The effects of oppositely sloping boundaries with Ekman dissipation in a nonlinear baroclinic system

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SUMMARY

A modified Eady channel model with oppositely sloping top and bottom Ekman layers is used to examine analytically and numerically the effect of the slope, $\Delta$, with dissipation, $\delta$, on baroclinic flows in linear and nonlinear systems. The spectral wave solution is truncated up to six components.

In the linear system, when $\delta = 0$, waves are dispersive and move westward owing to $\Delta$. The maximum value of $\Delta$ for which unstable waves may exist, $\Delta_{\text{max}}$, is equal to the slope of the isentropes of the basic state. $\Delta$ exhibits only a weak stabilizing influence on short waves, while exhibiting a strong stabilizing influence on long waves when it exceeds a critical value. When $\delta \neq 0$, a small $\delta$ has a destabilizing effect on short waves by shifting the short-wave cut-off in the shorter wave direction and extending $\Delta_{\text{max}}$ to a larger value. However, the combined effect of $\Delta$ and $\delta$ strongly stabilizes long waves. The most unstable wavenumber may be shifted to a higher one by increasing $\Delta$, in both inviscid and viscous cases. The relation between the most unstable wavenumber and $\Delta$ is similar to some annulus experiments with oppositely sloping boundaries. It is also similar to the relation between the most unstable wavenumber and $\beta$ in $\beta$-plane models.

In a nonlinear system without wave–wave interaction, $\Delta$ stabilizes the flow even for small $\delta$ and reduces the domain of oscillation and aperiodic flow regimes, while enlarging the domain of single-wave steady state. It also qualitatively agrees with the results from $\beta$-plane dissipative systems. Within the single-wave steady state regime, the preferred nonlinear wavenumber may also be shifted to a higher one by increasing $\Delta$. When wave–wave interaction is allowed, a variety of flow regimes is observed in four truncated systems. There are single-wave steady states, multi-wave steady states where wave dispersion or structural oscillation may occur, multi-wave oscillation, and aperiodic flows. In contrast to the system without wave–wave interaction, the domain of oscillation is enlarged. A ‘frequency-locking’ mechanism is found in multi-wave steady states, where all three conditions required for a resonant triad are strictly satisfied.

1. INTRODUCTION

The effect of sloping boundaries on baroclinic instability has been studied in two-layer and Eady models. Bretherton (1966), using a two-layer inviscid model with parallel but sloping upper and lower boundaries, discussed the linear stability of a quasi-geostrophic perturbation to a baroclinic zonal flow in terms of potential vorticity. A short-wave cut-off was found with a positive slope (towards the north); a long-wave cut-off was possible only when the slope was negative. Hide (1969) and Hide and Mason (1975) calculated the growth rate and phase speed of baroclinic waves for an inviscid system with parallel sloping end-walls as well as with oppositely sloping end-walls. There was no long-wave cut-off and no dispersion in the system with parallel end-walls sloping in the same sense as the isotherms. However, a long-wave cut-off and dispersion were seen in the system with oppositely sloping end-walls. The properties in the latter system were similar to those observed in $\beta$-plane models. Blumsack and Gierasch (1972, hereafter referred to as BG), using a linear 2-D Eady model with sloping bottom and flat top,

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studied the influence of the slope on growth rate, wavelength, propagation direction, and the heat fluxes associated with baroclinic waves on Mars. They found that stability occurs only when the ratio of the slope of the ground to that of the isentropes is less than 1. Mechoso (1980, hereafter referred to as M) studied the baroclinic instability of flows along sloping boundaries in both two-layer and Eady models. He found that, when only the bottom was sloping, sufficiently large values of the absolute value of the slope could stabilize a wave. However, the instability of long waves is enhanced by a small positive slope, and that of shorter waves by a small negative slope.

In barotropic flow, the vorticity gradient due to $\beta$ may be shown to be formally equivalent to a variation in depth of the rotating container. In baroclinic flow, the oppositely sloping boundaries are no longer completely equivalent to the $\beta$ effect. However, Mason (1975), by performing annulus experiments with oppositely sloping boundaries, indicated that if the boundaries sloped at equal but opposite angles, an effect similar to the latitudinal variation of the Coriolis parameter in a planetary atmosphere, the $\beta$ effect, was introduced. For these baroclinic waves, propagation properties similar to those of Rossby waves occurred. The stability properties were also affected by sloping boundaries in laboratory experiments.

The $\beta$ effect has been broadly studied in a linear framework. The most pronounced effects of $\beta$ on linear baroclinic waves are to make the waves dispersive and inhibit long-wave instability. The $\beta$ effect on finite-amplitude baroclinic wave flow both with and without wave–wave interaction has also been theoretically examined by many authors (Romea 1977; Yoden 1979; Pedlosky 1981; Boville 1982; Vallis 1983; Mak 1985; Pedlosky and Polvani 1987; and others). Romea found that a small but finite amount of dissipation destabilized the system in a $\beta$-plane model. Vallis studied the effects of $\beta$ and baroclinicity on the predictability of quasi-geostrophic flow. He found that the baroclinic instability of long waves was inhibited by a $\beta$, and energy entered the system at a higher wavenumber. He also found that the $\beta$ effect increased the wavenumber range over which significant nonlinear energy transfer occurred. In Mak’s two-layer model, where both $\beta$ effect and wave–wave interaction existed, multi-wave steady states were observed. Pedlosky and Polvani also observed a steady mixed-wave regime in their two-layer $\beta$-plane model with internal dissipation and with wave–wave interaction of two slightly unstable waves. This steady mixed-wave regime was not found in Pedlosky and Frenzen’s (1980) $f$-plane model.

There is some controversy about the $\beta$ effect. Yoden (1979), using a two-layer quasi-geostrophic spectral model, studied dynamical properties of nonlinear baroclinic waves and the effect of $\beta$ on these waves. He claimed that the flow regimes were not drastically changed by inclusion of the $\beta$ effect. In contrast to Yoden’s findings, Pedlosky (1981) found that the $\beta$ effect did change the flow regimes. The chaotic behaviour of baroclinic waves in an $f$-plane model found by Pedlosky and Frenzen was strongly suppressed by the $\beta$ effect. Thus, he claimed that the effect of $\beta$ reduced the domain of vacillation to a small region of parameter space.

Analyses related to oppositely sloping boundaries and comparisons of the results with the $\beta$ effect are few, especially in a nonlinear framework. It is not clear yet to what extent the effect of oppositely sloping boundaries is similar to, or different from, the $\beta$ effect on linear or nonlinear baroclinic waves. It is the purpose of the present work to study analytically and numerically these effects with Ekman layers in linear and nonlinear frameworks, and to compare these effects with those due to $\beta$ wherever possible. Hopefully, this study will offer some evidence as to whether further study of the ' $\beta$ effect' is appropriate in simple Eady-type models or laboratory annulus experiments with oppositely sloping top and bottom boundaries. Section 2 briefly introduces the governing
equations. Section 3 solves the linear instability problem, and discusses the effects of the slope, $\Delta$, with and without Ekman dissipation on linear waves. Section 4 shows the dependence of flow regimes on $\Delta$, based on numerical experiments in a 6-wave system without wave–wave interaction, and verifies analytically the dependence on $\Delta$ of the preferred wavenumber of a single-wave steady state. Section 5 shows various flow regimes in systems with wave–wave interaction but different truncation levels. Section 6 discusses some physics of a special flow regime with a multi-wave steady state and a ‘frequency-locking’ mechanism. Section 7 is a summary and gives a justification for using an appropriate low-order model to study some annulus phenomena.

