes whilst buoyancy modes remain undistorted (and no spurious modes are introduced). Horizontally propagating acoustic modes (the Lamb modes) may be removed by applying the lower boundary condition $\omega = 0$ at $p = p_o$, where $p_o$ is a constant (=1016 mb, say). Miller (1974) and Miller and White (1984) analyse these aspects in detail.

The MP model has been used in many studies of cumulonimbus and other convective systems; see, for example, Moncrieff and Miller (1976), Miller and Betts (1977), Thorpe and Miller (1978), Thorpe et al. (1982) and Brugge and Moncrieff (1985). The appearance of the reference temperature $T_s(p)$ in the definition of $\bar{w}$ (and elsewhere—see section 2) is not a worrying feature in these studies since each is concerned with situations in which the initial state is horizontally homogeneous. Variations of the temperature on pressure surfaces are thus due solely to the convection itself, and these turn out to be small. Should the MP model be used in mesoscale simulations—or, as speculatively suggested by Miller and White (1984), in larger scale modelling—the use of $T_s(p)$ instead of the true local temperature, $T$, would be less comfortable. In frontal zones, for example, considerable variations of temperature may occur on pressure surfaces, and a single reference state might not be typical of different parts of the computational domain.

A modification of the MP model to represent large temperature variations on pressure surfaces is clearly desirable, and this paper describes such a modification. The MP model is briefly reviewed in section 2. The proposed extension is presented in section 3, and some of its properties analysed in section 4. Closing remarks are to be found in section 5.
2. MILLER AND PEARCE'S MODEL

With the neglect of all forcing and cloud physics processes, the MP model consists of the following approximate forms of the horizontal and vertical momentum equations and the continuity equations (2)–(5) and the exact thermodynamic equation (6) for a perfect gas:

$$
\frac{Du}{Dt} + \frac{\partial \phi'}{\partial x} = 0 \quad (2)
$$

$$
\frac{Dv}{Dt} + \frac{\partial \phi'}{\partial y} = 0 \quad (3)
$$

$$
\frac{R}{g} \frac{D}{Dt} \left( \frac{\omega T_s}{p} \right) + \frac{g T'}{T_s} + \frac{g p}{RT_s} \frac{\partial \phi'}{\partial p} = 0 \quad (4)
$$

$$
\partial u/\partial x + \partial v/\partial y + \partial \omega / \partial p = 0 \quad (5)
$$

$$
\frac{DT}{Dt} - \omega RT / pc_p = 0. \quad (6)
$$

Here

$$
D/ Dt = \partial / \partial t + u \partial / \partial x + v \partial / \partial y + \omega \partial / \partial p. \quad (7)
$$

In Eqs. (2)–(7) (and throughout this note) all differentiations with respect to time $t$ and the horizontal coordinates $x$ and $y$ are carried out at constant pressure; $u$ and $v$ are the velocity components in the $x$ and $y$ directions; $c_p$ is the specific heat at constant pressure; and $\phi$ is the geopotential, $g z$, $z$ being the geometric height. $\phi'$ and $T'$ are the departures of $\phi$ and $T$ from the reference profiles $\phi_s(p)$ and $T_s(p)$:

$$
\phi = \phi_s(p) + \phi'; \quad T = T_s(p) + T'. \quad (8)
$$

The reference state is chosen to be in hydrostatic balance, so that

$$
\frac{d \phi_s}{dp} + RT_s / p = 0. \quad (9)
$$

Time integration of Eqs. (2)–(6) exploits the elliptic diagnostic equation for $\phi'$ which results when Eq. (5) is applied to differentiated forms of Eqs. (2)–(4) (Johnson 1978; Brugge and Moncrieff 1985):

$$
\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \frac{\partial}{\partial p} \left[ \frac{\partial \phi'}{\partial p} \right] = 2J_3 - \frac{\partial}{\partial p} \left[ \frac{g r_s T'}{T_s} - \frac{\omega^2 dr_s}{r_s dp} \right] \quad (10)
$$

in which

$$
J_3 = \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) + \left( \frac{\partial v}{\partial y} \frac{\partial \omega}{\partial p} - \frac{\partial \omega}{\partial y} \frac{\partial v}{\partial p} \right) + \left( \frac{\partial \omega}{\partial x} \frac{\partial u}{\partial p} - \frac{\partial u}{\partial x} \frac{\partial \omega}{\partial p} \right) \quad (11)
$$

and

$$
r_s = g \rho_s = gp / RT_s. \quad (12)
$$

(Boundary conditions on the solution of (10) are obtained using Eqs. (2), (3), (4) and (6) together with appropriate specifications of $u$, $v$, $\omega$ and $T'$ at the boundaries.)

