Frontal circulations in the presence of small viscous moist symmetric stability and weak forcing

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SUMMARY

The Sawyer–Eliassen (S–E) equation for frontal circulation is extended to include the effects of eddy viscosity and small negative moist potential vorticity (MPV) under the condition of weak or, at least, not very strong frontogenetic forcing. When MPV is positive, viscosity can be neglected and this equation becomes the conventional S–E equation. When forcing is absent and MPV is strongly negative, this equation degenerates into the equation of linear viscous conditional (moist) symmetric instability (CSI). By using an idealized distribution of frontogenetic forcing which attenuates exponentially away from the position of maximum forcing, the extended S–E equation is solved both analytically and numerically for frontal circulations where frontogenetic forcing and negative MPV coexist (but the negative MPV is not so strong as to initiate viscous CSI). The solutions contain slantwise banded substructures on the warm side of the region of maximum forcing. The intensity, structure and scale of the bands are controlled by the competition between the frontogenetic forcing, negative MPV and eddy viscosity with the following characteristics:

(a) For a horizontally concentrated forcing whose attenuation length \( L_s \) (i.e. the length over which the forcing is reduced by a factor of \( e^t = 23.1 \)) away from the region of maximum forcing) is less than \( L_s / 2 \) (where \( L_s \) is the semi-geostrophic Rossby radius of deformation), the substructure is single-banded. The band occurs even before MPV becomes negative. As MPV decreases and becomes negative, the band becomes narrow and intense.

(b) For a widespread forcing whose attenuation length \( L_s \) is larger than \( L_s \), the substructure is multi-banded. The multiple bands may occur only if MPV is negative enough, and the first occurring multi-bands are characterized by wide bands of moist ascent with narrow and weak dry descents embedded between the moist bands. As MPV becomes further negative, the moist bands become narrow, intense and widely spaced.

(c) For a moderately widespread forcing whose attenuation length \( L_s \) is between \( L_s / 2 \) and \( L_s \), the substructure changes from wide and weak multiple bands to a narrow and strong single band as MPV becomes increasingly negative.

(d) As the coefficient of eddy viscosity increases within a moderately wide range (5–100 m² s⁻¹), the band (or bands) become weak and wide.

The theoretical findings are explained physically and compared with observations.

1. INTRODUCTION

The banded organization of frontal precipitation has received increased attention since the early 70s (Browning et al. 1973). Among theoretical explanations of these rainbands, moist symmetric instability, or conditional symmetric instability (CSI, here the word ‘conditional’ means that the moist processes are associated with upward motion), has been elucidated by Bennett and Hoskins (1979), Emanuel (1983) and Xu (1986). The possible importance of CSI in the formation of frontal rainbands has been supported by many observational studies (e.g. Bennetts and Sharp 1982; Parsons and Hobbs 1983). On the other hand, it was also shown recently by Sanders and Bosart (1985), observationally, and by Emanuel (1985) theoretically, that an intense band of frontal precipitation could evolve from a frontal circulation in the presence of small moist symmetric stability, or say, small positive moist potential vorticity (MPV). Numerical calculations of this situation by Thorpe and Emanuel (1985) demonstrated that latent heating could readily be concentrated on the mesoscale for a moderately small positive MPV, with the moist ascent most intense near and on the warm side of the axis of maximum geostrophic frontogenesis. These studies suggested that when a rainband occurred very close to the surface front, the formative mechanism could involve both of the above factors—frontogenetical forcing and weak moist symmetric stability manifested by small positive MPV. In this case, a combination of frontogenetic forcing and (inviscid)
CSI instability manifested by negative MPV could also have occurred in the real atmosphere. This latter mechanism has not been studied theoretically and there is a gap between the theories of CSI and frontal circulation in the presence of small moist symmetric stability.

There have been a number of observational studies in which close correlations between negative MPV (or inviscid CSI) and frontal rainbands are found. However, not many of these studies paid particular attention to the combined role of negative MPV and frontogenetic forcing, perhaps because in many of these cases the banded regions were not near the synoptic surface fronts. Sanders (1986) recently reexamined in detail a case for which negative MPV was previously found to be responsible for rainband formation (Seltzer et al. 1985). He found that the major band of ascent aloft was also closely related to frontogenetic forcing, and that the absence of obvious surface fronts in the rainband region could not be taken as indicating the unimportance of a frontal mechanism. This is clearly a case which falls into the above-mentioned gap in our theoretical understanding. In this paper, an effort is made towards filling this gap.

In the following section we derive an extended Sawyer–Eliassen (S–E) equation for frontal circulations in the presence of negative MPV and turbulent eddy diffusivity (eddy viscosity). In section 3 we solve the extended S–E equation analytically, by using the bulk eddy viscosity formulation of Xu (1986), and examine the single-band and double-band substructures for a wide parameter range. Numerical solutions for the extended S–E equation are obtained with a cubic-spline finite element method in section 4. Section 5 examines the self-consistency of the model and explains the theoretical findings. Comparisons of the model results with observations are given in section 6.

2. BASIC STATE AND GOVERNING EQUATIONS

Consider a basic state of unidirectional moist flow described by the following five parameters:

\[
F^2 = f(f + \eta), \quad S^2 = (g/\Theta_o)\partial_z \Theta_o = f\partial_z V_o, \quad N_d^2 = (g/\Theta_o)\partial_z \Theta_o, \\
S_w^2 = (\Gamma_w/\Gamma_d)(g/\Theta_o)\partial_z \Theta_e, \quad N_w^2 = (\Gamma_w/\Gamma_d)(g/\Theta_o)\partial_z \Theta_e
\]

where \(\eta = \partial_z V_o, V_o \) and \(\Theta_o\) are the basic wind and potential temperature satisfying the thermal wind relationship, \(\Theta_e\) is the basic equivalent potential temperature, and \(\Gamma_w\) and \(\Gamma_d\) are the moist and dry adiabatic lapse rates, respectively. We assume \(S_w^2 = S^2\), i.e. we neglect the horizontal variation of the basic moisture field in the moist ascent, and that the five parameters in (1) are functions of time and space \((x,z)\) on the synoptic or subsynoptic scale. The Coriolis parameter \(f\) is assumed constant.

The basic dry and moist potential vorticities (with factor \(g/\Theta_o\)) are

\[
q_d = N_d^2 F^2 - S^4 \quad \text{and} \quad q_w = N_w^2 F^2 - S^4
\]

respectively. When the moist potential vorticity (MPV) is negative \((q_w < 0)\), the basic state is unstable to inviscid moist symmetric circulations (Bennetts and Hoskins 1979). This type of instability is called moist symmetric instability, or conditional symmetric instability (CSI). (In this paper we do not consider evaporative cooling effects, so the moist processes are associated only with upward motion and the word 'moist' is equivalent to 'conditional'.) The CSI theory (Xu 1986, referred to as X86 henceforth) predicts that the most unstable inviscid CSI modes tend to have infinitely narrow zones of moist ascent, which suggests the importance of eddy viscosity. When the (bulk) effect of eddy viscosity is taken into consideration, the theory predicts that viscous CSI may occur only if MPV
FRONTAL CIRCULATIONS

is negative enough to overcome the effect of eddy viscosity, and the most unstable viscous CSI modes have narrow moist ascents of finite width. In order to examine how this prediction is affected by a frontogenetic forcing, we consider that the background field deviates from (1) by a small part \((u_g, v_g, \theta_g)\) which contains geostrophic stretching and/or shear deformation and satisfies the following non-divergence and thermal wind relationships:

\[
\alpha = -\partial_x u_g = \partial_y v_g \quad s^2 = (g/\Theta_o)\partial_x \theta_g = f \partial_z v_g \quad s^2 = (g/\Theta_o)\partial_y \theta_g = -f \partial_z u_g. \tag{3}
\]

A pure stretching (shearing) deformation corresponds to \(s^2 = s'^2 = 0\) \((\alpha = 0)\).

Ageostrophic perturbations can be induced by CSI and/or frontogenetic forcing. For a small deviation of geostrophic flow from the basic state (1), the forcing should be weak or, at least, not very strong. In this case, if the negative MPV is small and the viscous CSI is weak or absent, then the ageostrophic perturbation is expected to be small. We write the ageostrophic field as

\[
(u', v', w', g\theta'/\Theta_o, p'/\rho_o) = (u, v, w, \theta, p). \tag{4}
\]

The assumed smallness of (4) will be checked later against the solutions. We assume that (4) is two-dimensional, i.e. a function of \((t, x, z)\), representing a frontal circulation.

The total flow field consists of (1)–(4). Substituting the total flow field into the viscous Boussinesq equations, dropping the small nonlinear terms comprising the ageostrophic components in (4) and geostrophic deviation field in (3) and neglecting the viscous and time-derivative terms for the smooth and slowly varying geostrophic deviation field, we obtain

\[
Du - fv + \delta \partial_x p = 0 \tag{5a}
\]

\[
Df v + F^2 u + S^2 w = f[\nabla \cdot v - \partial_t - u_g \partial_x - \alpha]V_o = A \tag{5b}
\]

\[
D\theta + S^2 u + N^2 w = (g/\Theta_o)\nabla \cdot v - \partial_t - u_g \partial_x \Theta_o - s^2 V_o = B \tag{5c}
\]

\[
\partial_x u + \partial_x w = 0 \tag{5d}
\]

where

\[
N^2 = \begin{cases} \frac{N^2}{w} & \text{for } w > 0 \\ \frac{N^2}{\delta} & \text{for } w \leq 0. \end{cases} \tag{6}
\]

\(D = \partial_t - \nabla \cdot v\), \(\nabla = (\partial_x, \partial_z)\), \(v\) is the coefficient of eddy viscosity, and \(\delta = 1\) (or 0) is a trace index. If the basic state is unstable to viscous CSI and/or (4) contains inertial–gravity waves, then \(\delta = 1\) and (5)–(6) is an extension of the linear CSI equations (X86) with given geostrophic forcing terms on the right-hand side. If the basic state is stable to viscous CSI perturbations, then we choose \(\delta = 0\), i.e. drop the fast time variation terms and ageostrophic pressure terms, which filters the inertial–gravity waves consistently. In this paper we study only the latter case \((\delta = 0)\). With \(\delta = 0\), the four ageostrophic components \((u, v, w, \theta)\) and two tendency terms of the thermal wind field \((V_o, \Theta_o)\) can be solved from the equations (5a–e) and the thermal wind relationship.