2. THE GOVERNING EQUATIONS

An Eady-type model having oppositely sloping boundaries with uneven Ekman dissipation is used. The governing equations for the perturbation geostrophic streamfunctions, $\widehat{\phi}$, and for the mean flow corrections, $\overline{\phi}$, are similar to those used by Weng et al. (1986, hereafter referred to as WBM), except that for the wave field an extra linear term, representing the effect of oppositely sloping boundaries, is added here in the boundary conditions at $z = \pm 1/2$. Thus, the governing equations for $\widehat{\phi}$ are written:

$$\begin{align*}
\widehat{\phi}_{zz} + \delta \nabla^2 \widehat{\phi} &= 0 \\
\widehat{\phi}_x &= 0, \quad \text{at } y = 0, 1 \\
\widehat{\phi}_{zl} + \lambda z \delta \widehat{\phi}_{xz} - (\lambda + 2z \Delta_S) \widehat{\phi}_x - 2z \gamma \delta \nabla^2 \widehat{\phi} + J(\widehat{\phi}, \overline{\phi}_z) - J(\overline{\phi}, \overline{\phi}_z) + J(\phi, \overline{\phi}_z) + J(\phi, \overline{\phi}_z) &= 0, \quad \text{at } z = \pm 1/2
\end{align*}$$

(1)

where $\Delta$ is the nondimensional slope of the boundaries, to be defined later. Other parameters share the same definitions as those given by WBM: i.e. $\lambda$ is the vertical shear, set to 1 in our calculation and dropped in later formulations; $S$ the stratification parameter; $\delta$ the dissipation parameter; $\gamma$ the asymmetry of Ekman layers, set to 1 and $\rho$ at $z = -1/2$ and $1/2$, respectively, where $\rho$ is arbitrarily set to 0.1. The slopes of the top and bottom boundaries are equal but of opposite directions, as shown in Fig. 1. The ratio of the dimensional height of the topography to the depth of the channel, $h_B/H$, is small and of the order of the Rossby number $\tilde{Ro}$. The nondimensional height of the topography is small:

$$\eta = h_B/(\tilde{Ro}H) = O(1)$$

(2)
so that the sloping boundary conditions may reasonably be imposed at \( z = \pm 1/2 \). The slope parameter, \( \Delta \), is defined as

\[
\Delta = d\eta/dy = \text{constant.}
\]  

(3)

The spectral solution of \( \Phi \) contains up to six wave components:

\[
\Phi = \sum_{n=1}^{3} \sum_{l=1}^{2} \left[ A_{nl}(t) \cosh(2\mu_{nl}z) + B_{nl}(t) \sinh(2\mu_{nl}z) \right] e^{i\nu x} \sin(\pi y) + cc
\]

(4)

and that of \( \Phi \) contains two components:

\[
\Phi = -\Phi_p \sum_{p=1}^{2} \left[ M_p(t) \cosh(2\omega_p z) + N_p(t) \sinh(2\omega_p z) \right] \sin(p\pi y)
\]

(5)

where

\[
\mu_{nl} = \frac{1}{2} \left[ (n^2 + l^2 \pi^2)S \right]^{1/2} \quad \text{and} \quad \omega_p = \frac{1}{2} p \pi S^{1/2}
\]

(6)

and \( n \) and \( l \) are the zonal and meridional wavenumbers, respectively, \( p \) indicates the meridional mode of the mean flow correction, and \( cc \) stands for complex conjugate. The justification for using such a spectral solution will be discussed in section 7.

The system of amplitude equations is obtained after substituting the spectral solutions into the boundary conditions at \( z = \pm 1/2 \). It has the form

\[
\begin{align*}
\hat{A}_{nl} &= L_{nl}^\Delta + (WM)_{nl}^\Delta + \alpha_W(WW)_{nl}^\alpha \\
\hat{B}_{nl} &= L_{nl}^B + (WM)_{nl}^B + \alpha_W(WW)_{nl}^B \\
\hat{M}_p &= L_p^M + (WM)_p^M \\
\hat{N}_p &= L_p^N + (WM)_p^N
\end{align*}
\]

(7)

where \( L, WM, \) and \( WW \) stand for linear, wave–mean flow interaction and wave–wave interaction terms, respectively, and \( \alpha_W \) is a tracer which is set to 1 or 0 for the flow with or without wave–wave interaction.

3. Linear Analysis

The linear terms for the wave field include the effect of \( \Delta \) as well as that of asymmetric Ekman dissipation, i.e.

\[
\begin{align*}
L_{nl}^\Delta &= (-a_{1nl} + ia_{6nl})A_{nl} + (a_{2nl} + ia_{3nl})B_{nl} \\
L_{nl}^B &= (-a_{4nl} + ia_{7nl})B_{nl} + (a_{2nl} + ia_{5nl})A_{nl}
\end{align*}
\]

(8)

where the \( a_{\cdot\cdot} \) are determined by the parameters and are given in the appendix. The coefficients \( a_{1nl} \) and \( a_{4nl} \) represent the stabilizing effect of Ekman dissipation \( \delta \); \( a_{2nl} \) represents the destabilizing effect due to the asymmetry between the top and bottom Ekman layers; \( a_{3nl} \) and \( a_{5nl} \) represent the baroclinic forcing due to the vertical shear; \( a_{6nl} \) and \( a_{7nl} \) represent the effect of the slopes. The effects of \( \Delta \) do not explicitly appear in the linear terms of mean flow corrections or in any of the nonlinear terms. However, as will be seen later, its implicit effect on nonlinear flows is obvious. For simplicity, the subscript \( nl \) will be dropped in the linear analysis. Thus, the system becomes

\[
\hat{A} = L^\Delta, \quad \hat{B} = L^B.
\]

(9)
We let

\[ B = \Lambda A, \quad A = |A| e^{-i\delta} + cc \]  

where \( \Lambda \) is a complex constant which reflects the linear wave structure, and can be determined by the coefficients of linear terms (Weng and Barclon 1987); \( c \) is the phase speed. To isolate the effect of \( \Delta \) from that of \( \delta \), we consider the inviscid and viscous cases separately.

(a) Inviscid case (\( \delta = 0 \))

By using (10), the linear system (9) can be simplified for the inviscid case as

\[ \hat{A} = \frac{1}{2}(a_6 + a_7) \pm \frac{i}{2} \sqrt{I} A = (nc_i - inc_r)A \]  

where

\[ I = -(a_6 - a_7)^2 - 4a_3a_4 \]  

and \( c_r \) and \( c_i \) are the real and imaginary parts of the phase speed:

\[ c_r = -\frac{\Delta S}{4\mu T}(1 + T^2) \]  

\[ c_i = \frac{1}{4\mu T} [4(\mu - T)(1 - \mu T)T - \Delta^2 S^2(1 - T^2)^2]^{1/2}. \]  

Equation (13) shows that the waves are dispersive and move west owing to \( \Delta \). For instability to occur, \( c_i \) must be positive, which gives the criterion

\[ \Delta < \Delta_c = \frac{4}{S(1 - T^2)} \sqrt{(\mu - T)(1 - \mu T)T}. \]  

For given \( S \) and wavenumber \((n, l)\), the wave will be unstable only when \( \Delta \) is less than its critical value \( \Delta_c \). For the wave spectrum, the maximum value for which marginally unstable waves may exist occurs when

\[ \Delta_{e_{max}} \leq 1 \]  

at

\[ \mu_{max} = (1 + T^2_{max})/2T_{max} = 1.032. \]  

Since \( S^{-1} \) represents the slope of the isentropes of the basic state, (16) implies that no instability will occur when the slope of the boundaries exceeds that of the isentropes. (It is also true for dimensional slopes.) This criterion is the same as that found by BG and M in their models with a sloping bottom and a flat top. It seems that the slope of the upper boundary does not affect the stability criterion. For a given \( S \), (17) determines the total wavenumber of the marginally unstable wave, \((n^2 + l^2\pi^2)\). Thus, if \( S \) is small enough, there may be several marginally unstable waves with a similar total wavenumber. As shown by (16), \( \Delta_{e_{max}} \) increases with decreasing \( S \).

Although the criterion for instability is not affected by the existence of the upper sloping boundary, the growth rate and phase speed are affected by this extra sloping boundary. The \( \Delta \) effect stabilizes both long and short waves, but this stabilization is achieved in a different way, as is shown analytically in the appendix.

Figure 2 presents the marginal curves for five values of \( S (0.40, 0.30, 0.25, 0.20 \) and 0.15) in the \( n-\Delta \) plane when \( l = 1 \) and \( \delta = 0 \). For each \( S \), it is seen that \( \Delta \) strongly stabilizes very long waves as it exceeds a certain value \( \Delta_{e_{op}} \), which depends on \( S \) and is
given in the appendix. However, \( \Delta \) slightly stabilizes short waves even when large. The most unstable wavenumber is slightly increased with increasing \( \Delta \). These results are consistent with those observed by Mason (1975) in the annulus experiments with oppositely sloping end-walls. He found that in the case with oppositely sloping boundaries the effects of the slopes on stability cannot be predicted by energetic arguments applied to the parallel boundary cases or the case with flat upper boundary and sloping lower boundary.

The way that \( \Delta \) slightly stabilizes short waves is very different from that shown in BG and M's models with a single sloping bottom where the stabilizing effect of the slope on short waves was almost as strong as that on long waves. However, this effect of \( \Delta \) is consistent with \( \beta \) which has a slight stabilizing effect on short waves (see Fig. 10.1 in Holton (1972)). The stabilizing effect of \( \Delta \) on long waves is also similar to the \( \beta \) effect, but is different from that in BG and M's models, where even a small slope may cause a long-wave cut-off (see the appendix).