Miller (1974) derived Eqs. (2)–(5) by transforming the familiar height-coordinate equations to $p$ coordinates and then neglecting various small terms. By using a scaling and power series expansion method, Miller and White (1984) gave a formal justification of Eqs. (2) and (4)–(6) for flow independent of one horizontal coordinate. Their derivation assumed that the departures $\phi'$ and $T'$ from the reference state profiles (see Eq. (8)) were due solely to processes occurring on the convective scale. They identified the quantity
\[ \alpha = \frac{p_0 \, d\theta_s}{\theta_s \, dp} \]  

(13)

as the key parameter whose smallness \((\alpha \ll 1)\) validates the approximations made by Miller (1974). (In Eq. (13), \(\theta_s = \theta_s(p)\) is the potential temperature profile corresponding to \(T_s(p) = T_s(p_0/p)^{\kappa}\), where \(\kappa = R/c_p.\)

Equations (2)–(6) imply an energy equation in the form

\[ \frac{D}{Dt} \{ \frac{1}{2}(u^2 + v^2 + \tilde{w}^2) + c_p T \} = -\nabla_p \cdot (v \phi) - \frac{\partial}{\partial p} (\omega \phi) \]  

(14)

in which \(\nabla_p = (\partial/\partial x, \partial/\partial y)\) and \(v\) is the horizontal velocity \((u, v, 0)\). Equation (14) is an extension of the familiar hydrostatic form to include the contribution of the approximate vertical velocity \(\tilde{w} = -\omega RT_s/gp\);—see Eq. (1)) to the specific kinetic energy. Miller and White (1984) noted a 2-dimensional case of Eq. (14), and also examined a \(\theta\)-coordinate transformation of the corresponding versions of Eqs. (2)–(6).

Johnson (1978) carried out a detailed study of the vorticity properties of the MP model, and (amongst a number of important results) established the existence of an analogue of Ertel’s potential vorticity theorem. For the adiabatic forms (2)–(6) the analogue is

\[ \frac{D}{Dt} \left\{ \tilde{Z} \cdot \tilde{\nabla} \theta \right\} = 0 \]  

(15)

with

\[ \tilde{Z} = \left( \frac{\partial \tilde{w}}{\partial y} - \frac{\partial v}{\partial h_s}, \frac{\partial u}{\partial h_s}, \frac{\partial w}{\partial x} - \frac{\partial v}{\partial y} \right) \]  

(16)

\[ \tilde{\nabla} \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial h_s} \right) \]  

(17)

\[ \frac{\partial}{\partial h_s} = -g \rho_s \frac{\partial}{\partial p} = - \frac{gp}{RT_s} \frac{\partial}{\partial p} \]  

(18)

and

\[ \theta = T(p_0/p)^{\kappa}. \]  

(19)

The quantity \(h_s\) defined by Eq. (18) may be considered as the hydrostatic height, \(\phi_s/g\), defined by the reference state.