Substituting \(\partial_x D(5c) - \partial_x D(5a)\) into \(\partial_x (5d) - \partial_x (5b)\) gives the following equation for the streamfunction \(\psi\) \((u = -\partial_z \psi\) and \(w = \partial_x \psi)\):

\[
[\partial_z (F^2 + D^2)\partial_x - \partial_z S^2 \partial_x - \partial_x S^2 \partial_x + \partial_x (N^2 + D^2)\partial_x] \psi = Q \tag{7}
\]

where

\[
Q = 2f\partial (V_o, u_g)/\partial (x, z) = 2[\alpha S^2 - \eta s^2] \tag{8}
\]
is the geostrophic forcing. When the basic state (1) is dry and stable, the differential operator in (7) is strongly elliptic and the ageostrophic circulation is smooth spatially, so \( D = 0 \) and (7) degenerates into the conventional S–E equation. As assumed in (1), (3) and (6), the forcing and parametric coefficients in (7) are functions of space. But the leading-order spatial variation for the parametric coefficients is the change in \( N^2 \) represented in (6). To facilitate the analytical approach in the following section, we will treat the parametric coefficients in (7) as constants in the dry and moist regions, respectively.

We may call (6)–(8) the extended Sawyer–Eliassen (S–E) equation, which is different from the extension by Thorpe and Nash (1984). In their formulation, the convective heating was parametrized and eddy viscosity was considered only for the geostrophic components, so the differential operator for \( \psi \) was exactly as in the conventional S–E equation. If the heating in their model is related to upward motion locally rather than parametrized with the vertical motion at the top of PBL, then their equation will be essentially the same as that of Emanuel (1985, E85 henceforth) or Thorpe and Emanuel (1985). In this case, if the MPV decreases toward zero, the solution of the ageostrophic circulation will become singular with the moist ascent concentrated into an infinitely thin slantwise sheet. The rapid variation of the ageostrophic motion in the vicinity of this thin singular sheet indicates the existence of an internal boundary layer (IBL) along the (slantwise) interface between the moist ascent and dry subsidence. Within this IBL the eddy viscous terms for the ageostrophic components, i.e. the highest derivative \( D \) terms in (5) or (7), become important. Thus, in order to eliminate the inviscid singularity and obtain a uniformly (in both dry and moist regions) valid solution in the presence of zero or negative MPV, we have to retain these \( D \) terms regardless of the smallness of \( \nu \). The above physical argument can be justified by IBL-scale analysis and the details are given in Xu (1989, X89 henceforth).

3. ANALYTICAL APPROACH

(a) Bulk eddy viscosity formulation and coordinate transformation

To solve for \( \psi \) analytically from (6)–(8), we use the bulk eddy viscosity formulation (24) of X86:

\[
- \nabla \cdot \nu \nabla = \nu (\pi/H)^2 \left[ 1/(l_0 \sin \varphi) \right]^2 + 1
\]

where \( H \) is the depth of the domain, \( l_0 \) is the non-dimensional (scaled by \( H \)) horizontal width of the narrowest (for multi-bands) moist ascent, \( \varphi \) is the slope angle (with respect to the horizontal) of the moist ascent, and thus \( l_0 \sin \varphi \) is the nondimensional spatial width of the moist ascent. We will mostly use \( \nu = 20 \text{m}^2\text{s}^{-1} \) for \( H = 10 \text{km} \) (or \( \nu = 5 \text{m}^2\text{s}^{-1} \) for \( H = 5 \text{km} \)), which gives the same non-dimensional value as in the study of the viscous pure CSI problem (X86). (Note that (9) is different from (24) of X86 by a factor of \( \pi^2 \), but \( \nu = 200 \text{m}^2\text{s}^{-1} \) was used in (24) of X86). We will also consider a variation of \( \nu \) within the range 10–10^2m^2s^-1 for \( H = 10 \text{km} \) (or 2.5–25 m^2s^-1 for \( H = 5 \text{km} \)).

We introduce the following coordinate transformation:

\[
\xi = (x - z/k)H, \quad \zeta = z/H \quad \text{with} \quad k = -(F^2 + K^2)/S^2
\]

where the vertical coordinate \( \zeta \) is slantwise with a slope determined by the balance between the inertial force and bulk eddy viscosity (cf. (14)–(16) of X86). Substituting (9)–(10) into (7) gives the following non-dimensional equation

\[
\left[ \partial^2_\xi + [B/(F^2 + K^2)] \partial^2_\zeta \right] \psi = [F^2/(F^2 + K^2)]Q
\]

(11)
where

\[
B = \begin{cases}
B_w = N_w^2 + K^2 - S^4/(F^2 + K^2) = [q_w + \delta_w(K)]/[F^2 + K^2] & \text{for } w > 0 \\
B_d = N_d^2 + K^2 - S^4/(F^2 + K^2) = [q_d + \delta_d(K)]/[F^2 + K^2] & \text{for } w \leqslant 0
\end{cases}
\]

\[\delta_w(K) = K^2(N_w^2 + F^2 + K^2) \text{ and } \delta_d(K) = K^2(N_d^2 + F^2 + K^2).\]

Here \(Q\) is normalized by its amplitude \(Q_0\), so that \(Q \sim Q_0/\psi\). \(\psi\) is non-dimensional and the scaling is \(\psi = (Q_0 H^2/F^2)\). If viscosity vanishes, then \(K \to 0\), \(B \to q/F^2\) where \(q = q_d\) (as \(w \leqslant 0\)) or \(q_w\) (as \(w > 0\)), and the coefficient of the horizontal derivative term in (11) gives \(B/(F^2 + K^2) \to q/F^4\). Since \((F^2 + K^2)/B_d \approx q^4/q_d \ll 1\) and \(F^2/\sqrt{q_d}\) is the intrinsic aspect ratio of dry semi-geostrophic space, it is convenient to define the semi-geostrophic Rossby radius of deformation \(L_s\) and viscous semi-geostrophic Rossby radius of deformation \(L_v\) as

\[
L_s = H\sqrt{(q_d/F^2)} = L_R \sqrt{(1 - 1/Ri)}, \quad L_v = H\sqrt{(B_d/(F^2 + K^2))}
\]

where \(L_R = HN_d/F\) is the conventional Rossby radius of deformation and \(Ri = N_d^2 F^2/S^4\) is the dry Richardson number. For the parameter range used in this paper, \(K^2 \ll F^2\) and the relative error for \(L_v = L_s\) is less than 5\%.

In the dry region (11) can be normalized as

\[
[\partial^2_{\xi} + r^2 \partial^2_{\eta}]\psi = [F^2/(F^2 + K^2)]Q \quad \text{for } \partial \psi/\partial \chi \leqslant 0
\]

where the new coordinate \(\chi = H\xi/L_v\). In the moist region (11) becomes

\[
[\partial^2_{\xi} - r^2 \partial^2_{\eta}]\psi = [F^2/(F^2 + K^2)]Q \quad \text{for } \partial \psi/\partial \chi > 0
\]

where

\[-r^2 = B_w/B_d = [q_w + \delta_w(K)]/[q_d + \delta_d(K)].\]

Note that \(\delta_d(K) \approx K^2 N_d^2 \ll q_d\), so \(B_d \approx q_d\). However, since \(q_w\) is small (negative), \(\delta_w(K)\) has a significant contribution to \(B_w\), which renders \(r^2\) 20–30\% smaller than \(-q_w/q_d\) within the parameter range used in this paper. This implies that the bulk eddy viscosity modifies the moist flow significantly but gives little impact to the dry flow.

The forcing in (11)–(14) is normalized by its amplitude \(Q_0\), so \(|Q| \ll 1\) and the forcing distribution can be assumed as

\[
Q = e^{-\pi |\xi|/\xi_b} = e^{-\pi |\xi|/\xi_p}
\]

where \(\xi_b\) (or \(1/b\)) is the non-dimensional distance scaled by \(H\) (or by \(L_v\)) over which the forcing attenuates by a factor of \(e^2 = 23.1\) along the horizontal direction away from the region of maximum forcing. We call the dimensional value \(L_b = H\xi_b = L_v/b\) the attenuation length of the forcing. The forcing distribution (15) is as (19) of E85, except that here the maximum forcing line (i.e. the coordinate \(\xi\)) is slightly steeper than the absolute momentum coordinate used by E85. (Note that \(K^2 \ll F^2\) and \(k = -F^2/S^2\) in (10).)