(b) Viscous case (\( \delta \neq 0 \))

When the oppositely sloping boundaries have asymmetric Ekman layers, the combined effect of \( \Delta \) with \( \delta \) on wave stability is more complicated. The linear system (9) leads to the equation of the marginal curve for the viscous case

\[
\delta_c^2 = \frac{1}{4\rho a^2} (-\mathcal{A} \Delta^2 + \mathcal{B} \Delta + \mathcal{C})
\]  

(18)

where \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) are positive and derived in the appendix. Figure 3 shows marginal curves in the \( \log \delta^{-4}-\Delta \) plane for wavenumbers 1–4 with the lowest \( y \) mode when \( S = 0.20 \). All the second \( y \) modes are stable. For each wave, \( \delta_c \) decreases with increasing \( \Delta \). The stabilizing effect of \( \Delta \) becomes more obvious when \( \Delta \) is increased to a certain value, which depends on both \( S \) and \( \delta \). This effect may be seen from (18). When \( \Delta \) is
small, the first two terms largely cancel and therefore have little influence on $\delta_c$. When $\Delta$ is large, the first term dominates the second so that $\delta_c$ is greatly reduced.

Figure 4 shows the marginal curves for five values of $\delta$ (0, 0.01, 0.025, 0.05 and 0.1) in the $\Delta$-$n$ plane when $S = 0.20$ and $l = 1$. The destabilizing effect of a small $\delta$ may be seen by comparing the viscous and inviscid cases. A small $\delta$ extends $\Delta_{\epsilon_{\text{max}}}$ as well as the short-wave cut-off. However, a large $\delta$ ($\delta = 0.1$) does stabilize short waves. The Ekman
layers always stabilize very long waves, even with a small amount of dissipation. Therefore, $\Delta_c$ decreases with increasing $\delta$. For $\delta > 0.05$, $\Delta_c$ no longer exists, which means that the combined effect of the oppositely sloping boundaries with Ekman dissipation strongly stabilizes long waves to such a degree that a long-wave cut-off may exist even when $\Delta$ is small.

<table>
<thead>
<tr>
<th>Wave</th>
<th>$\Delta$</th>
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<tbody>
<tr>
<td>$nc_{11}$</td>
<td>0.1673 0.1645 0.1565 0.1426 0.1212 0.0889 0.0329 — — — —</td>
</tr>
<tr>
<td>$nc_{12}$</td>
<td>0.3245 0.3215 0.3123 0.2966 0.2734 0.2407 0.1951 0.1265 — — —</td>
</tr>
<tr>
<td>$nc_{13}$</td>
<td>0.3854 0.3831 0.3760 0.3641 0.3467 0.3233 0.2925 0.2522 0.1982 0.1197 0.0106</td>
</tr>
<tr>
<td>$nc_{14}$</td>
<td>0.2476 0.2457 0.2396 0.2290 0.2136 0.1926 0.1653 0.1306 0.0897 0.0514 0.0254</td>
</tr>
<tr>
<td>$nc_{15}$</td>
<td>— — — — — — — — — — — —</td>
</tr>
<tr>
<td>$n_{mw}$</td>
<td>3 3 3 3 3 3 3 3 3 3 3 4</td>
</tr>
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</table>

Table 1 gives linear growth rates of wavenumbers 1–5 ($nc_i$) and the most unstable wavenumber ($n_{mw}$) for $\delta = 0.01$ and $\delta = 0.05$ when $S = 0.20$. As $\Delta$ increases, the growth rates of all waves are reduced, and the most unstable wavenumber is shifted higher when $\Delta$ is large enough for a wavenumber transition.

In the following two sections we will study the combined effect of oppositely sloping boundaries with asymmetric Ekman dissipation on nonlinear flows without and with wave–wave interaction.

4. NONLINEAR FLOWS WITHOUT WAVE–WAVE INTERACTION

Two series of parameter settings with different dissipations ($\delta = 0.01$ and 0.05) for $\Delta = 0.5$ at $S = 0.20$ are chosen to perform numerical experiments by integrating the system (7), to investigate the flow regimes in a 6-wave system with all waves given in (4). These parameter settings are the same as those for Table 1 where all the second-lowest $y$ modes and the zonal wavenumbers higher than 4 are linearly stable. Write

$$A_{nl} = |A_{nl}|e^{i\theta_{nl}} \quad B_{nl} = |B_{nl}|e^{i\eta_{nl}}$$

(19)

and set the initial values to be the same for all experiments, i.e. $|A_{nl}(0)| = 0.001$ and $\theta_{nl}(0) = 0$, while $|B_{nl}(0)|$ and $\eta_{nl}(0)$ are given by linear wave structures expressed by (10). The results are shown in Fig. 5. At most parameter settings, single-wave steady states are observed and the preferred wavenumbers of these steady wave states increase with increasing $\Delta$. The relationship between the preferred wavenumber and $\Delta$ is similar to that between the dominant wavenumber and $\beta$ found by Vallis (1983), i.e. wave energy
entered higher wavenumbers when $\beta$ existed. The chaotic behaviour at $\Delta = 0$ is suppressed by even a small $\Delta$ ($\Delta = 0.5$ in Fig. 5(a)). This phenomenon is similar to that found by Pedlosky (1981) when he compared the results of his $\beta$-plane model with those obtained by Pedlosky and Frenzen's (1980) $f$-plane model. He found that the $\beta$ effect, although a linearly destabilizing effect for small friction, could lead to more regular and

![Graphs showing wave amplitudes vs. $\Delta$](image)

Figure 5. The dependence of numerically solved wave amplitudes at mid-depth mid-channel, $|A_{ij}|$, on $\Delta$ when $S = 0.20$ in the 6-wave nonlinear system without wave-wave interaction for (a) $\delta = 0.01$ and (b) $\delta = 0.05$. The ordinate of a dot represents the amplitude of a steady wave; the length of a dashed line segment represents the range of amplitude variation of an aperiodic vacillating wave.

less chaotic amplitude behaviour, i.e. the presence of even a weak $\beta$ effect could lead to the elimination of chaotic behaviour and the preference for simple steady finite-amplitude waves. Thus effects of both $\Delta$ and $\beta$ are to reduce the domain of vacillation to a small region of parameter space.

In a nonlinear system without sloping boundaries and wave-wave interaction, the preferred wavenumber for a single-wave steady state is lower than the linearly most unstable wavenumber and solely determined by external parameters. The procedure for analytically selecting the preferred wavenumber was described in Weng and Barcilon (1988, WB). For the readers' convenience, the main steps in the procedure are briefly outlined. WB considered a triad with three adjacent zonal wavenumbers, $n_0 - 1$, $n_0$, $n_0 + 1$, with the lowest y mode. A nonlinear equation system similar to (6) for these three waves without wave-wave interaction was solved analytically for three single steady wave states at a given parameter setting. Then the linear stability of each single steady wave state was tested. If the steady wave state of the middle wavenumber, $n_0$, was stable, this was most likely the preferred wavenumber. If the steady state of an outer wavenumber, $n_0 - 1$ or $n_0 + 1$, was stable, one was not sure whether that wavenumber
was the true preferred wavenumber or just a spurious one introduced by the choice of this particular triad, for the true preferred wavenumber might not be in the triad and, therefore, out of the testing range. Thus, the next step was to form a new triad where the middle wavenumber of the new triad was equal to the lowest (highest) wavenumber of the old triad. If the single steady wave state of the middle wavenumber in the new triad was stable, this wavenumber was selected as the preferred wavenumber; otherwise, the procedure of forming a new triad and testing three single steady wave states had to be repeated, until a single-wave steady state with the middle wavenumber in a new triad was found to be stable, and selected as the preferred wavenumber at the given parameter setting. This procedure was applicable only to the parameter regions where single steady wave states may occur.

<table>
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<th>0-0</th>
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<th>1-5</th>
<th>2-0</th>
<th>2-5</th>
<th>3-0</th>
<th>3-5</th>
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<td>3</td>
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<td>0-0294</td>
<td>0-0352</td>
<td>0-0350</td>
<td>0-0303</td>
<td>0-0314</td>
<td>0-0307</td>
<td>0-0274</td>
<td>0-0203</td>
<td></td>
</tr>
<tr>
<td>δ 0-05 Number</td>
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<td>2</td>
<td>2</td>
<td>2</td>
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In order to see whether a similar mechanism for wavenumber selection exists in the presence of the slopes, following the above procedure, we analytically solve for single-wave steady states and select the preferred wavenumbers at the same parameter settings as in Table 1 and Fig. 5. The results are given in Table 2. The good agreement between Table 2 and Fig. 5 shows that the preferred wavenumber of a single nonlinear steady wave in the presence of sloping boundaries is also selected by a mechanism similar to that discussed by WB. Comparison between Table 1 and Table 2 indicates that the difference between the linearly most unstable wavenumber and the preferred nonlinear wavenumber decreases with increasing Δ. Larger Δs have a stronger stabilizing effect on long waves and cut off very long waves. This Δ effect may prevent a nonlinear preferred wavenumber from transiting to the same lower wavenumber as that in the case with flat boundaries. There is no preferred wavenumber selected at Δ = 0 for δ = 0-01 because no stable steady wave is found at this parameter setting.