The Lagrangian conservation law (15) is readily understood in mathematical terms by noting that the material derivative (7) can be written alternatively as

\[ D/Dt = \partial/\partial t + u \partial/\partial x + v \partial/\partial y + \tilde{w} \partial/\partial h_s. \]  

(7A)

Equations (2)–(4) can therefore be expressed in the vector form

\[ (\partial/\partial t + \tilde{u} \cdot \tilde{\nabla}) \tilde{u} - (\sigma T'/T_s) \mathbf{k} + \tilde{\nabla} \phi' = 0 \]  

(20)

in which \(\mathbf{k}\) is unit vector in the upward vertical direction and \(\tilde{u} = (u, v, \tilde{w})\). All the elements of \(\tilde{\nabla}\) commute with one another (because \(\rho_s\) is a function of pressure only—see Eq. (18)) so standard vector differential identities may be applied in taking the curl \((\tilde{\nabla} \times)\) of Eq. (20). A vorticity equation results from this operation, and use of Eqs. (5), (6)—together with further standard identities—then gives Eq. (15) in a way formally analogous to the usual derivation of Ertel’s theorem from the momentum equation for nonhy-
drostatic, Boussinesq flow. The formal similarity between the equations of the MP model and the Boussinesq equations in height coordinates has been discussed by Moncrieff (1985).

The conservation laws (14) and (15) provide strong evidence for the dynamical consistency of the MP model. Reference state functions are nevertheless implicit in Eq. (14) (through the definition of $\bar{\nu}$) and are conspicuous in Eq. (15) as well as in the vertical momentum equation (4).

3. AN EXTENDED VERSION OF MILLER AND PEARCE'S MODEL

The unapproximated vertical component of the momentum equation (for motion on an $f$ plane) is

$$\frac{Dw}{Dt} + g + \frac{1}{\rho} \frac{\partial p}{\partial z} = 0$$  \hspace{1cm} (21)

in which $w$ is the true vertical velocity $Dz/Dt$. Equation (21) may be written as

$$\frac{1}{g} \frac{\partial \Phi}{\partial p} \frac{Dw}{Dt} + \frac{\partial \Phi}{\partial p} + \frac{RT}{p} = 0.$$  \hspace{1cm} (22)

Upon subtracting the reference state relation (9), and rearranging, Eq. (22) becomes

$$\frac{Dw}{Dt} + g \frac{\partial p}{\partial \Phi} \left( \frac{\partial \Phi'}{\partial p} + \frac{RT'}{p} \right) = 0.$$  \hspace{1cm} (23)

Substitution for $\partial p/\partial \Phi$ from Eq. (22) then gives

$$\frac{Dw}{Dt} - \frac{gp}{RT} \left( 1 + \frac{1}{g} \frac{Dw}{Dt} \right) \left( \frac{\partial \Phi'}{\partial p} + \frac{RT'}{p} \right) = 0.$$  \hspace{1cm} (24)

Equation (24) is more complicated than Eq. (21) (to which it is precisely equivalent) but gives useful insight into the range and nature of possible approximations. With the non-critical approximation $Dw/Dt \ll g$ (see Miller 1974) and with

$$w \approx \bar{w} \equiv - \omega/\rho g = - \omega RT/gp$$  \hspace{1cm} (25)

Eq. (24) reduces to

$$\frac{R}{g} \frac{D}{Dt} \left( \frac{\omega T}{p} \right) + \frac{gT'}{T} \frac{GP}{RT} \frac{\partial \Phi'}{\partial p} = 0.$$  \hspace{1cm} (26)

Equation (26) is the proposed extended version of the MP vertical momentum equation (4). It is consistent with the spirit of the MP model, but does not include the further replacement of $T$ by $T_v$, and so seems a gentler approximation than the MP form.

The extended model consists of Eqs. (2)–(6) with (26) replacing (4). For convenience these forms are listed below, and (with mesoscale applications in mind) Coriolis terms are included in the horizontal component equations; $f$, the Coriolis parameter, will be assumed constant.

$$\frac{Du}{Dt} - f\nu + \partial \Phi'/\partial x = 0$$  \hspace{1cm} (27)

$$\frac{Dv}{Dt} + f\nu + \partial \Phi'/\partial y = 0$$  \hspace{1cm} (28)

$$\frac{R}{g} \frac{D}{Dt} \left( \frac{\omega T}{p} \right) + \frac{gT'}{T} \frac{GP}{RT} \frac{\partial \Phi'}{\partial p} = 0.$$  \hspace{1cm} (29)
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0
\]  
(30)

\[
\frac{\partial T}{\partial t} - \kappa \omega T/p = 0.
\]  
(31)

In the linearized problems studied by Miller (1974) and Miller and White (1984), Eq. (29) reduces to the linearized MP form. The extended model therefore supports no vertically propagating acoustic modes; their absence may also be inferred from the form of the implied energy equation (see Eq. (34), below).