The boundary conditions for (13)–(14) are

\[
\psi = 0 \text{ on } \xi = 0,1 \quad \text{and } \psi \to 0 \text{ as } \xi \to \pm \infty.
\]

The mass continuity and continuity of the forcing \(B\) in (5d) require the following matching conditions on the interfaces:

\[
\psi \text{ and } \lambda^2 \partial \psi/\partial \chi \text{ continuous across } \chi = \chi_i, \chi_i + \delta_i
\]

where \(\lambda^2 = B/B_d = 1\) (or \(-r^2\)) in the dry (or moist) region, \(\chi_i\) is the position of the left
(cold side) boundary of the \(i\)th moist ascent and \(\delta_i\) is the width of the \(i\)th moist ascent. Here we assume that the interfaces can be prescribed approximately by slantwise lines parallel to the coordinate \(\xi\). This assumption will be verified later. The matching conditions at \(\chi = 0\) are

\[
\psi \text{ and } \frac{\partial \psi}{\partial \chi} \text{ continuous across } \chi = 0.
\]  

(b) Single-band solutions

When \(-r^2 < 0\), the solution of (13)–(18) with a single moist ascent has the following form

\[
\psi = \sum_{\text{odd}} [G(n) \ e^{ab\chi} + a_1(n) \ e^{an\chi}] \sin(n\pi\chi) \quad \text{for } -\infty < \chi \leq 0
\]  

\[
\psi = \sum_{\text{odd}} [G(n) \ e^{-ab\chi} + a_2(n) \ \cosh(n\pi\chi) + a_3(n) \ \sinh(n\pi\chi)] \sin(n\pi\chi)
\]  

\(\text{for } 0 < \chi \leq \chi_1\)

\[
\psi = \sum_{\text{odd}} [C(n) \ e^{-ab\chi} + a_4(n) \ \cos(\mu_n\chi') + a_5(n) \ \sin(\mu_n\chi')] \sin(n\pi\xi)
\]  

\(\text{for } 0 < \chi' = \chi - \chi_1 \leq \delta_1\)

\[
\psi = \sum_{\text{odd}} [G(n) \ e^{-ab\chi} + a_6(n) \ e^{-an\chi}] \sin(n\pi\chi) \quad \text{for } 0 < \chi' \equiv \chi - \chi_1 - \delta_1 < \infty
\]

where \(\mu_n = n\pi/r\) and

\[
G(n) = \begin{cases} 
4[F^2/(F^2 + K^2)]/[n\pi^3(b^2 - n^2)] & \text{for } b \neq 1 \\
-2|\chi| [F^2/(F^2 + K^2)]/(n^2\pi^2) & \text{for } b = 1
\end{cases}
\]

\[
C(n) = -4[F^2/(F^2 + K^2)]/[n\pi^3(r^2b^2 + n^2)].
\]

The coefficients \(a_1(n)\)–\(a_6(n)\) can be determined explicitly by substituting (19) into the matching conditions (17)–(18) and the details are given in the appendix. The structure of a typical solution is shown in Fig. 1.

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**Figure 1.** Single-band frontal circulation for \(b = 5\) and \(r = 0.54\). The width and position of moist ascent are \(\delta_1 = 0.24\) (\(L_uH = 170\) km for \(H = 10\) km) and \(\chi_1 = 0.048\) (\(\xi_uH = 34\) km), respectively. The solution is also the preferred \(S\) mode for the external parameter setting: \(q_0/q_1 = -0.4\) (or \(N_x^2/N_x = 0.3\)), \(Ri = 2\), \(F/N_u = 0.01\), \(\xi_u = 4.5\pi\) (\(L_u = 140\) km), \(\nu^* = \nu/(FH^2) = 0.02\) (\(\nu = 20\) m s\(^{-1}\) for \(F = 10^{-8}\) s\(^{-1}\) and \(H = 10\) km). Contour intervals are 0.008 (which is nondimensional and scaled by \(Q_H F^2 / F^3\), so the dimensional value is 800 m s\(^{-1}\) for \(Q_H = 10^{-10}\) s\(^{-2}\)).
So far (19) has not been checked with the condition (6), which requires that upward motion should occur in the moist region with downward motion elsewhere. This condition can be satisfied approximately if it is satisfied for the leading component \((n = 1)\) of the vertical modes, because all the higher components \((n > 1)\) are much smaller than the leading component. The situation is as in E85 and we will only consider the leading component \((n = 1)\).

The structure of solution (19) in \((\xi, \chi)\) space is described completely by the two internal parameters \((\chi_1, \delta_1)\) and two external parameters \((r, b)\). For a given pair of the external parameters \((r, b)\), we need to check the existence of the solution, or, say, check whether (6) is satisfied by the leading component \((n = 1)\) of (19). The shaded areas (with either solid or dashed boundaries) and dashed lines (which are actually very thin shaded areas) in Fig. 2(a) are the domains where solutions exist in the parameter subspace \((r, \delta_1)\) with fixed \(b = 5\) and \(\chi_1 = 0\). Along a fixed level of \(r = \) constant, a solution exists wherever the level intersects the shaded domains. As shown, the width \(\delta_1\) of the moist ascent of the solution is not unique. When \(\chi_1\) increases from 0 to 0.1, the shaded areas change their shapes continuously from Fig. 2(a) to Fig. 2(b). Thus, the location \(\chi_1\) of the moist

\[
\begin{align*}
\text{(a)} \\
\text{(b)}
\end{align*}
\]

Figure 2. Domains for the existence of single-band solution in the parameter subspace \((r, \delta_1)\), as indicated by the largest heavily hatched area (domain S), small lightly hatched area, and dashed lines (which are actually very thin domains). The parameter settings are: (a) \(b = 5\) and \(\chi_1 = 0\); (b) \(b = 5\) and \(\chi_1 = 0.1\). The three thick curves show the variations of \(r\) with \(\delta_1\) for fixed values of MPV at: \(q_w/q_d = -0.1\), \(-0.3\) and \(-0.5\), respectively, while the other external parameters are fixed as in Fig. 1.
ascent of the solution is also not unique. When $\chi_1$ further increases and becomes larger than the attenuation length $1/b$, the small shaded domains vanish and the largest shaded domain shrinks into a thin layer along the lower boundary $r = r_o(\delta_1)$—the critical curve of the bulk viscous CSI.

According to (A.11) of X86, the pure CSI mode with single moist ascent of width $\delta_1$ will become unstable if $-r^2$ becomes lower than the critical value $-r_0^2$, where $r_0 = r_o(\delta_1)$ is the smallest positive root of $r_o = \tan(\pi\delta_1/2r_o)$. The single-band solution (19) can exist stably only if $-r^2$ is above the critical value $-r_0^2$, so $r_o = r_o(\delta_1)$ is the lower boundary for the largest shaded domain in Fig. 2. However, the upper boundary $r = r_1(b, \chi_1, \delta_1)$ of the largest shaded domain changes with $\chi_1$ and $b$. Note that $\mu\delta_1 = \pi\delta_1/r$ is the total wavenumber for the horizontal structure of the moist ascent. For a solution in the largest shaded domain (with solid boundaries), it is found that $\mu\delta_1 < \pi/2$, and thus the moist ascent has a single core. For a solution in the small shaded domains (with dashed boundaries or indicated by dashed lines), $\mu\delta_1 > \pi/2$, and thus the moist ascent has more than one core. Because the bulk eddy viscosity formulation (9) is proposed for moist ascents of single-core type, we will only consider the solutions in the largest shaded domain, or say, domain S for short. The solutions in domain S are called S modes.

In the physical space, the independent external and internal non-dimensional parameters are

$$ (q_w/q_d, Ri, F/N_d, \xi_b, \nu^*) \quad \text{and} \quad (\xi_1, l_o) \quad (20) $$

respectively. Here $q_w/q_d$ is the normalized MPV (or MPV for short), $Ri$ is the Richardson number defined in (12), $\nu^* = \nu/(FH^2)$, and $(\xi_b, \xi_1, l_o) = (L_w/H)(1/b, \chi_1, \delta_1)$, so that $\xi_b$ is as in (15), $l_o$ is as in (9), and $H/L_w \approx H/L_e$—the intrinsic aspect ratio of the dry semi-geostrophic space. Now we need to examine the existence of S modes in the physical parameter space (20). First, from the definition of $-r^2$ in (14), we note that $r$ is a function of both the basic state parameters and the bulk eddy viscosity $K$. Since $K$ depends on the width of the moist ascent, $r$ is not a purely external parameter. The bulk eddy viscosity $K$ and slope $k = \sin \varphi$ can be obtained together with $L_w$ as implicit functions of $(q_w/q_d, Ri, F/N_d, \nu^*, l_o)$ from (9)–(10) and (12). In other words, when the external parameters $(q_w/q_d, Ri, F/N_d, \nu^*, l_o)$ are fixed, $(L_w, K)$ and thus $r$ can be determined uniquely as functions of $l_o$ or $\delta_1$. The thick curves in Fig. 2 are the variations of $r$ with $\delta_1$ for differently fixed external parameters. Each curve is associated with a given value of MPV (i.e. $q_w/q_d$). When the moist ascent becomes wide, the bulk eddy viscosity $K$ decreases and $-r^2$ approaches the inviscid limit $q_w/q_d$ rapidly. Thus, the asymptotic level of each curve corresponds to the square root of the negative of the MPV. As MPV decreases and becomes negative, the associated curve moves downward into the figure from the upper-right corner. When the negative MPV becomes large enough, the curve intersects domain S indicating the occurrence of S modes. If MPV further decreases and the curve touches the lower boundary $r = r_o(\delta_1)$ of domain S, then the viscous CSI mode will become marginally unstable at the tangent point.

Figure 3 shows the marginal occurrence curves of S modes and critical curve of CSI modes with $F/N_d = 0.01, Ri = 2.0$ and $\nu^* = 0.002$. If $\nu^*$ and/or $Ri$ increase (decrease), then these curves will shift downward and rightward (upward and leftward). From Fig. 3 we can see that as MPV decreases to $q_w/q_d = -0.12$, an S mode occurs first at $(\xi_1, l_o) = (2.0, 22.0)$. When $q_w/q_d < -0.12$, the S mode becomes non-unique because $(\xi_1, l_o)$ can vary within a certain range. As MPV decreases to $q_w/q_d = -0.53$, the marginally unstable CSI mode occurs at $l_o = 14.0$. When $q_w/q_d < -0.53$, the unstable CSI mode becomes non-unique because $l_o$ can vary within a certain range. In this case the most unstable,
i.e. the fastest growing, CSI mode is traditionally considered to be the physically preferred one, which is determined uniquely. Following this idea, when MPV is lower than $-0.12$ in Fig. 3, we assume that the physically preferred S mode is the strongest one, which gives the maximum upward transport of the moist air mass.

(c) Preferred single-band solutions

Because the bulk eddy viscosity (9) is used, the above single-band solutions satisfy only two very basic interface conditions, i.e. the mass continuity and continuity of the forcing as shown in (17). Like many other types of bulk viscosity, the along-interface velocity ($\partial \psi / \partial \chi$) is discontinuous across the interfaces. This discontinuity is unrealistic and renders the width and location of the moist ascent not uniquely determined. As shown in section 4, when the differential form of eddy viscosity formulation is retained, Eq. (7) and the associated boundary and interface conditions allow only one solution for a given set of external parameters in (20), so the non-uniqueness is a spurious feature resulting from the bulk approximation of eddy viscosity.