It is seen from Table 2 and Fig. 5 that within the range of Δ for each wavenumber, the wave amplitude may increase as Δ increases for small and moderate Δ; it may decrease as Δ increases for large Δ, until the amplitude of that wavenumber vanishes. Then, a transition to a higher wavenumber may be observed, if that higher wavenumber is provided for in the model. Figure 6 shows the northward eddy heat flux of a single-wave steady state at mid-depth and mid-channel for different Δ when the parameters are the same as those for Fig. 5. The eddy heat flux is calculated by

$$
\bar{vT} = 4\eta|A||B|\sin\Theta
$$

(20)

where Θ = η − θ is the phase difference between B and A. In general, the heat flux corresponding to each wavenumber increases with Δ and then there is an abrupt drop
Figure 6. The dependence of analytically solved eddy heat flux, $\bar{\nabla}^2 T$, at mid-depth mid-channel, on $\Delta$ at the same parameter settings as for Fig. 5 for single-wave steady states.

when the wavenumber transits to the next higher integral value. This phenomenon was discussed by Pfeffer and Barcilon (1978) in their weakly nonlinear analysis and numerical simulation of annulus experiments when such variation was caused by increasing rotation rate.

5. **Nonlinear flow with wave-wave interaction**

Several distinct flow regimes are observed in numerical experiments at different truncation levels when wave-wave interaction is present. For example, Fig. 7 gives the amplitudes or amplitude excursions of the streamfunction of the wave components, $|A_n|$ ($n = 1, 2, 3; \ell = 1, 2$), in the $6$-wave system at $z = 0$ when wave-wave interaction is considered for the same two series of parameter settings as those for Fig. 5. Figure 8 gives the amplitudes or amplitude excursions of the streamfunction of a triad in a $3$-wave system, with waves $(1, 1), (2, 1)$ and $(1, 2)$ at $z = 0$ for the same parameters as those for Fig. 7(a). The results are summarized based on numerical experiments with time integration long enough ($t = 3000, 6000$ or $9000$) so that the final states are in equilibrium or do not exhibit qualitative changes. There are single-wave steady states, multi-wave steady states, multi-wave vacillation and aperiodic flows. In the flow regimes with different zonal wavenumbers, wave dispersion is also involved.

It is useful to define a flow more quantitatively. Following WBM, we classify a flow regime by its number of spatial and temporal degrees of freedom. In most cases, each wave component has similar behavior with others for a given parameter setting. Thus, we may use the nomenclature of degree of spatial freedom to account for the total number of wave components that survive. That is, if a wave has $i$ surviving wave components, we consider the flow to have $i$ spatial degrees of freedom. Similarly, if a wave has $m$ frequencies, whether in phase speed or amplitude or structure, we consider the flow to have $m$ temporal degrees of freedom. Thus, the flow regime may be designated as $F_i^m$. 
A single-wave steady state has only one wave component and one frequency with which the wave travels along the channel. This state is designated as $F_1$. This kind of flow is observed in Fig. 7(b) at $\Delta = 0$ and 4.5.

A multi-wave steady state is defined such that each wave component is steady, i.e. each wave component has its own time-independent amplitude and phase speed, while the total wave field may vary with time owing to interference among these travelling
steady waves. The variation of the total wave field may have the characteristics of
structural vacillation or those of wave dispersion.

There are different definitions for wave dispersion. Hide and Mason (1975) classified
wave dispersion as one kind of vacillation. Pfeffer and Fowlis (1968) considered wave
dispersion as a different phenomenon from vacillation. However, both viewpoints agreed
that wave dispersion resulted from the interference of waves with different zonal wave-
numbers having different phase speeds and with rather steady wave amplitudes. Here
we distinguish wave dispersion and vacillation from the energy point of view. A vacillation
must be involved in some kind of energy exchange, potential and/or kinetic, between
the total wave and the mean flow, while a pure wave dispersion is not involved in such
an energy exchange. If there are at least a pair of waves with the same zonal wavenumber
but different $y$ or $z$ structure, the resultant wave flow may exhibit the characteristics of
structural or amplitude vacillation. If all the waves have different zonal wavenumbers,
even though they may also have different $y$ or $z$ structure, the resultant wave flow will
exhibit the characteristics of wave dispersion. For example, Madden (1983) studied the
interference of stationary and travelling waves of the same longitudinal scale. He found,
based on time variation of heat and momentum fluxes etc., that the interference between
the waves caused some of the observed time variations in the large-scale circulation. As
defined above, the time variation should be a vacillation phenomenon, not a dispersion
one, because the waves with different vertical structures have the same longitudinal
scale. If these waves have different longitudinal scales, the interference would result in
dispersion, without energy conversion between the wave and mean flow fields, although
the flow field may show some kind of variation in time. Pedlosky and Polvani (1987)
found a steady mixed-wave regime with two waves having different zonal wavenumbers
and constant amplitudes and phase speeds. They indicated that the time dependence of the total solution is not trivial since the waves move past each other so that at any fixed point the observed wave disturbance amplitude would oscillate with both frequencies. However, as we discussed earlier, although the total wave field may vary with time, such a variation may not be considered as a vacillation, because there is no energy exchange between the resultant wave and the mean flow. Thus, that is just a wave dispersion phenomenon.

In most cases in our experiments with multi-wave steady states, the flows may exhibit structural vacillation and wave dispersion. Examples of a pure wave dispersion due to interference among multiple steady waves with different zonal wavenumbers are seen at $\Delta = 4.5$ and 5 in Fig. 7(a), where there are three surviving wave components, $(1, 1), (3, 1)$ and $(2, 2)$. Since these waves have different zonal wavenumbers, the interaction among them will not result in time-dependent heat and momentum fluxes under our periodic assumption in the zonal direction. Thus, although the constructive and destructive interference may result in a kind of oscillation, from the energy point of view, it is not a vacillation observed in annulus experiments; rather, it is a wave dispersion similar to that observed by Pfeffer and Fowlis (1968) in annulus experiments. Since the flow has 3 wave components with 3 different phase speeds, it has 3 spatial degrees of freedom and 3 temporal degrees of freedom, and is therefore designated as $F^3_3$; in general, such a flow may be designated as $F^m_n$. Thus, wave dispersion does not increase the number of temporal degrees of freedom of the flow.

The multi-wave steady state with wave components $(1, 1), (2, 1)$ and $(1, 2)$ observed at $\Delta = 2.5$ and 3-0 in Fig. 8 exhibits different behaviour. Since there is a pair of waves, $(1, 1)$ and $(1, 2)$, having the same zonal wavenumber but different $y$ structure, the interaction between these two wave components results in time-dependent heat and momentum fluxes, and is therefore characterized by a structural vacillation of wavenumber 1. Thus, the flow has one more temporal degree of freedom than in the case of wave dispersion discussed above, although both have the same spatial degrees of freedom. This flow may be designated as $F^3_1$. In fact, this structural vacillation is accompanied by wave dispersion between wavenumbers 1 and 2. However, as discussed earlier, wave dispersion does not increase the number of temporal degrees of freedom of the flow.

The feature that the structural vacillation in a multi-wave steady state in a 3-wave system has one more temporal degrees of freedom than the spatial degrees of freedom is robust for a multi-wave steady state even with more than 3 waves. Table 3 shows frequencies of each wave and the periods of structural vacillation calculated by $P = 2\pi/|\sigma_{n1} - \sigma_{n2}|$ in four wave systems with different truncations when $S = 0.20$, $\delta = 0.01$ and $\Delta = 2.5$. The 3-wave system contains waves $(1, 1), (2, 1)$ and $(1, 2)$; the 4-wave system contains yet another wave, $(3, 1)$; the 5-wave system contains still another wave, $(2, 2)$; the 6-wave system contains all waves given in (4). For example, there are 3 pairs of waves in the 6-wave system, although the amplitude of the wave component $(3, 2)$ is too small to be recognized. The two waves of each pair have the same zonal wavenumber but different $y$ structures. All these waves travel with different but constant phase speeds and constant amplitudes. However, the interference of the two waves in each pair results in a structural vacillation with the same period, $P = 39.47$. Thus, the flow should be designated as $F^7_6$ or, in general, as $F^m_{n+1}$, which is the same as that derived from the 3-wave system. Thus, there is an extra temporal degree of freedom in vacillation compared with dispersion when the systems have the same spatial degrees of freedom. This extra temporal degree of freedom in vacillation results from nonlinear interaction, and is shown by the variation of zonally averaged heat and momentum fluxes, and therefore, by the energy cycle of the flow.
TABLE 3. FREQUENCIES OF THE WAVE COMPONENTS, $\sigma_{ni}$, AND THE PERIOD OF THE STRUCTURAL VACILLATION, $P = 2\pi/|n|a_1 - a_2|$, OF A MULTI-WAVE STEADY STATE AT $\Delta = 2.5$, $S = 0.20$ AND $\delta = 0.01$ IN THE 3-, 4-, 5- AND 6-WAVE SYSTEMS