Equations (27)–(31) are independent of the choice of hydrostatic reference state \( \phi_s(p) \) (see Eq. (9)). Equations (27)–(29) could indeed be written with \( \phi' \) and \( T' \) replaced by \( \phi \) and \( T \), but the subtraction of reference state profiles \( \phi_s(p) \) and \( T_s(p) \) would almost certainly be desirable in computational practice.

4. Properties of the Extended MP Model

It is readily shown that Eqs. (27)–(31) imply a diagnostic equation for the geopotential deviation \( \phi' \):

\[
\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \frac{\partial}{\partial p} \left( r^2 \frac{\partial \phi'}{\partial p} \right) = 2J_3 + f \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial p} \left( \frac{rgT'}{T} - \frac{\omega^2}{\gamma p} \right).
\]  
(32)

Here

\[
r = g\rho = gp/RT
\]  
(33)

and \( \gamma = c_p/c_v \) is the ratio of the principal specific heats. \( J_3 \) is defined in Eq. (11.). Equation (32) is elliptic everywhere, and is hardly more complicated than the MP form (10). The thermodynamic equation is needed in the derivation of (32), but not of (10). Boundary conditions on \( \phi' \) may be obtained using (27), (28), (29) and (31) together with suitable specifications of \( u, v, \omega \) and \( T \) at the boundaries (essentially as in the MP case).

The energy equation implied by Eqs. (27)–(31) is

\[
\frac{D}{Dt} \left\{ \frac{1}{2}(u^2 + v^2 + \bar{w}^2) + c_p T \right\} = - \nabla_p \cdot (v\phi) - \frac{\partial}{\partial p} (\omega \phi).
\]  
(34)

(\( \bar{w} \) is defined in Eq (25.).) Equation (34) is a simple and satisfactory generalization of the MP form (14).

In examining the vorticity and potential vorticity properties of Eqs. (27)–(31) it is convenient to work in terms of a reversed pressure coordinate \( p_m \) and the corresponding ‘vertical velocity’ \( \omega_m \), where

\[
p_m = p_0 - p
\]  
(35)

and

\[
\omega_m = -\omega.
\]  
(36)

As before, \( p_0 \) is the pressure level at which the lower boundary condition is applied; thus \( p_m \geq 0 \) everywhere. The axes of \( x, y \) and \( p_m \) increasing form a right-handed system when \( x \) and \( y \) are in their usual orientation in the horizontal plane. The components of the vorticity vector \( \mathbf{Z} (\xi, \eta, \zeta) \) are defined as

\[
\xi = -\frac{\partial}{\partial y}(\omega_m/r^2) - \frac{\partial \omega}{\partial p_m}
\]  
(37)

\[
\eta = \frac{\partial u}{\partial p_m} - \frac{\partial}{\partial x}(\omega_m/r^2)
\]  
(38)

\[
\zeta = f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.
\]  
(39)
According to these definitions, 

\[ \nabla \cdot \mathbf{Z} = 0 \]  

(40)

with 

\[ \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial p_m). \]  

(41)

In view of the identity 

\[ \frac{D}{Dt} \left( \frac{1}{r} \right) = \frac{\omega_m}{\gamma pr} \]  

(42)

Eq. (29) may be written as 

\[ \frac{D}{Dt} \left( \frac{\omega_m}{r^2} \right) - \frac{\omega_m^2}{\gamma pr^2} \frac{RT}{p} + \frac{\partial \phi'}{\partial p_m} = 0. \]  

(43)

From Eqs. (27), (28), (30) and (43), equations for \( D\xi/Dt, D\eta/Dt \) and \( D\zeta/Dt \) are readily obtained. With \( \mathbf{u} = (u, v, \omega_m) \) they may be written in vector form as 

\[ D\mathbf{Z}/Dt - (\mathbf{Z} \cdot \nabla) \mathbf{u} + (R/p) \mathbf{k} \times \nabla T + (1/\gamma p) \mathbf{k} \times \nabla (\omega_m^2/r^2) + (1/r^3) \nabla r \times \nabla \omega_m^2 = 0. \]  