To eliminate the spurious non-uniqueness, we assume that the physically preferred S mode is the one which gives the maximum upward transport of the moist air mass, i.e.

$$\Delta \psi = \max_{(\xi_1, l_o)} [\psi(\xi_1 + l_o) - \psi(\xi_1)] \quad \text{at} \quad \zeta = 0.5$$

for a given set of the external parameters in (20). The solution in Fig. 1 is also an example of the preferred S modes.

The preferred S modes have been identified in a broad range of MPV with fixed $Ri = 2$ and $\xi_b = 4.5\pi$. The results are presented in Fig. 4. The thick curve in Fig. 4(a) shows that the amplitude $\Delta \psi$ of the preferred S mode increases as MPV decreases. The dashed vertical line indicates the critical value of MPV for the onset of viscous CSI. When MPV approaches this critical value, the amplitude $\Delta \psi$ increases rapidly. The circles denote the amplitudes of the numerical solutions obtained in section 4. The thin curve is the amplitude of the preferred double bands (see the next subsection). Clearly, in this case, the double bands are not the most preferred modes.

Figure 4(b) shows how the width $l_o$ (thick curve) and central position $\xi_1 + l_o/2$ (thin curve) of the moist ascent of the preferred S mode change with MPV. When MPV decreases, the band becomes narrow and shifts slightly towards the region of maximum forcing. The values of $N^+_w/N^+_d = (q_w/q_d)(1 - Ri^{-1}) + Ri^{-1}$ are also labelled in Fig. 4(b).
The thick curve approaches the asterisk, which represents the onset point of viscous CSI, suggesting a continuous transition of structure between the preferred S mode and CSI mode. The squares and triangles denote, respectively, the widths of moist ascents and locations of moist ascent centres for the numerical solutions obtained in section 4.

Figures 4(c, d) are as Fig. 4(b) except that $Ri = 4$ and $4/3$ (instead of $Ri = 2$), respectively. When MPV is fixed and $Ri$ decreases (i.e. both the basic shear $S^2$ and moist stratification $N^2$ increase), the moist ascent of the preferred S mode becomes wide and shifts slightly away from the region of maximum forcing. The amplitude variation for the
preferred S modes in Figs. 4(c, d) (not shown) is very similar to the thick curve in Fig. 4(a).

In Figs. 1–4 we set \( b = 5 \) (\( \xi_b = 4.5\pi \)) which is about the same value (\( b = 5.1 \)) as in E85. This value may represent the concentrated forcing observed by Sanders and Bosart (1985, see their Fig. 6f). However, in many other cases of frontal rainbands (e.g., Browning et al. 1973), the frontogenetic forcing seemed widespread. In this case, we may choose \( \xi_b = 50\pi \) (\( b = 0.45 \)). With this setting of forcing attenuation length, we found that no single-band solution existed unless the condition (6) was modified to allow dry upward motion some distance away (to the warm side) from the moist ascent. Under this modified condition, the ‘preferred’ single-band S modes can be found and an example is shown in Fig. 5. As shown, the dry subsidence to the warm side of the moist ascent covers only a limited region. Further away from the first moist ascent the air is forced to rise, suggesting the occurrence of the second band. Multi-band solutions are examined in the next subsection.

![Figure 5](image)

Figure 5. As Fig. 1 but for \( \xi_b = 50\pi \) (\( L_b = 785 \) km for \( H = 5 \) km) and \( q_w/q_c = -0.3 \). The internal parameters are \( \delta_1 = 0.3 \) (\( L_0H = 105 \) km) and \( \chi_1 = 0.33 \) (\( \xi_1H = 115 \) km). Note that, with \( H = 5 \) km, \( \nu^* = 0.002 \) corresponds to \( \nu = 5 \text{m}^2\text{s}^{-1} \) and the contour interval 0.008 corresponds to 200 \( \text{m}^2\text{s}^{-1} \).

\[(d) \text{ Multi-band solutions}\]

Multi-band solutions are obtained by adding more moist bands to (19), so there are more pairs of internal parameters \((\chi, \delta)\) (see (17)). For example, the double-band solution has the following form

\[
\psi = \sum_{\text{odd}} [G(n) e^{n\chi} + a_1(n) e^{n\pi \xi}] \sin(n\pi \xi) \quad \text{for} \ -\infty < \chi \leq 0
\]

\[
\psi = \sum_{\text{odd}} [G(n) e^{-n\chi} + a_2(n) \cosh(n\pi \chi) + a_3(n) \sinh(n\pi \chi)] \sin(n\pi \xi)
\]

\[\text{for} \ 0 < \chi \leq \chi_1\]

\[
\psi = \sum_{\text{odd}} [C(n) e^{-n\chi'} + a_4(n) \cos(\mu_0 \chi') + a_5(n) \sin(\mu_0 \chi') \sin(n\pi \xi)]
\]

\[\text{for} \ 0 < \chi' = \chi - \chi_1 \leq \delta_1\]

\[
\psi = \sum_{\text{odd}} [G(n) e^{-n\chi} + a_6(n) \cosh(n\pi \chi') + a_7(n) \sinh(n\pi \chi')] \sin(n\pi \xi)
\]

\[\text{for} \ 0 < \chi' = \chi - \chi_1 - \delta_1 \leq \chi_2 - \chi_1 - \delta_1\]
\[ \psi = \sum_{\text{odd}} [C(n) e^{-\chi \beta} + a_1(n) \cos (\mu_n \chi') + a_2(n) \sin (\mu_n \chi')] \sin(n \pi \xi) \]  
for \( 0 < \chi' = \chi - \chi_2 < \delta_2 \) \tag{21e} 

\[ \psi = \sum_{\text{odd}} [G(n) e^{-\chi \beta} + a_{i0}(n) e^{-n \pi \xi}] \sin(n \pi \xi) \]  
for \( 0 < \chi' = \chi - \chi_2 - \delta_2 < \infty \) \tag{21f}

where \( \mu_n, G(n) \) and \( C(n) \) are as in (19). The coefficients \( a_1(n) - a_{i0}(n) \) are given in the appendix. Here, again, we consider only the leading order component \( (n = 1) \) of (21). An example is shown in Fig. 6, which is also the preferred double-band solution for the same external parameter setting as in Fig. 5.

In principle, the preferred double-band solution for a fixed set of external parameters should be sought in the four-dimensional parameter space \((\chi_1, \delta_1, \chi_2, \delta_2)\). However, since the numerical results in the next section indicate that the moist ascents for a multi-band solution have approximately the same width, we may choose \( \delta_1 = \delta_2 = Hl_0/L_v \) and reduce the number of internal parameters. In this case, it is convenient to use the three internal parameters \((l_0, \xi_2, \Delta \xi)\), where \( \Delta \xi = \xi_2 - \xi_1 = (\chi_2 - \chi_1) L_0/H \) is the band spacing (scaled by \( H \)), \( l_0 \) and \( \xi_1 \) are as in (20). The preferred double band is the one which gives the maximum total upward transport of the moist air, i.e.

\[ \Delta \psi = \max_{(l_0, \xi_1, \Delta \xi)} [\psi(\xi_1 + l_0) - \psi(\xi_1) + \psi(\xi_1 + \Delta \xi + l_0) - \psi(\xi_1 + \Delta \xi)] \quad \text{at} \quad \xi = 0.5 \]

under the condition (6). It is found that when the forcing is concentrated (typically \( b > 2 \), i.e. \( L_0 < L_g/2 = 35H \) as in Fig. 1) the single band is preferred over the multi-bands (see the thin curve in Fig. 4(a)). When the forcing is widespread (\( b < 1 \), i.e. \( L_0 > L_g \)), the multi-bands become preferred. When the forcing is moderately widespread (\( 1 < b < 2 \), i.e. \( L_g/2 < L_b < L_g = 70H \)), the situation becomes complicated. For example, when \( L_b = 53H \), there is a transition of the preferred mode from double bands to a single band at \( q_0/q_d = -0.34 \) (see Fig. 7).

Figure 8(a) shows the amplitude of the preferred double bands (thick curve) and the amplitude of the preferred single band (thin curve) as functions of MPV for \( \xi_b = L_b/H = 50 \pi \). The double bands are more preferred than the single band, although both curves increase rapidly as MPV approaches the critical value of viscous CSI (i.e. the dashed vertical line). The band spacing \( \Delta \xi \) for the preferred double bands is shown by
the dashed curve in Fig. 8(a), while the width \( l_0 \) and central position \( \xi_1 + l_0/2 \) of the first band are shown by the thick and thin curves in Fig. 8(b), respectively. These curves indicate that as MPV decreases, the moist ascents of the preferred double bands become narrow and shift away from each other. Note that \( \xi_1 + l_0/2 \) in Fig. 8(b) is about 3 times as large as in Fig. 4(b), so the band (or bands) shift towards the warm side as the forcing becomes more widespread. As MPV approaches the onset point of the viscous CSI (the asterisk in Fig. 8(b)), the double bands evolve into two loosely coupled single-band CSI modes. The numerical results obtained in section 4(c) are also plotted in Fig. 8. The comparisons are generally good, except for the band spacing. Here it is necessary to

Figure 8. (a) As Fig. 4(a), except that \( \xi_0 = 50\pi \) (as in Figs. 5–6), the thick (thin) curve is for the preferred double-band (single-band) solutions, and the dashed curve is for the band spacing \( \Delta \xi \) of the preferred double-band solutions. The circles and squares indicate the amplitudes and band spacings of the numerical triple-band solutions, respectively. (b) As Fig. 4(b), except for \( \xi_0 = 50\pi \) and for the preferred double-band solutions in (a).
point out that when \( L_0 > L_e \), like the single-band solution in Fig. 5, the double-band solutions in Figs. 6 and 8 satisfy only the modified condition (6). In other words, dry upward motion (though extremely weak and too weak to be shown in Fig. 6) still occurs to the warm side some distance away from the second moist band. Actually, we can show (see section 5(b)) that for \( L_0 > L_e \) the (leading mode \( n = 1 \)) solution satisfying the exact condition (6) should contain an infinite number of moist bands. But the band amplitude attenuates rapidly as the band location moves further away from the region of maximum forcing and only the first few bands are significant. Thus, the results obtained for the double-band solutions in this subsection will largely remain the same if more bands are added to the solutions.