<table>
<thead>
<tr>
<th>Wave</th>
<th>Truncation level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td>(n, l)</td>
<td>$\sigma_{ni}$</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>0.3618</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>0.2120</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>0.5738</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>0.6995</td>
</tr>
<tr>
<td>(3, 2)</td>
<td></td>
</tr>
</tbody>
</table>

A multi-wave vacillation is the flow state in which all wave components exhibit amplitude vacillation separately, with a single period or multiple periods in amplitude. The multiple periods in amplitude may be caused by period-doubling bifurcation of the wave component itself, or by the wave–wave interaction within triads. The flow is more complicated but we may still be able to designate their temporal degrees of freedom. First, $i$ waves have $l$ temporal degrees of freedom in phase speed. Second, different wave components may have different temporal degrees in their amplitudes. We will consider the highest temporal degrees of freedom in amplitude of the wave components, $j$, to represent the temporal degrees of the total flow in its amplitude. Third, unlike the flow with multi-wave steady state where all pairs of waves result in the same structural period, the flow in a multi-wave vacillation state having $k$ pairs of waves may have up to $k$ structural periods. For example, Fig. 9 illustrates the amplitude evolution of the six wave components and two zonal flow components at $\Delta = 4.0$ of Fig. 7(a). The flow has 6 non-zero wave components, thus $i = 6$. There is a single-period amplitude vacillation in $|A_{12}|$ and $|A_{22}|$, and double-period amplitude vacillation in $|A_{11}|$, $|A_{21}|$ and $|A_{31}|$, thus $j = 2$. There are three pairs of waves with the same zonal wavenumber but different $y$ structures, thus $k = 3$, although $|A_{32}|$ is too small to be seen. Therefore, the flow may be designated as $F_{6}^{1}$ or, in general, as $F_{l+m}^{i+k}$. An aperiodic flow has, theoretically, infinite temporal degrees of freedom in wave amplitude. Practically, we may consider a flow to be aperiodic even when $j$ is finite. The flows observed at $\Delta = 0.2$ in Fig. 7(a) and those at $\Delta = 0.5$ and 2.5 in Fig. 7(b) are examples of aperiodic flows, or geostrophic turbulence observed in annulus experiments. These flows may be designated as $F_{6}^{1}$ or, in general, $F_{l}^{i}$.

The flow regimes in Figs. 7(a) and (b), as well as those at the same parameter settings as for Fig. 7(a) but in differently truncated systems, are summarized in Table 4 by using the above-discussed notations. The effect of $\Delta$ on nonlinear flow with wave–wave interaction is obvious but complicated. A single-wave steady state at $\Delta = 0$ may become a multi-wave aperiodic flow by the addition of a small $\Delta$ (see Fig. 7(b)). In contrast, when wave–wave interaction is absent, an aperiodic flow at $\Delta = 0$ may be stabilized by a small $\Delta$ and become a single-wave steady state (see Fig. 5(a)). Thus, when wave–wave interaction is allowed, a small $\Delta$ may increase the spatial degrees of freedom. It agrees with the results found by Vallis (1983) that $\beta$ increases the range of wavenumbers.
Figure 9. Amplitude evolution of the 6-wave system at $\Delta = 4.0$, $\delta = 0.20$ and $\delta = 0.01$ in a multi-wave mixed vacillation regime.
over which significant nonlinear energy transfers occurred. However, a sufficiently large $\Delta$ may decrease the spatial degrees of freedom for all the cases we studied. This is consistent with the linear result shown in Fig. 4 and Table 1.

There is no systematic change in temporal degrees of freedom with $\Delta$. The temporal degrees of freedom alternate increase and decrease with increasing $\Delta$ until a steady wave state with fewer spatial degrees of freedom or even a wave-free state is observed at large $\Delta$. The single-wave steady state observed at $\Delta = 5$ for $\delta = 0.01$ in the 6-wave system (Fig. 7(a)) does not exist in the 3-wave system with waves (1, 1), (2, 1) and (1, 2) (Fig. 8). This difference is due to different truncation levels. As discussed in section 3, a large $\Delta$ strongly suppresses very long waves and shifts the most unstable wave to higher wavenumbers. Thus, the fact that a wave-free state occurs at a large $\Delta$ in a low-order system containing only very long waves may not mean that the flow attains a real wave-free state. The flow may exhibit a single-wave steady state in another 3-wave system, or a multi-wave steady state in a higher level truncated system, containing wave (3, 1) which may survive at a given parameter setting.

Comparing the series of flow regimes in differently truncated systems, we found more temporal degrees of freedom, as well as more spatial degrees of freedom, in higher level truncated systems. However, the main characteristic that the temporal degrees of freedom alternately increase and decrease with increasing $\Delta$ remains in all four truncated systems. The flow behaviours in the 5- and 6-wave systems are especially close. This is because the amplitude of wave (3, 2) is so small that the wave does not play any significant role in determining the flow's evolution and its final state. We also found that all the flows exhibit a multi-wave steady state around $\Delta = 2.5$ and 3.0 for $\delta = 0.01$. Such a state is also observed around $\Delta = 3.0$ and 3.5 for $\delta = 0.05$. This flow state is not an accidental phenomenon, since it will not disappear if a differently truncated system is used, or the parameters slightly changed, as long as the parameter settings are still within the region for such a state to exist. We will examine closely some physics of this state in the next section.

6. MULTI-WAVE STEADY STATES AND THE "FREQUENCY-LOCKING" MECHANISM

Here we focus our attention on the oscillation in a multi-wave steady state regime. This kind of oscillation has been studied in laboratory experiments. Buzyna et al. (1984) measured, in an amplitude oscillation regime, two incommensurate frequencies, which were interpreted as being associated with a constant angular drift of two patterns with the same zonal wavenumber but different phase speeds, corresponding to the two different frequencies. The amplitudes of these two zonal waves are constant and unequal. It is not difficult to show that the passage of two such zonal waves relative to each other
will result in constructive and destructive interference and a combined waveform whose amplitude and phase speed oscillate with the frequency difference of these two waves. This interference mechanism for vacillation has also been studied theoretically by Lindzen et al. (1982), Moroz and Brindley (1982), Barcilon and Drazin (1984), Weng and Barcilon (1987), and others. In fact, this kind of vacillation is also involved in nonlinear interaction, as far as heat and momentum fluxes are concerned. However, in order to distinguish it from the vacillation due to instability of a single steady wave, we may simply consider this kind of vacillation being caused by ‘interference’.

Most of the above-mentioned studies primarily focus on amplitude vacillation, which is caused by the interference of two steady waves with the same zonal wavenumber and different vertical structures. Weng and Barcilon (1987) showed in their 2-wave system that the interference of two steady or quasi-steady waves with the same zonal wavenumber but different meridional structures may cause structural vacillation if they have different phase speeds. A similar phenomenon is observed in our truncated systems here. In order to generalize the phenomenon, we choose a multi-wave steady state observed in the 6-wave system where 4 resonant triads or 3 pairs of waves coexist at $S = 0.2$, $\delta = 0.01$ and $\Delta = 2.5$.

Figure 10 shows the total wave streamfunction fields of these steady waves in the $x$–$y$ plane at three levels, $z = -0.25$, $0.0$ and $0.25$ for the period $t = 1700$–$1820$ with an interval of $dt = 20$. The wave shape is complicated but vacillates over time at a constant frequency, i.e. it repeats its shape at regular time intervals. The period is the same as that given in Table 3, i.e. $P = 39.47 = 2dt$. There are three vacillation cycles during this period. At $t = 1700$, $1740$, $1780$ and $1820$, the intensity of the disturbance is stronger in the north ($y > 0.5$) than in the south ($y < 0.5$), and the phase line is tilted in the NW–SE direction; while at $t = 1720$, $1760$ and $1800$, the opposite is true. The phase line of the disturbance is slightly tilted westward with height, showing baroclinic processes existing to overcome Ekman dissipation. The time variation shows that barotropic processes also take place. The intensity of the disturbance increases with height. The wave patterns at different heights are similar, propagating westward with the same apparent speed.