(44)

The \( \omega_m^2 \) terms in Eq. (44) arise because of the definitions (37) and (38) of the horizontal vorticity components \( \xi \) and \( \eta \), and through the term \( -\omega_m^2/\gamma pr^2 \) in Eq. (43). Since 

\[ \nabla \ln r = \nabla \ln \rho = -\nabla \ln \theta - k/\gamma p \]  

(45)

it follows that 

\[ \nabla \theta \cdot ((1/\gamma p) \mathbf{k} \times \nabla (\omega_m^2/r^2) + (1/r^3) \nabla r \times \nabla \omega_m^2) = -(2\omega_m^2/\gamma pr^3)k \cdot (\nabla r \times \nabla \theta). \]  

(46)

But 

\[ k \cdot (\nabla r \times \nabla \theta) = k \cdot (\nabla_p r \times \nabla_p \theta) = 0 \]  

(47)

since \( r = gp(p_o/p)^r/R \theta \). Hence the right side of (46) vanishes. Upon noting also that 

\[ \nabla \theta \cdot (\mathbf{k} \times \nabla T) = k \cdot (\nabla_T \times \nabla \theta) = k \cdot (\nabla_p T \times \nabla_p \theta) = 0 \]  

and after some straightforward manipulation, it is seen that Eq. (44) implies 

\[ \frac{D}{Dt} \{ \mathbf{Z} \cdot \nabla \theta \} = 0. \]  

(48)

The extended MP model implies a p-coordinate analogue of Ertel's potential vorticity conservation law.

Since \( k \cdot (\nabla r \times \nabla \theta) = 0 \) (see Eq. (47)) there are infinitely many other definitions of \( \mathbf{Z} \) which imply Lagrangian conservation laws of the form of (48). For example, the quantities 

\[ \begin{align*}
\xi_N &= r^{-N} \frac{\partial}{\partial y} \left( \frac{\omega_m/r^{2-N}}{r} \right) - \frac{\partial u}{\partial p_m} \\
\eta_N &= \frac{\partial u}{\partial p_m} - r^{-N} \frac{\partial}{\partial x} \left( \frac{\omega_m/r^{2-N}}{r} \right)
\end{align*} \]  

(49)

where \( N \) is any real number, obey 

\[ \begin{align*}
\xi_N &= \xi + (N\omega_m/r^3) \partial r/\partial y \\
\eta_N &= \eta - (N\omega_m/r^3) \partial r/\partial x
\end{align*} \]  

(50)
The vector $Z_N$, defined as $(\xi_N, \eta_N, \zeta)$, may therefore be expressed as
\[ Z_N = Z - (N \omega_m/r^3) k \times \nabla r \]
and so
\[ Z_N \cdot \nabla \theta = Z \cdot \nabla \theta. \] (51)

An important member of this particular class of quantities is that obtained when $N = 1$. $\xi_1$ and $\eta_1$ may be written as
\[ \xi_1 = (\partial \hat{w}/\partial y - \partial v/\partial h)/\rho g \]
\[ \eta_1 = (\partial u/\partial h - \partial \hat{w}/\partial x)/\rho g \] (52)
in which $\hat{w}$ is defined in Eq. (25) and $h$ is the hydrostatic height defined by the actual density field $\rho(x,y,p,t)$:
\[ \partial/\partial h = - \rho g \partial/\partial p = \rho g \partial/\partial p_m. \] (53)

Since
\[ \nabla \theta = \nabla_p \theta + (k/\rho g) \partial/\partial h \]
it follows that
\[ Z \cdot \nabla \theta = Z_1 \cdot \nabla \theta = \hat{Z} \cdot \hat{\nabla} \theta/\rho g \] (54)
in which
\[ \hat{Z} = (\partial \hat{w}/\partial y - \partial v/\partial h, \partial u/\partial h - \partial \hat{w}/\partial x, f + \partial v/\partial x - \partial u/\partial y) \] (55)
and
\[ \hat{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial h). \] (56)

According to Eq. (54), $Z \cdot \nabla \theta$ is proportional to a quantity that is formally similar to the height-coordinate expression of Ertel’s potential vorticity as implied by the unapproximated equations of motion.