4. NUMERICAL SOLUTIONS

(a) Variational formulation and finite element method

Since the derivation of the variational principle for the extended S–E Eq. (7) and detailed proofs of existence, uniqueness and stability of the solutions can be found in X89, here we only need to state some of the results which are important to the numerical solutions. It is shown in X89 that the solution will be stable to viscous CSI if the eddy viscosity is strong and/or the negative MPV is small, more specifically, if the following functional

\[
\Phi_e(\psi) = \int \left[ (D \partial_z \psi)^2 + (D \partial_x \psi)^2 + F^2(\partial_z \psi)^2 + N^2(\partial_x \psi)^2 - 2S^2 \partial_x \psi \partial_z \psi \right] dx dz
\]  

(22)

is positive definite under the constraint (6) for \( \psi \in W^0 \), where \( W^0 \) is the completion (in the norm \( \int \int |D \nabla \psi|^2 \, dx dz \)) of the functional space of all smooth functions \( \psi \) satisfying the following boundary conditions:

\[
\begin{align*}
\psi & = \partial_z \psi = D \partial_z \psi = 0 \quad \text{on} \quad z = 0, H \\
\psi & = \partial_x \psi = D \partial_x \psi = 0 \quad \text{on} \quad x = \pm L \text{ or as } x \to \pm \infty.
\end{align*}
\]  

(23)

Note that \( D \) is defined in (5) but with \( \delta = 0 \). Clearly, (23) is for nonslip rigid boundaries: \( u = w = v = 0 \) on \( z = 0, H \); and \( u = w = \theta = 0 \) on \( x = \pm L \). It is also shown in X89 that the following functional, if \( \Phi_e(\psi) \) is positive definite,

\[
\Phi(\psi) = \Phi_e(\psi)/2 + \int \int \psi Q \, dx dz
\]  

(24)

has at least one minimum under the constraint (6) and the function \( \psi \in W^0 \) which minimizes (24) under the constraint (6) is a generalized solution of (6)–(7) and (23) with the following interface conditions:

\[
\psi, \nabla \psi, D\psi, D \nabla \psi, D^2 \partial_z \psi, (D^2 + N^2) \partial_x \psi \text{ continuous.}
\]  

(25)

The first four conditions in (25) are due to the continuity of \((u, w, v, \theta)\) across the interface. The last two conditions are due to the continuity of the forcing \((A, B)\) in (5). Note from (6) that \( N^2 \) is not continuous across the interface.

Numerically, the generalized solution is obtained in a truncated functional space \( \mathcal{W}^n \) (i.e. a subspace of \( W^0 \)) constructed by the two-dimensional cubic-spline basis functions on rectangular finite elements. Each element node \((i,j)\) is associated with \( 3 \times 3 = 9 \) basis functions and an element is associated with \( 4 \times 9 = 36 \) basis functions:

\[
B_{ij}^{\alpha \beta}(x, z) = B_{ij}^{\alpha}(x)B_{ij}^{\beta}(z), \quad i, j = 1, 2 \quad \text{and} \quad \alpha, \beta = 0, 1, 2
\]
where $B_{i}^{\alpha}(x)$ and $B_{j}^{\beta}(z)$ are the one-dimensional basis functions in $x$ and $z$ coordinates, respectively. The basis function $B_{i}^{\alpha}(x)$ and its derivatives of order $\alpha' = 0, 1, 2$ at the nodes $x = x_{i'}$ ($i' = 1, 2$) satisfy

$$\partial_{x}^{\alpha'} B_{i}^{\alpha}(x_{i'}) = \delta_{ii'} \delta_{\alpha \alpha'}$$  \hspace{1cm} (26)$$

where $\delta_{ii'}$ and $\delta_{\alpha \alpha'}$ are Kronecker deltas. The situation for $B_{j}^{\beta}(z)$ is similar. To construct these basis functions, we need to divide each element into 9 subelements (see Fig. 9). In this way, for each $B_{i}^{\alpha}(x)$ (with fixed $i$ and $\alpha$) over $[x_{1}, x_{2}]$, there are 12 constraints: 6 of them are given by (26) at the two nodes ($x = x_{1}$ and $x = x_{2}$) and the remaining 6 constraints are due to the continuity of $\partial_{x}^{\alpha} B_{i}^{\alpha}(x)$ ($\alpha' = 0, 1, 2$) at the two internal knots (i.e. $x = x_{1} + \Delta x/3$ and $x = x_{1} + 2\Delta x/3$ in Fig. 9). Thus $B_{i}^{\alpha}(x)$ can be determined uniquely by the cubic-spline Hermite interpolation, because each cubic polynomial contains 4 parameters over each interval of $\Delta x/3$ in Fig. 9 and the total number of parameters is $4 \times 3 = 12$ for each basis function $B_{i}^{\alpha}(x)$ over $[x_{1}, x_{2}]$. The detailed formulation for $B_{i}^{\alpha}(x)$ can be found in Yuan (1985).

![Figure 9. Rectangle element and its 9 subelements.](image)

The expression for the truncated solutions $\psi \in \mathcal{W}^{0}$ is

$$\psi(x, z) = \tilde{\psi}_{i}^{\alpha \beta} B_{i}^{\alpha \beta}(x, z)$$  \hspace{1cm} (27)$$

where the summation convention (implied by double indices) is used and $\tilde{\psi}_{i}^{\alpha \beta}$ is the value of $\partial_{x}^{\alpha} \partial_{z}^{\beta} \psi$ at the node $(x, z) = (x_{i}, z_{i})$. For each node there are 9 components of $\psi_{i}^{\alpha \beta}$ corresponding to $\alpha, \beta = 0, 1, 2$. The basic state parameter $N_{w}^{0}$ equals either $N_{w}^{2}$ or $N_{d}^{2}$ within each subelement, so a slantwise interface between the moist and dry regions is approximated by a staircase boundary between the moist and dry subelements (see Fig. 10(a)). By substituting (27) into (22) with a prescribed distribution of moist subelements and setting the variation $\delta \Phi(\tilde{\psi}_{i}^{\alpha \beta}) = 0$ with respect to all $\tilde{\psi}_{i}^{\alpha \beta}$ (except for those with given boundary values), we obtain a linear algebraic system for $\psi_{i}^{\alpha \beta}$. Once $\psi_{i}^{\alpha \beta}$ are calculated, (27) gives the truncated generalized solution. This solution, however, may not satisfy condition (6). If not, then adjust the moist region according to the vertical velocity field averaged over each subelement and iterate the above procedure until condition (6) is satisfied.

As mentioned earlier, the solution will be stable to viscous CSI if the functional (22) is positive definite. In the truncated space $\tilde{\mathcal{W}}^{0}$, this stability condition is equivalent to the definite positiveness of the matrix associated with $\psi_{i}^{\alpha \beta}$ in the above linear algebraic system. Thus, the viscous CSI stability can be easily (without an extra computation) checked as the solution is computed. All the numerical solutions presented in this paper are stable to viscous CSI.
To reduce the effect of the lateral boundary, the computation domain is chosen to be wide enough: $2L = 10L_R$, where $L_R = HN_d/F = 100H$ is the Rossby radius of deformation (see (12)). To increase the resolution in the central region of the domain where the frontogenetic forcing is large and banded substructure occurs, the domain is divided non-uniformly in the horizontal direction by 16 columns of elements (48 columns of subelements). In the vertical direction there are 8 equally divided layers of elements (24 layers of subelements). The open (half-width) subelements on each side of Fig. 10(a) are outside the central region $[-L_R, L_R]$, beyond which there are 5-5 more columns of subelements (not shown).

Figure 10. (a) Moist region and (b) streamfunction of the numerical solution with the same external parameter setting as in Fig. 1. Moist subelements are denoted by dark boundary lines. Plus signs at the centres of subelements indicate upward motion. Dashed line shows the axis of maximum forcing which is also along an absolute momentum surface. The horizontal scale is indicated by $L_R = HN_d/F = 100H$. Contour interval is 0-008 (as in Fig. 1).

(b) Single-band solutions

For the numerical solutions, the following form of frontogenetic forcing is used:

$$Q = \exp\{-\pi |x - z/k_o|/L_b\}$$

(28)

where $k_o = -F^2/S^2$ is the slope of the absolute momentum surface of the basic state and $L_b = H\zeta_b$ is the attenuation length as in (15). Within the parameter range used in this paper, we have $K^2 \ll F^2$ for the analytical solutions, so the slope $k$ defined in (10) is very close to $k_o$ and (15) is about the same as (28).

Figure 10 shows the moist region and streamfunction of the numerical solution obtained with the same external parameter setting as in Fig. 1. Moist subelements are plotted with dark boundary lines. If the vertical velocity averaged over a subelement is positive, then a plus sign is plotted at the subelement centre. As shown in Fig. 10(a), the moist subelements match the plus signs very well. (There are a few cases, e.g., Figs. 11 and 18, in which one or two moist subelements do not match the plus signs. We stopped the iteration at that point, because the un-matched one or two subelements began to switch back and forth between moist and dry conditions and the solution $\psi$ remained almost the same. In this case, the iteration was considered convergent within the accuracy of the resolution.)

The dashed line in Fig. 10 indicates the position of maximum forcing, which is along an absolute momentum surface. The moist ascent in the numerical solution is slightly steeper than the absolute momentum surface and tilts up into the warm side. The moist ascent and maximum forcing for the analytical solution in Fig. 1 are along the coordinate $\zeta$ which is also slightly steeper than the absolute momentum surface. Both the numerical
and analytical solutions have their moist ascents slanted between the absolute momentum surface and moist isentropic surface (not shown). Thus the moist inertial–buoyancy energy of the basic state is released efficiently in both solutions.