In order to understand what kind of processes are involved in Fig. 10, it is helpful to analyse the heat and momentum fluxes of the total wave at different locations in the channel. In the present model, with three zonal wavenumbers and two $y$ modes, the total heat flux is

$$\overline{uT} = \sum_{n=1}^{3} \sum_{p=1}^{3} \sum_{l=1}^{2} \sum_{q=1}^{2} v_{nl} T_{pq} = \sum_{n=1}^{3} \sum_{p=1}^{3} \sum_{l=1}^{2} \sum_{q=1}^{2} \frac{\partial\phi_{nl}}{\partial x} \frac{\partial\phi_{pq}}{\partial z}$$

where

$$\phi_{nl} = [A_{nl} e^{i\theta_{nl} \cosh(2\mu_{nl} z)} + |B_{nl}| e^{i\alpha_{nl} \sinh(2\mu_{nl} z)}] e^{i\pi x \sin(l\pi y) + cc}$$

$$\phi_{pq} = [|A_{pq} e^{i\beta_{pq} \cosh(2\mu_{pq} z)} + B_{pq} e^{i\alpha_{pq} \sinh(2\mu_{pq} z)}] e^{i\pi x \sin(p\pi y) + cc}.$$  

(22)

Since $v_{nl} T_{pq} \propto (e^{z/(x+p)} + e^{-z/(x-p)})$ and the assumption of periodicity in $x$ has been made, the wave–wave interaction between different wavenumbers ($p \neq n$) does not make any contribution to the total heat flux. (It should be kept in mind that although wave–wave interaction has no direct influence on total heat flux, it does have an indirect influence by influencing wave amplitudes and phases.) Thus, here we need only consider the inter-
action between different $y$ modes with the same zonal wavenumber ($p = n$). This portion of the heat flux is written as

$$\overline{v_{nl} F_{nq}} = 4n \mu_{nq} \sin(l\pi y) \sin(q\pi y) \times$$

$$\times [ |A_{nl}| |A_{nq}| \cosh(2\mu_{nq}z) \sinh(2\mu_{nq}z) \sin(\eta_{nq} - \theta_{nl}) +$$

$$+ |B_{nl}| |B_{nq}| \sin(2\mu_{nq}z) \cosh(2\mu_{nq}z) \sin(\eta_{nq} - \eta_{nl}) +$$

$$+ |A_{nl}| |B_{nq}| \cosh(2\mu_{nq}z) \cosh(2\mu_{nq}z) \sin(\eta_{nq} - \theta_{nl}) +$$

$$+ |B_{nl}| |A_{nq}| \sinh(2\mu_{nq}z) \sinh(2\mu_{nq}z) \sin(\theta_{nq} - \eta_{nl})].$$  (23)

When $q \neq l$, since the two $y$ modes of wavenumber $n$ have different phase speeds, the phase differences between these two modes, $\theta_{nq} - \theta_{nl}$, $\eta_{nq} - \eta_{nl}$, $\eta_{nq} - \theta_{nl}$ and $\theta_{nq} - \eta_{nl}$, may be time-dependent. Their sinusoidal variations with time cause a time
variation of $v_{nl} T_{nl}$. In the present case, either $l$ or $q$ must be 1 and the other must be 2 so that $\sin(l\pi y) \sin(q\pi y)$ is antisymmetric about the mid-channel ($y = 0.5$). Thus, $v_{nl} T_{nl}$ vanishes at $y = 0.5$, and is out of phase for $y < 0.5$ and $y > 0.5$ at all times.

When $q = l$, the heat flux component due to self-interaction of wave $n$ with the $\sin(l\pi y)$ mode, (23) is simplified to

$$v_{nl} T_{nl} = 4n \mu \sin^2(l\pi y)|A_{nl}| |B_{nl}| \sin(\eta_{nl} - \theta_{nl}). \quad (24)$$

$v_{nl} T_{nl}$ is symmetric about $y = 0.5$. For $l = 1$, the maximum of $v_{nl} T_{nl}$ occurs at $y = 0.5$, while for $l = 2$, two maxima occur, at $y = 0.25$ and $0.75$. Since the wave component is steady, i.e. the barotropic and baroclinic parts of the wave component have constant amplitudes, $|A_{nl}|$ and $|B_{nl}|$, and constant phase difference between them, $\eta_{nl} - \theta_{nl}$, the portion of the heat flux, $v_{nl} T_{nl}$, is time-independent. To overcome Ekman dissipation, the steady wave component must have a structure such that $\eta_{nl} - \theta_{nl}$ must be positive, i.e. the phase of the baroclinic part of the wave must exceed that of the barotropic part of the wave. In other words, the wave component must tilt westward with height to extract potential energy from the mean flow to compensate the energy loss due to dissipation.

Based on (23) and (24), it is seen that these steady waves make three types of contribution to the total heat flux $\bar{vT}$:

- **H1**: Due to self-interaction of each steady wave component with the first $y$ mode. This is time-independent, and symmetric about $y = 0.5$. The maximum is at $y = 0.5$.
- **H2**: Due to self-interaction of each steady wave component with the second $y$ mode. This is also time-independent, and symmetric about $y = 0.5$. However, there are two maxima, at $y = 0.25$ and $0.75$.
- **H3**: Due to wave–wave interaction between two wave components with the same zonal wavenumber but different $y$ structures. This is time-dependent, and antisymmetric about $y = 0.5$. The maximum time variations occur at $y = 0.25$ and $0.75$. The contribution vanishes at $y = 0.5$.

These three types of contribution can be verified by numerical results as shown in Fig. 11. The time evolution of total wave heat flux $\bar{vT}$ during $t = 1000-2000$, at the same parameter setting as that for Fig. 10, is calculated at $y = 0.75$, 0.5 and 0.25. At $y = 0.5$, $\bar{vT}$ is a small positive constant. All the second modes vanish at $y = 0.5$ so that only the self-interactions of first $y$ modes made their contributions to $\bar{vT}$, i.e. $\bar{vT}$ contains only H1. At $y = 0.25$ and 0.75, $\bar{vT}$ exhibits regular oscillation with period $P = 39.47$, but is out of phase between the south ($y = 0.25$) and the north ($y = 0.75$). The oscillation is caused by the contribution of H3. However, $\bar{vT}$ does not oscillate around the zero value, which would be the case if there were only the contribution of H3. The time-averaged values of $\bar{vT}$ at both $y = 0.25$ and 0.75 are positive, and due to contributions from H1 and H2. Globally, at all points in time, the $y$-averaged value of $\bar{vT}$ is a positive constant, which shows that the total wave always extracts potential energy from the mean flow to overcome the Ekman dissipation so that the wave and the mean flow can maintain an equilibrium. Locally, the situation is different at different $y$. The wave may extract potential energy from the mean flow in the south ($\bar{vT} > 0$), while transporting the same amount of potential energy back to the mean flow in the north ($\bar{vT} < 0$), and vice versa. The total wave field may not be in static energy balance at different parts of the channel; wave energy is just redistributed in the $y$ direction. For example, at $t = 1700$, an intense wave centre is observed in the north and a weak wave field in the south, as shown in Fig. 10. It is also seen from Fig. 11 that at this moment.
$\bar{v}T$ has passed its minimum at $y = 0.25$ and maximum at $y = 0.75$. The wave in the north has extracted less potential energy from the mean flow than before so that it is decaying; meanwhile, the wave in the south has transferred less potential energy to the mean flow than before so that it is growing. This tendency continues until $t = 1715$, when $\bar{v}T$ attains its maximum in the south and minimum in the north. At $t = 1720$, an intensive wave centre is observed in the south and a weak wave flow in the north. This situation is just opposite to that at $t = 1700$. The intensity of the wave flow may keep changing so that it
vacillates with time with period $P = 39.47$ and is out of phase between north and south. Meanwhile, the intensity of the mean flow also vacillates with time at the same period and is out of phase between north and south.

This characteristic wave energy redistribution in the $y$ direction without changing the total amount of energy is similar to the basic characteristics of structural vacillation observed in annulus experiments of Pfeffer et al. (1980a, b). The dominant characteristic of this type of vacillation is the radial fluctuation of the wave energy centres, which appear alternately on either side of mid-radius of the annulus. However, Pfeffer et al. did not report significant heat flux time variation in structural vacillation, because their measurements were made at mid-channel. According to the above analyses, the contribution of $H_3$, which causes $vT$ away from the mid-channel to vacillate with time, was zero.