The equivalence (54) is useful for physical interpretation, but it should be noted that the elements of $\hat{\nabla}$ (see Eq. (56)) do not all commute with one another. From Eq. (53) it is readily shown that
\[ \left[ \frac{\partial}{\partial n} \frac{\partial}{\partial h} \right] = \frac{1}{\rho} \frac{\partial p}{\partial n} \frac{\partial}{\partial h} \] (57)
where $n = x, y$ (or $t$) and $\partial/\partial n$ is taken at constant $p$. This non-commutation has two important consequences.

First, although Eqs. (27)–(29) may be written in vector form as
\[ (\partial/\partial t + \hat{u} \cdot \hat{\nabla}) \hat{u} = \left( g T'/T \right) k + \hat{\nabla} \phi' = 0 \] (58)
in which $\hat{u} = (u, v, \hat{w})$, it is not straightforward to derive an equation for $D \hat{Z}/Dt$ since the standard vector differential identities must be augmented by terms which take account of (57). Similar complications occur in deriving the conservation law
\[ \frac{D}{Dt} \left( \frac{\hat{Z} \cdot \hat{\nabla} \theta}{\rho} \right) = 0 \] (59)
from the equation for $D \hat{Z}/Dt$. In fact, it is exquisitely tedious to derive Eq. (59) in this way; by far the simpler proof is to proceed via Eqs. (48) and (54), as above.
Second, and more fundamental, the ‘vorticity’ \( \mathbf{\hat{z}} \) is divergent with respect to \( \mathbf{\hat{v}} \):

\[
\mathbf{\hat{v}} \cdot \mathbf{\hat{z}} = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial y} \frac{\partial u}{\partial h} - \frac{\partial \rho}{\partial x} \frac{\partial v}{\partial h} \right).
\]  

(60)

The operator \( \mathbf{\hat{v}} \) is thus seen to be essentially different from the operator \( \mathbf{\hat{v}} \) defined by (17) for the original MP model. It appears that the vorticity and the gradient operator in the modified MP model are best defined by Eqs. (37)–(39) and (41).

5. Discussion

The existence of the conservation laws (34) and (48) argues strongly for the dynamical consistency of the extended MP model proposed in section 3 (Eqs. (27)–(31)). The conservation laws indeed suggest that Eqs. (27)–(31) may possess a Hamiltonian substructure (see Salmon 1988). Conversely, it is probable that a slick derivation of the Ertel property (48) could be constructed if the relevant Hamiltonian were known (and could be properly manipulated). These aspects require further study.

Since the proposed extension of the MP model differs from the original only in the replacement of the reference state temperature \( T_r(p) \) by the true temperature \( T \), it might be considered a trivial extension. Algebraically, this does not appear to be the case, at least as regards the vorticity and potential vorticity dynamics (see section 4)—although a Hamiltonian treatment might give a different perspective. It is worth recalling the case of quasi-geostrophic theory (see Hoskins and Pearce (1983), appendix A2) in which the use of a reference state temperature profile is a necessary requirement if good conservation properties are to be retained. The MP model is not subject to any such irritating restriction, and this is a significant result.

It is straightforward to transform the extended MP model to \( \sigma \) coordinates \( (\sigma = p/p_* \), where \( p_* \) is the surface pressure); the procedure is much the same as that applied by Miller and White (1984) to the MP model itself, and will not be described here. A computational penalty of using \( \sigma \) coordinates would be the presence of Lamb waves because the lower boundary condition would no longer be \( \omega = 0 \) at \( p = p_o \). However, in view of the absence of vertically propagating acoustic modes and the quality of the implied conservation properties, \( \sigma \)-coordinate forms of the extended MP model seem worth considering for use in mesoscale numerical modelling (at a horizontal grid length of about 10 km and a vertical grid length of 1 km or less).

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