Figures 11 and 12 are as Fig. 10, except for \( q_w/q_d = -0.2 \) and 0.01, respectively. Note that the streamline intervals in Figs. 10, 11 and 12 are 0.008, 0.004 and 0.002 (the dimensional unit is \( Q_w H^2 F^2 \)), respectively. As MPV decreases and becomes negative, the moist ascent becomes narrow, intense and increasingly steeper than the absolute momentum surface. These characteristics are also seen from the analytical solutions in the previous section. In Fig. 12, \( q_w/q_d = 0.01 \) is the same small positive value used in Fig. 5 of E85. The inviscid moist ascent in Fig. 5 of E85 is of semi-infinite width, although it attenuates rapidly towards the warm side. However, the moist ascent in our Fig. 12 is of finite width. The dry subsidence to the warm side of the moist ascent is a very weak compensation branch of the closed circulation driven by eddy viscosity and the strong gradient of the forcing (manifested by the small value of \( L_b/L_s = 0.2 \)). The numerical solutions for \( q_w/q_d = 0 \) and 0.01 (not shown) are almost as in Fig. 12, so the viscous frontal circulation changes continuously as MPV decreases from positive to negative.

![Figure 11](image1)

**Figure 11.** As Fig. 10, except that \( q_w/q_d = -0.2 \) and the contour interval is 0.004.

![Figure 12](image2)

**Figure 12.** As Fig. 10, except that \( q_w/q_d = 0.01 \) and the contour interval is 0.002.
(c) Multi-band solutions

The numerical solution for $q_w/q_d = 0$ and $\xi_b = L_b/H = 50\pi$ is shown in Fig. 13. Because the forcing gradient is small and MPV is not negative, there is no substructure in Fig. 13. As MPV becomes negative and decreases to $q_w/q_d = -0.2$ (Fig. 14), the moist region splits into three bands in the upper and middle levels owing to the intrusion of two narrow and weak dry subsidences. As MPV decreases further down to $q_w/q_d = -0.3$ (Fig. 15), the dry subsidences become wide and deep down to the lower levels while the moist bands become narrow and intense. Note that the streamline intervals are 0-008 in Figs. 13 and 14, 0-004 in Fig. 11, and 0-002 in Fig. 12, so with a fixed value of MPV, the more widespread the forcing, the stronger the overall circulation. The intensity of the circulation increases with the volume-integral of the forcing.

As shown in Fig. 8, the amplitudes, widths and locations of the (first two) moist bands in the numerical solutions are close to the analytical solutions. The band spacings are less comparable. Note that the analytical solutions in Fig. 8 are double bands satisfying the modified condition (6). However, the structure of the numerical solution changes from about $1 + 0.6 + 0.4 = 2$ bands (Fig. 14) to 3 bands (Fig. 15) as MPV decreases from $q_w/q_d = -0.2$ to $-0.3$. This may explain why the dashed curve in Fig. 8(a) becomes significantly higher than the squares as MPV decreases from $q_w/q_d = -0.2$ to $-0.3$.

When the forcing is moderately widespread ($\xi_b = L_b/H = 17\pi$) and MPV is at the transitional point $q_w/q_d = -0.34$ (Figs. 7 and 16), the numerical solution contains only one major moist band while the other two moist bands are extremely weak. As revealed by other numerical solutions (not shown), when MPV decreases through the point $q_w/q_d = -0.34$, the dry subsidences between the moist bands become increasingly wide, push the second and third bands towards the warm (right) side and finally wipe them out completely. In this way, the multiple bands change into a single band.

5. Discussion

(a) Self-consistency of the model

We have shown that when the frontogenetic forcing and negative MPV coexist, banded substructures may occur in a frontal circulation before the basic state becomes unstable to the viscous CSI perturbations. The amplitudes of these bands are proportional to the amplitude of the frontogenetic forcing, so the model can be self-consistent as long as the forcing is weak and MPV is not close to the critical value for the onset of viscous CSI.

Now the question is whether the frontogenetic forcing required by the self-consistency of the model would be too small to make the model useful for real-data applications. To check this problem we note that the frontogenetical forcing observed by Sanders and Bosart (1985) and Sanders (1986) for two heavy snowband events in intensifying cyclones over the north-east United States had the maximum value about $Q_0 = 2 \times 10^{-11} s^{-3}$. The frontogenetic forcings in these two cases are strong and highly concentrated. Thus, we may choose $Q_0 = 10^{-11} s^{-3}$ as a typical value. With this $Q_0$ and the external parameter settings: $N_d = 10^{-2} s^{-1}$, $F = 10^{-4} s^{-1}$, $S^2 = 0.7 \times 10^{-6} s^{-2}$ (or $Ri = 2.0$), $q_w/q_d \geq -0.4$ (or $N_\theta^2/N_d^2 \geq 0.3$), $H = 10$ km, $\nu = 20$ m$^2$s$^{-1}$, we obtain the maximum ageostrophic wind: $u = 1$ m/s and $w = 7$ cm/s for the single-band solutions in Figs. 1 and 10, and $u \leq 2$ m/s and $w \leq 5$ cm/s for the multi-band solutions in Figs. 6 and 15. The magnitude of the basic thermal wind implied by the above external parameter setting is at least $O(V_0) = 0.5 HS^2/f = 35$ m/s, so the assumed smallness of the ageostrophic wind is satisfied. Note that the amplitudes $\Delta \psi$ for the above solutions are about 0.1 (see Figs. 4 and 8). If we choose $O(u/V_0) \approx O(2\Delta \psi/(V_0 H)) < 0.2$ to be the criterion for the model
Figure 13. As Fig. 10 but for $\xi = 50\pi$ and $q_w/q_d = 0$. Contour interval is 0.008 (as in Figs. 5–6).

Figure 14. As Fig. 13 but for $q_w/q_d = -0.2$.

Figure 15. As Fig. 13 but for $q_w/q_d = -0.3$. Note that the external parameter setting is the same as in Fig. 6.

Figure 16. As Fig. 10 but for $\xi_0 = 17\pi$ and $q_w/q_d = -0.34$ (the transitional point in Fig. 7).
self-consistency, then based on the above parameter setting the amplitude $\Delta \psi$ should be smaller than $0.5/\sqrt{Ri} = 0.35$. As shown in Figs. 4, 7 and 8, $\Delta \psi$ remains smaller than 0.35 until MPV decreases down to about $q_w/q_d = -0.5$ which is very close to the onset point of viscous CSI.

The model self-consistency can be similarly examined for different settings of $Ri$. For example, the amplitudes (not shown) of the solutions in Fig. 4(c) ($Ri = 4.0$ and $O(V_o) = 25 \text{ m/s}$) remain less than $0.5/\sqrt{Ri} = 0.25$ until $q_w/q_d < -0.23$. The amplitudes (not shown) of the solutions in Fig. 4(d) ($Ri = 4.3$ and $O(V_o) = 43 \text{ m/s}$) remain less than $0.5/\sqrt{Ri} = 0.43$ until $q_w/q_d < -1.1$. All these results show that for a broad range of $Ri$ the model can be self-consistent until MPV is very close to the onset point of viscous CSI.

Calculations have been done for different settings of $\nu^*$ (0.01–0.001). Two examples are shown in Figs. 17 and 18 where $\nu^* = 0.008$, $q_w/q_d = -0.6$ ($N_w^2/N_d^2 = 0.2$), and $\xi_b = 4.5\pi$ and $50\pi$ respectively. The rest of the external parameters are fixed as in Figs. 10–16. Note that the streamline contours are 0.002 in Fig. 17, 0.004 in Fig. 18, and 0.008 in Figs. 10 and 15, so the moist ascent becomes weak and wide as $\nu^*$ becomes large. In general, our calculations indicate that although the band width and intensity change with $\nu^*$, the dependence of the band occurrence and band structure on the basic state parameters (mainly MPV, Ri and $L_h/L_o$) remains qualitatively the same. For a moderately broad range (0.01–0.001) of $\nu^*$, the model can be self-consistent until MPV is very close to the onset point of viscous CSI.

It is necessary to point out that the model resolution is four times greater than the subelement meshes, because the basis functions are second-order smooth and, as mentioned earlier, each element node is associated with 9 basis functions. With this resolution the numerical diffusion is much smaller than the physical turbulent eddy diffusivity ($\nu^* = 0.01–0.001$) used in the model. Actually, the solutions were computed first by iterations with a coarse-element model. Each coarse element covers four elements

Figure 17. As Fig. 10, except that $q_w/q_d = -0.6$, $\nu^* = 0.008$, and the contour interval is 0.002.

Figure 18. As Fig. 17, except that $\xi_b = 50\pi$ and the contour interval is 0.004.
of the refined model. When the upward motion matched the moist subelements in the coarse-element model, the iteration was continued with the refined model. It is found that the streamfunctions finally obtained in the refined model are nearly the same as in the coarse-element model, indicating that the model resolution is high enough.

(b) Physical explanations of the model results

To explain the model results, we decompose the analytical solution (19) into two parts \( \psi = \psi_o + \psi_1 \), where \( \psi_o \) is the dry solution of (13) without moist effects and \( \psi_1 = \psi - \psi_o \) is the part due to moist effects. The leading mode \( (n = 1) \) of \( \psi \), has the following form

\[
\psi_o = G(1)[e^{-\pi b|x|} - (b/\pi)e^{-\pi|x|}]\sin(\pi \xi) \tag{29}
\]

where \( G(1) \) is given in (19). If \( b > 1 \) (or \( b < 1 \)), then \( G(1) > 0 \) (or \( G(1) < 0 \)) and the second (or first) term is dominant in (29), consequently \( \psi_o \) attenuates exponentially as \( e^{-\pi b|x|} \) (or \( e^{-\pi|x|} \)) as \( |x| \to \infty \). Thus, the dimensional attenuation length (denoted by \( L_w \)) for \( \psi_o \) and its associated vertical velocity \( w_o = \partial \psi_o / \partial x \) (> 0 to the warm side) is

\[
L_w = \max(L_b, L_v) = \max(L_b, L_s) \tag{30}
\]

where \( L_b \) is the attenuation length of the forcing defined in (15), \( L_v \), and \( L_s \) are the viscous and inviscid semi-geostrophic Rossby radii of deformation defined in (12), respectively. Since \( L_s \approx L_v \) (the relative error is within \( \pm 5\% \) for \( Ri \geq 1.25 \)) and \( L_s \) can be easily calculated from (12), we use \( L_s \) to replace \( L_v \) in the following discussions. As implied by (30), \( w_o \) represents the large-scale lifting. Two examples of the above decomposition for the cases of \( L_b < L_s \) and \( L_b > L_s \) are shown in Figs. 19(a, b), where the vertical velocity \( w \) (thick curve) at \( z = 0.5H \) for the solution in Fig. 1 or 5 is decomposed into \( w_o \) (dashed) and \( w_1 = \partial \psi_1 / \partial x \) (thin solid).