Similarly, the momentum flux of the total wave can be shown to be

$$\overline{uv} = \sum_{n=1}^{3} \sum_{p=1}^{3} \sum_{l=1}^{2} \sum_{q=1}^{2} u_{nl} v_{pq} = -\sum_{n=1}^{3} \sum_{p=1}^{3} \sum_{l=1}^{2} \sum_{q=1}^{2} \frac{\partial \phi_{nl}}{\partial y} \frac{\partial \phi_{pq}}{\partial x}. \quad (25)$$

Again, the contribution for $p \neq n$ is zero. Unlike the case of $\overline{vT}$, the contribution of self-interaction of a wave to $\overline{uv}$ is also zero. Thus, we need only consider the interaction between wave components with the same zonal wavenumber but different $y$ modes. The contribution from each zonal wavenumber is

$$\overline{u_{n1}v_{n2}} + \overline{u_{n2}v_{n1}} = n\pi[3 \sin(\pi y) - \sin(3\pi y)] \times$$

$$\times \left[|A_{n1}||A_{n2}| \sin(\theta_{n2} - \theta_{n1}) \cosh(2\mu_{n1}z)(\cosh(2\mu_{n2}z) +
+ |B_{n1}||B_{n2}| \sin(\eta_{n2} - \eta_{n1}) \sinh(2\mu_{n1}z) \sinh(2\mu_{n2}z) +
+ |A_{n1}||B_{n2}| \sin(\eta_{n2} - \theta_{n1}) \cosh(2\mu_{n1}z) \sinh(2\mu_{n2}z) +
+ |B_{n1}||A_{n2}| \sin(\theta_{n2} - \eta_{n1}) \sinh(2\mu_{n1}z) \cosh(2\mu_{n2}z)| \right]. \quad (26)$$

The momentum flux of each wave component is symmetric about the mid-channel. For each wave, there is a maximum at $y = 0.5$; the time variations in the south and the north are in phase. The evolution of the total momentum flux $\overline{uv}$ during $t = 1000-2000$, at the same parameter settings, is calculated at $y = 0.75, 0.5$ and $0.25$, and shown in Fig. 12. At all three $y$, $\overline{uv}$ vacillates around zero and is in phase. At each time point, $\overline{uv}$ is symmetric about $y = 0.5$, where the largest excursions occur. Thus, when $\overline{uv}$ is divergent in the south, causing easterly mean flow acceleration, it is convergent in the north, causing westerly mean flow acceleration there; and vice versa (figures are not shown). When the easterly mean flow is strong enough in the north (south), a high (low) may be blocked in north (south) of that easterly mean flow. For example, at $t = 1700$, $\overline{uv}$ has just passed its minimum values ($<0$) for all three $y$. Thus, $\partial \overline{uv}/\partial y < 0$ (convergent) in the south and $\partial \overline{uv}/\partial y > 0$ (divergent) in the north. This tendency continues until about $y = 1715$ when $\overline{uv}$ attains its maximum at all $y$. During the period $t = 1700-1715$, easterly mean flow increases in the south and decreases in the north so that the high centre in the north at $t = 1700$ is weakening while a low centre in the south is intensifying, to be cut off as shown at $t = 1720$. In fact, in the present case both heat flux and momentum flux may play a role in the blocking of highs and lows.
It is also interesting to note that the wavenumbers and frequencies of each interactive triad strictly satisfy the three requirements for resonance discussed by Pedlosky (1979, Eq. (3.26.21)), i.e.

\[
\begin{align*}
n_i + n_j + n_k &= 0 \\
l_i + l_j + l_k &= 0 \\
\sigma_i(n_i, l_i) + \sigma_j(n_j, l_j) + \sigma_k(n_k, l_k) &= 0
\end{align*}
\]
where we have allowed the $\eta$, $\lambda$ and $\omega$ to take on both positive and negative values. We now check (27) based on the data given in Table 3. In the simplest 3-wave system, where only one interactive triad exists, we have

$$\sigma_{11} + \sigma_{12} = \sigma_{21}$$

which satisfies (27). The conditions in (27) are also satisfied in the 4-, 5- or 6-wave systems, where two, three or four interactive triads exist. For example, in the 6-wave system, all the interactive triads satisfy condition (27), i.e.

$$\begin{align*}
\sigma_{11} + \sigma_{12} &= \sigma_{21} \\
\sigma_{11} + \sigma_{21} &= \sigma_{32} \\
\sigma_{12} + \sigma_{21} &= \sigma_{31} \\
\sigma_{11} + \sigma_{22} &= \sigma_{31} 
\end{align*}$$

(29)

Also, it is found that the three frequency differences between the two waves of each pair of waves with the same zonal wavenumber but different $y$ structures are the same, i.e.

$$|\sigma_{n1} - \sigma_{n2}| = \text{constant}$$

(30)

which ensures that the structural oscillations with different zonal wavenumbers have the same period $P$. In the 4-, 5- and 6-wave systems where 1, 2 and 3 pairs of waves exist, $P = 39.17$, 39.44 and 39.47, respectively. Thus, the period can be calculated from any one of the pairs in a system.

The frequency relations expressed by (27) and (30) indicate a ‘frequency-locking’ mechanism in multi-wave steady states. Owing to the frequency locking, the apparent propagation speed of the pattern of the total wave field is the same at all heights, as shown in Fig. 10, although the wave components have different vertical structures. The frequency relations of (27) and (30) are not satisfied by the wave components at any multi-wave oscillation state, no matter what the truncation level is. Thus, the ‘frequency locking’ is a unique mechanism for a flow with a multi-wave steady state to maintain each wave component having a constant amplitude and phase speed. However, such a delicate ‘frequency locking’ does not exist for all values of $R$. Only certain values of $R$ can maintain frequencies to satisfy (27) and (30) and result in resonant triad(s). Beyond those values, other flow regimes may occur. Based on our limited results, the values of $R$ for multi-wave steady states are about half of $S^{-1}$ (the slope of isentropes), or slightly larger, which is larger for larger $R$ if other parameters are fixed. In order to exhibit the ‘frequency-locking’ mechanism represented by both (27) and (30), it is required that the system contains at least two interactive triads and two pairs of waves with the same zonal wavenumber and different $y$ structures. Among the present tested systems, the 3- and 4-wave systems are able to show only the mechanism presented by (27), but not (30); while the 5- and 6-wave systems are able to show both. It is expected that an even higher-order model may also show a similar behaviour of the multi-wave steady state and the ‘frequency-locking’ mechanism as shown by the 6-wave system.

The ‘frequency-locking’ mechanism does not occur in our linear system. In fact, this mechanism is a result of nonlinear tuning among the interactive waves and mean flow at some favourable parameter settings. The sloping boundaries exert a dispersive effect on waves, which is in favour of the nonlinear tuning to attain such a degree that a multi-wave steady state can be maintained by their frequencies being locked to satisfy (27) and (30). A similar multi-wave steady state was also observed by Mak (1985) in his $\beta$-plane model with wave–wave interaction. Although it was not reported whether or not the
frequencies of the triad satisfy (27) and (30), it is expected from the present results that a similar ‘frequency-locking’ mechanism may be responsible for the multi-wave steady state observed in the β-plane model, because the β has a similar dispersive effect on waves.

7. DISCUSSION

This work studies the effects of oppositely sloping boundaries, Δ, with Ekman dissipation, δ, on linear and nonlinear baroclinic waves with and without wave–wave interaction, in an Eady-type truncated model, and compares these effects with β effect wherever possible.

In the linear framework, the stability criteria are found. Wave instability may occur only when Δ is less than the slope of the isentropes of the basic state. Δ slightly stabilizes short waves, while greatly stabilizing long waves when Δ exceeds a critical value. The most unstable wavenumber may be shifted higher by increasing Δ. The waves are dispersive and move westward. Δ affects both stability and phase speed in a manner similar to the β effect. When the sloping boundaries have Ekman layers, the combined effect is more complicated, and is also different for short and long waves. For short waves, it has a destabilizing effect for small dissipation while having a stabilizing effect for large dissipation. For long waves, it always has a stabilizing effect.

In the nonlinear framework, the combined effects of Δ and δ with and without wave–wave interaction are quite different. When wave–wave interaction is absent, vacillation and aperiodic flows may be stabilized to a single-wave steady state by Δ so that the domain of vacillation is reduced. The preferred wavenumber of a single-wave steady state may transit to a higher number as Δ increases. The northward heat flux of the preferred wavenumber increases for small and moderate Δ and decreases with large Δ. When wave–wave interaction is included, there are richer flow regimes than those without wave–wave interaction. There exist single-wave steady states, multi-wave steady states, multi-wave vacillation and aperiodic flows; and Δ enlarges the vacillation domain.

In multi-wave steady states, the interference among the steady waves may result in either wave dispersion or vacillation. Vacillation and dispersion may be distinguished by whether or not there is a time-dependent energy conversion between the wave and mean flow fields.