Note that the dry subsidence \( (w_1 < 0) \) induced by the moist ascent attenuates horizontally away from the moist ascent and the attenuation length is \( L_s \). Thus, when the forcing is concentrated \( (L_b < L_s) \) as shown in Fig. 19(a), the induced dry subsidence \( (w_1 < 0) \) suppresses the large-scale lifting \( (w_o > 0) \) over the entire warm-side dry region. However, when the forcing is widespread as shown in Fig. 19(b), the large-scale lifting is suppressed by dry subsidence only in the vicinity of the moist ascent. Further away from the moist ascent the dry subsidence attenuates rapidly and becomes weaker than the large-scale lifting, so the large-scale lifting will initiate a second moist band if the moisture supply is abundant. For the same reason, the dry subsidence induced by the second moist ascent will again attenuate faster than \( w_o \), indicating the occurrences of the third moist band, fourth moist band, and so on. Thus, under the exact condition (6), the analytical solution should have an infinite number of moist bands. As the band number

![Figure 19](attachment:figure19.png)

Figure 19. (a) Horizontal distribution (thick profile) of vertical velocity at middle level \( (z = H/2) \) for the single-band solution in Fig. 1. Thin-dashed profile is for the vertical velocity \( w_o = \partial \psi_o / \partial x \) induced purely by the frontogenetic forcing without latent heating. Thin-solid profile is for the vertical velocity \( w_1 = \partial \psi_1 / \partial x \) induced by latent heating. (b) As (a) but for the solution in Fig. 5.
increases and the band lies further away from the region of maximum $w_o$, its moist ascent diminishes rapidly. Only the first two or three bands are significant.

Since the basic state is stable to viscous CSI, the moist bands (or band) are supported partially by the large-scale lifting and partially by latent heating in the moist region. The band circulations are also resisted by eddy viscosity and by the inertial force and negative buoyancy in the dry region. Thus, the structure of the preferred bands (or band) is controlled by the following mechanisms:

(a) To be assisted favourably by the large-scale lifting $w_o$ the bands need to be wide and occupy as much as possible of the region of maximum $w_o$.

(b) To be supported effectively by latent heating the bands need to be narrow and widely spaced with relatively strong moist ascent and weak dry subsidence; this leads to fast latent heat release in the moist region and slow deposition of perturbation buoyancy energy back to the basic state in the dry subsidence region (see X86).

(c) To be less resisted by eddy viscosity the bands need to be wider and less densely packed.

For a given forcing, mechanism (a) acts at a fixed level of intensity, so the band structure is controlled by the competition between mechanisms (b) and (c), as with the situation for viscous CSI modes. Mechanism (b) tends to produce an infinitely narrow and intense band, like the inviscid CSI. This tendency is counteracted by the eddy viscosity. The preferred band width is determined by the balance between mechanisms (b) and (c) (with (a) fixed). A decrease of MPV will activate mechanism (b) and move the balance to a new level, rendering the bands narrower, more intense, and more widely spaced. However, unlike the viscous CSI, the balance between mechanisms (b) and (c) is assisted by mechanism (a), so it is not at the same level as for the viscous CSI. The contribution of mechanism (a) is reflected by the fact that the bands are wider and occur at smaller negative MPV than the viscous CSI modes.

For a moderately widespread forcing ($L_q/2 < L_b < L_a$), the distribution of $w_o$ is somewhere between the two dashed curves in Figs. 19(a), (b) and the region of maximum $w_o$ is comparable to the effective coverage of a moist band and its induced dry subsidence. When the negative MPV is not strong, the band is weak and its induced dry subsidence attenuates to an insignificant level within a short distance (think that the amplitude of the thin solid curve in Fig. 19(b) is largely reduced). Consequently, more bands can be packed in the region of maximum $w_o$ to obtain the maximum support from the large-scale lifting, while being still far enough from each other to be less affected by eddy viscosity. In this case, the second (or third) band provides an extra transport of moist air mass without significantly suppressing the first band, so the multi-band substructure is preferred over a single band. However, when the negative MPV decreases, the major (first) band becomes strong. Its induced dry subsidence pushes all the other bands away from the region of maximum $w_o$, rendering the single-band substructure preferred (see Figs. 7 and 16).

The dependence of the preferred band structure on the Richardson number $Ri$ (see Figs. 4(b)–(d)) can be explained by the fact that smaller $Ri$ manifests stronger basic shear and more stable moist stratification (with fixed MPV) and, thus, the moist ascent is more tilted and wider. The band positions (thin curves) in Figs. 4(b)–(d) and 8(b) are found to be very close to the positions of the maximum $w_o$ (see Fig. 19), indicating that the band position is largely controlled by the above mechanism (a).
Figure 20. The airflow associated with an ana-cold front, from Fig. 11 of BP73. The thin lines are streamlines relative to the moving system while thick lines represent the cold-frontal zone and the top of the convective boundary layer. Regions of saturated ascent are stippled.

6. COMPARISONS WITH OBSERVATIONS

Figure 20 is the schematic model of the transverse circulation associated with an ana-cold front (in which warm air ascends above a surface-based wedge of cold air), reproduced from Fig. 11 of Browning and Pardoe (1973, BP73 henceforth). The thin lines are streamlines relative to the moving system while thick lines represent the cold-frontal zone and the top of the convective boundary layer. According to BP73 (see their Fig. 2(c)), the frontogenetic forcing was concentrated along the cold-frontal zone, so comparisons can be made with the single-band solution in Fig. 10 where the dashed line of maximum forcing is considered as the position of a cold front.

Note that the circulation in Fig. 20 is the total wind. The cross-frontal geostrophic wind relative to the system can be represented approximately by a westerly shear flow (nondimensional)

$$u_g = -\frac{\partial \psi_g}{\partial z} = \beta(z - 0.5), \quad \beta \equiv -(1 + \eta)s^2/Q_o$$

(31)

where $u_g$ is scaled by $Q_o H/F^2$, $\eta$ and $s^2$ are defined in (1) and (3), respectively. By using $\beta = 0.3$ (the dimensional value for $\partial u_g/\partial z$ is $3 \times 10^{-4} \text{s}^{-1}$) and superimposing $\psi_g$ on the
ageostrophic circulation in Fig. 10(b), we obtain Fig. 21. Figures 10(a) and 21 are comparable with Fig. 20 in the following aspects:

(i) strong low-level inflow from the warm side onto the cold front;
(ii) slant moist ascent (about 150 km wide) above the cold frontal zone;
(iii) elongated circulation along the cold frontal zone;
(iv) slantwise descent underneath the frontal zone (upper troposphere extrusion);
(v) middle-level dry subsidence ahead of the cold front (not shown in Fig. 20 but mentioned in BP73).

![Figure 21. Streamfunction for the total circulation $\psi + \psi_x$, where $\psi$ is the solution in Fig. 10(a) and $\psi_x$ is the cross-frontal geostrophic streamfunction in (31) with $\beta = 0.3$ (i.e. $3 \times 10^{-5} \text{s}^{-1}$ for $Q_a = 10^{-3} \text{s}^{-1}$, $H = 10 \text{km}$ and $F = 10^{-9} \text{s}^{-1}$).](image)

As shown by the solutions in Figs. 10–12, the middle-level dry subsidence will become weak and move further to the warm side away from the cold front if MPV increases and becomes positive. This suggests that negative MPV is a crucial factor in producing a significant dry subsidence ahead of the cold front. This dry subsidence is dynamically produced by mechanisms (a)–(c) and its warming effect will enhance the temperature inversion and humidity drop at the top of the PBL and thus suppress convection over a broad region ahead of the cold front. Consequently, the only place where the convection can make an easy break is at the surface cold front. As the low-level easterly inflow travels through the broad area underneath the dry subsidence, it further concentrates the moist supply and southerly along-frontal momentum. These processes could contribute to the intensification of the narrow convective band and its associated low-level jet in Fig. 20, although the primary forcing for the narrow cold-frontal rainband seems to be due to the interaction of the advancing cold front with the warm sector PBL flow. Clearly, such a highly nonlinear small-scale feature is beyond the description of our model solutions.

Figures 22(a), (b) are reproduced from Figs. 4(b) and 8 of Browning et al. (1973). Figure 22(a) shows the warm-frontal precipitation in a partly occluded frontal system which traversed the Isles of Scilly on 18 January 1971. The stippled areas indicate the extent of surface rain while the further extent of precipitation is slightly stippled and the leading and rear edges of the dense high-level cirrus canopy are hatched. Figure 22(b) is the time–height cross-section in which the cloudy areas are stippled. The bands were quasi-stationary and moved with the same speed as the cold front ($6 \text{ m s}^{-1}$ faster than the warm front) while sliding up over the warm front. The overall up-sliding of the moist conveyor belt over the broad surface-based wedge of cold air in Fig. 22(b) suggested that the frontogenetic forcing, or, at least, the large-scale lifting was widespread. The surface
Figure 22. (a) Warm-frontal precipitation in a partly occluded frontal system, from Fig. 4(b) of Browning et al. (1973). The stippled areas indicate the extent of surface rain while the further extent of precipitation is slightly stippled and the leading and rear edges of the dense high-level cirrus canopy are hatched. The time scale indicates when the corresponding parts of the pattern passed over the surface station (Isles of Scilly).

(b) Time–height cross-section, from Fig. 8 of Browning et al. (1973). Cloudy areas are stippled.
and 1000–500 mb thickness analysis in Fig. 3 of Browning et al. indicated that the forcing was widespread and the region of maximum forcing was between the surface cold front and warm front. Thus, we may compare Fig. 22 with our multi-band solution in Fig. 15. Again, the dashed line of maximum forcing in Fig. 15 can be considered as the position of a cold-front zone. Note that the rainbands in Fig. 22(b) are confined within a relatively shallow layer between the upper-level over-running dry air and the surface-based wedge of cold moist air, so we choose $H = 5\text{ km}$ in Fig. 15 for comparison.

The similarities between the solution in Fig. 15 and the observations in Fig. 22 are as follows.

(i) The bands appear as a quasi-stationary substructure embedded in a large-scale, thermally direct circulation.

(ii) The bands are about 100 km wide, spaced 200–250 km apart, slanted towards the cold front, and close to the absolute momentum surface (suggested by the background shear in Fig. 1 of Browning et al.).

(iii) Band I is about 100 km ahead (to the warm side) of the cold-front zone (the axis of the maximum forcing).

(iv) Band III is very weak, broad and rather uniform.

(v) The bands lose their identity if the potential energy (proportional to the negative MPV) becomes exhausted. This feature was reported in Browning et al., and is comparable to the structural change of the solutions from Figs. 15 to 13.

The major difference between our solution and the observations is that in Fig. 22 band II is strongest while in Fig. 15 the first band is the strongest. Note that band I in Fig. 22 is in the developing stage during which the moist air in the warm conveyor belt is just starting to rise, while band II is a matured one for which the moist air in the warm conveyor belt has risen above the lifting condensation level. These observed features depend on the saturation processes in association with the fine structure of the warm front and moist conveyor belt, which are absent in our idealized model. Besides, the frontogenetic forcing responsible for the large-scale lifting (convergence) along the warm front in Fig. 22 could be very different from our idealized forcing, especially in its vertical structure. Despite all the above differences, the observations seem to support our hypothesis that widespread forcing (which could be in the form of large-scale lifting as in Fig. 22) favours the production of multiple bands.

We summarize the dependence of band structure on the basic state parameters $(Ri^{-1}, N_w^2/N_b^2, L_b/H, \nu^*)$ in Figs. 23–24, which may be checked with observational data and/or numerical simulations in the future. In Fig. 23, $E_0$ is the marginal instability curve for viscous CSI with $\nu^* = 0.002$ and $F/N_d = 0.01$, just as curve E in Fig. 9 of X86. The dashed diagonal line corresponds to the marginal inviscid CSI (zero MPV). Curve $E_1$ (along a–b–c) and curve $E_2$ (along b–d) show the marginal occurrences of single band and multiple bands, respectively. Figure 24 is as Fig. 23 except for $\nu^* = 0.008$. With figures of this kind our theory predicts the following:

(i) If the parameter point $(Ri^{-1}, N_w^2/N_b^2)$ falls into the region to the right of both curve $E_1$ and curve $E_2$, then there will be no banded substructure in the frontal circulation.

(ii) Single-band (multi-band) substructure will occur if the parameter point moves leftward and/or upward and meets curve $E_1$ (curve $E_2$).

(iii) If the parameter point passes curve $E_1$ and moves toward curve $E_0$, then the single band will become narrow and intense. When the parameter point passes beyond curve $E_0$, the band changes into a growing viscous CSI band.
(iv) If the parameter point passes curve $E_2$ and moves toward curve $E_0$ without passing curve $E_1$, then the bands will become narrow and intense and remain multiple. When the parameter point passes beyond curve $E_0$, the bands change into growing viscous multiple CSI bands. However, if the parameter point passes curve $E_1$ before it reaches curve $E_0$, then the multiple bands evolve into a single band.

(v) Curves $E_1$ and $E_2$ change with $L_b/L_R$, but curve $E_0$ does not. The topological structure of the diagram is determined by the positions of points a, b, c, and d. The two transitional points b and c correspond to $L_b/L_0 = 1/2$ and 1, respectively. Their vertical coordinates ($Ri^{-1}$) are determined by $L_b/L_0$ as follows:

$$Ri^{-1} = 1 - (2L_b/L_R)^2$$ for point b,  
$$Ri^{-1} = 1 - (L_b/L_R)^2$$ for point c

(32)

where (12) is used. As $L_b$ becomes large (small), points a, b, c, and d shift leftward
(rightward) along the lower boundary line, thin dashed diagonal line, curve \( E_0 \), and upper boundary line, respectively. As indicated by (32), if \( L_b > L_R/2 \) (and \( L_R \)), then point \( b \) (and point \( c \)) will move down outside the diagram and the first-occurring substructure will be multi-banded (and not transfer into a single band) for all \( Ri > 1 \).

(vi) As shown by Figs. 23–24, the diagram changes with \( \nu^* \).

The above picture can be severely distorted by other complications in the real atmosphere. For example, if the moisture supply is very limited, then we may expect a single band even if the parameter point is in the multi-band region. On the other hand, a single band may change into multiple CSI bands as the parameter point passes curve \( E_0 \) and moves further leftward. According to our other numerical solutions (not shown), when a single-band solution becomes locally unstable to viscous CSI, the band may split into smaller multi-bands regardless of the concentrated forcing. These fine multi-bands can be much more intense than the overall circulation, suggesting the occurrence of viscous multiple CSI modes. In view of the fact that sub-bands have also been observed and reported in the literature, the above model-generated fine multi-bands seem to deserve further investigation.

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APPENDIX

**Coefficients in solutions (19) and (21)**

The coefficients in (19) and (21) are obtained from the matching conditions (17)–(18). Here we only give the results for \( b \neq n \) \((n = 1, 3, 5, \ldots)\) as follows:

\[
a_1(n) = -[B_1(n) + B_{j+1}(n)]/[E_j(n) + E_{j+1}(n)]
\]

\[
a_i(n) = a_1(n)E_j(n) + B_1(n) \quad \text{for } i = 2, 3, \ldots J
\]

where \( J = 6 \) for (19), \( J = 10 \) for (21), and

\[
E_2(n) = E_3(n) = 1, \quad E_4(n) = \cos(n\pi \chi_1) + \sinh(n\pi \chi_1)
\]

\[
E_5(n) = -E_4(n)/r, \quad E_6(n) = E_4(n) \cos(\mu_n \delta_1) + E_5(n) \sin(\mu_n \delta_1)
\]

\[
E_7(n) = r[E_4(n) \sin(\mu_n \delta_1) - E_5(n) \cos(\mu_n \delta_1)]
\]

\[
E_8(n) = E_6(n) \cosh(n\pi \chi_2') + E_7(n) \sinh(n\pi \chi_2')
\]

\[
E_9(n) = -[E_6(n) \sinh(n\pi \chi_2') + E_7(n) \cosh(n\pi \chi_2')]/r
\]

\[
E_{10}(n) = E_8(n) \cos(\mu_n \delta_2) + E_9(n) \sin(\mu_n \delta_2)
\]

\[
E_{11}(n) = r[E_8(n) \sin(\mu_n \delta_2) - E_9(n) \cos(\mu_n \delta_2)]
\]

\[
B_2(n) = 0, \quad B_3(n) = (2b/n)G(n)
\]

\[
B_4(n) = B_3(n) \sinh(n\pi \chi_1) + [G(n) - \zeta(n)] e^{-n b x_1}
\]
FRONTAL CIRCULATIONS

\begin{align*}
B_5(n) &= -\left[ B_3(n)/r \right] \cosh(n \pi \chi_1) + \left[ C(n) + G(n)/r^2 \right] (\pi b/\mu_n) e^{-\pi b \chi_1} \\
B_6(n) &= B_4(n) \cos(\mu_n \delta_1) + B_5(n) \sin(\mu_n \delta_1) + \left[ C(n) - G(n) \right] e^{-\pi b(\chi_1 + \delta_1)} \\
B_7(n) &= r \left[ B_4(n) \sin(\mu_n \delta_1) - B_5(n) \cos(\mu_n \delta_1) \right] + \left[ G(n) + r^2 C(n) \right] (b/n) e^{-\pi b \chi_1} \\
B_8(n) &= B_6(n) \cosh(n \pi \chi_2') + B_7(n) \sinh(n \pi \chi_2') + \left[ G(n) - C(n) \right] e^{-\pi b \chi_2'} \\
B_9(n) &= -\left[ B_6(n) \sinh(n \pi \chi_2') + B_7(n) \cosh(n \pi \chi_2') \right] / r + \left[ C(n) + G(n)/r^2 \right] (\pi b/\mu_n) e^{-\pi b \chi_2'} \\
B_{10}(n) &= B_8(n) \cos(\mu_n \delta_2) + B_9(n) \sin(\mu_n \delta_2) + \left[ C(n) - G(n) \right] e^{-\pi b(\chi_2 + \delta_2)} \\
B_{11}(n) &= r \left[ B_8(n) \sin(\mu_n \delta_2) - B_9(n) \cos(\mu_n \delta_2) \right] + \left[ G(n) + r^2 C(n) \right] (b/n) e^{-\pi b(\chi_2 + \delta_2)}.
\end{align*}

\( \mu_n, G(n) \) and \( C(n) \) are the same as given in (19), and \( \chi_2 = \chi_2 - \chi_1 - \delta_1 \) is the width of the dry region between the first and second moist ascents (see (17)). Note that the formulae for \( E_i(n) \) and \( B_i(n) \) are as for \( E_{i-4}(n) \) and \( B_{i-4}(n) \) respectively, so the above formulation can be easily extended recursively to any \( i > 11 \). For \( N \)-multiple bands (\( N \) is the number of the moist ascents), \( J = 2 + 4N \).

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