A ‘frequency-locking’ mechanism is found in multi-wave steady states, where the frequencies of wave components satisfy the frequency requirements for resonant triads, and the frequency differences of each pair of waves, with the same zonal wavenumber but different y structures, are the same. It is expected that the dispersive effect of the sloping boundaries, as well as of β, is in favour of such a ‘frequency-locking’ mechanism.

In annulus experiments studied by Pfeffer et al. (1980a, b), the Rossby number ranged between 0-01 and 0-07. If we consider an average value of Ro = 0-04, then the average value of Δ considered here, 2-5, corresponds to a physical slope of about 0-1, so that the slopes are small enough to be imposed at the boundaries. In Mason's (1975) experiments the slopes were 12° and 36°. Since he used a narrow annulus, the maximum topography heights were about 0-08H and 0-23H, which correspond to our Δ = 2 and 5-7, respectively. Thus, the present results may be compared with his 12° case but not his 36° case.

The results discussed here are obtained from a highly truncated spectral model, a so-called ‘low-order model’. The justifications for using such a model are as follows. First, this model is an extension of that of Weng et al. (1986) and Weng and Barcilon (1987). The results from their models qualitatively agreed well with the observations in
annulus experiments which have been discussed by Pfeffer et al. (1980a, b) and Buzyna et al. (1984), in the aspects of the evolutionary behaviour of amplitude and structural vacillation, wave structure, and the transitions between some flow regimes. In the amplitude and structural vacillation regimes of these experiments, a dominant zonal wavenumber with an amplitude of an order larger than those of its harmonics was often observed. The short waves or high-order harmonics in their annulus experiments seldom dominated over long periods of time. Second, in the experiments with oppositely sloping boundaries performed by Mason (1975), a dominant wavenumber was often observed. In some of his experimental runs, wave dispersion was seen, but in most of the experiments only a single regular wave was present. Third, in annulus experiments with a sloping bottom, Pfeffer, Kung and Buzyna (personal communication) found that the number of dominant waves is usually less than three. These annulus experiments have confirmed that the flows in the annulus with oppositely sloping boundaries or a sloping bottom may be of low-order character. Fourth, comparison among the numerical experiments with different truncation levels in the present work has shown that the truncation level should be determined by the phenomenon under investigation and its possible mechanism. For different phenomena we may use different truncations in order to catch the essence of these phenomena while minimizing computer time. For present purposes, the 5- and 6-
wave systems are suitable to describe some annulus phenomena with sloping boundaries and exhibit the 'frequency-locking' mechanism. However, it should be emphasized that the justification for using low-order models does not preclude the necessity of using a high-order model to describe a high-order phenomenon.

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APPENDIX

1. Coefficients in (7):

\[ a_{1,\nu l} = \frac{(1 + \rho)\delta (n^2 + l^2 \pi^2)}{4\mu_{\nu l} T_{\nu l}} \]
\[ a_{2,\nu l} = \frac{(1 - \rho)\delta (n^2 + l^2 \pi^2)}{4\mu_{\nu l}} \]
\[ a_{3,\nu l} = \frac{n(T_{\nu l} - \mu_{\nu l})}{2\mu_{\nu l} T_{\nu l}} \]
\[ a_{4,\nu l} = \frac{(1 + \rho)\delta (n^2 + l^2 \pi^2) T_{\nu l}}{4\mu_{\nu l}} \]
\[ a_{5,\nu l} = \frac{n(1 - \mu_{\nu l} T_{\nu l})}{2\mu_{\nu l}} \]
\[ a_{6,\nu l} = \frac{n\Delta S T_{\nu l}}{2\mu_{\nu l} T_{\nu l}} \]
\[ a_{7,\nu l} = \frac{n\Delta S T_{\nu l}}{2\mu_{\nu l}} \]
\[ T_{\nu l} = \tanh \mu_{\nu l}. \]
2. Linear instability criterion for the inviscid case:

For \( S = O(10^{-1}) \) and very long zonal wave with the lowest \( y \) mode, i.e. for \( \mu = [(n^2 + \pi^2)S]^{1/2}/2 < 1 \), \( T \) may be approximately expressed by

\[
T = \mu - \mu^3/3. \tag{A1}
\]

Then, \( \Delta_c \) in (15) becomes

\[
\Delta_c = \frac{n^2 + \pi^2}{2} \sqrt{\frac{8}{3} [1 - \frac{8}{3} (n^2 + \pi^2) S^2 \left( \frac{8}{3} \pi^2 S - 1 \right)]}. \tag{A2}
\]

For the minimum zonal wavenumber, i.e. for \( n = 0 \) and \( l = 1 \), it is likely that for \( S = O(10^{-1}) \),

\[
1 - \frac{4}{3} \pi^2 S^2 \left( \frac{8}{3} \pi^2 S - 1 \right) > 0. \tag{A3}
\]

Thus, when

\[
\Delta < \Delta_{c_0} = \frac{\pi^2}{3} \sqrt{\frac{8}{3} \left[ 1 - \frac{8}{3} \pi^2 \left( \frac{8}{3} \pi^2 S - 1 \right) \right]} \tag{A4}
\]

where \( \Delta_{c_0} \) is the value of \( \Delta_c \) for \( n = 0 \), the effect of \( \Delta \) on stability is not large enough to result in a long-wave cut-off, i.e. all long waves are unstable. The stabilizing effect of \( \Delta \) becomes prominent when \( \Delta > \Delta_{c_0} \), resulting in a long-wave cut-off. The existence of \( \Delta_{c_0} \) is due to the \( y \) structure of the wave. If the wave is \( y \)-independent, then \( \Delta_{c_0} = 0 \), and a long-wave cut-off will occur even for a small \( \Delta \), which was shown in the 2-D models of BG and M and in 2-D 2-layer \( \beta \)-plane models.

For short waves, i.e. for large \( \mu \), the approximation (A1) is not valid. The criterion (15) may be rewritten as

\[
1 - \mu_c T_c > \frac{(1 - T_c^2)^2}{T_c (\mu_c - T_c)} \frac{\Delta^2 S^2}{4} = P > 0 \tag{A5}
\]

where \( P \) is a small positive quantity, because \( \mu > T \) and \( T \to 1(T < 1) \) for large \( \mu \). In the Eady model without sloping boundaries and Ekman dissipation (Eady 1949), the marginal curve for the instability is given by

\[
\mu_c T_E = 1. \tag{A6}
\]

Therefore, the criterion (A5) for large \( \mu \) may be expressed as

\[
\mu_c T_E - \mu_c T_c > P > 0 \tag{A7}
\]

or

\[
\mu_c T_c < \mu_c T_E - P. \tag{A8}
\]

Thus, the \( \Delta \) effect shifts the short-wave cut-off away from Eady cut-off in the direction of longer waves by a small amount. The shorter the wavelength near the short-wave cut-off, the smaller the \( P \) owing to smaller \((1 - T_c^2)^2\), and the less the effect of sloping boundaries on the stabilities of these short waves.

3. Linear instability criterion for the viscous case:

From (8) to (10) we have

\[
c_t = -\frac{1}{2n} \left( a_6 + a_7 \right) - \frac{F}{2} L^{-1/2} \frac{1}{2n} \left( I^{1/2} - 1 \right) \tag{A9}
\]
where

\[ I = \frac{1}{4} \left( 1 - E + \sqrt{(1 - E)^2 + F^2} \right) \tag{A10} \]

and

\[ \begin{align*}
E &= \frac{1}{(a_1 + a_4)^2} \left\{ 4a_1a_4 + (a_6 - a_7)^2 - 4(a_3^2 - a_5a_3) \right\} \\
F &= -\frac{2}{(a_1 + a_4)^2} \left\{ (a_4 - a_1)(a_6 - a_7) + 2a_2(a_3 + a_5) \right\}.
\end{align*} \tag{A11} \]

For instability to occur, \( c_i \) must be positive, i.e. the inequality

\[ I > 1 \quad \text{or} \quad F^2 > 4E \tag{A12} \]

must be satisfied. The marginal curve on the \( \delta - \Delta \) plane is given by Eq. (18):

\[ \delta_c^2 = \frac{1}{4\rho a^2} \left( -s\Delta^2 + \beta \Delta + c \right) \]

where

\[ \begin{align*}
s\Delta &= 4n^2 S^2 \frac{(1 - T^2)^2}{(1 + T^2)^2} \\
\beta &= 4Sn^2 \frac{(1 - \rho)}{(1 + \rho)} \frac{(1 - T^2)^2}{(1 + T^2)^2} \left( \mu T + \frac{\mu}{T} - 2 \right) \\
c &= 4n^2 \left[ \frac{(\mu - T)(1 - T)}{T} + \frac{(1 - \rho)}{1 + \rho} \left( \frac{2T}{1 + T^2} - \mu \right)^2 \right].
\end{align*} \tag{A13} \]

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