Analytical model for the growth of the coastal internal boundary layer during onshore flow

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(Received 14 December 1988; revised 21 June 1989)

SUMMARY

A model of near-neutral and convective steady-state internal boundary layer evolution is presented. The model deals with the internal boundary layer that forms over land in coastal and lake areas during onshore winds. Near the ground, the growth of the internal boundary layer is controlled by friction velocity in accordance with surface layer theory. Further downwind, the growth is determined by the atmospheric stability and friction velocity within the internal boundary layer, and the temperature gradient in the air above. The wind profile inside the internal boundary layer is assumed to follow Obukhov similarity theory. An expression for the strength of the inversion that caps the layer is derived and used in the model. A comparison is carried out with independent experimental observations of internal boundary layer growth in the sea–land transition.

Kinematic heat flux through the top of the internal boundary layer is described by the formulation \(-\frac{1}{2}(w'\theta')_s = 0.2(w'\theta')_s + 2.5 u_{*s}^2 T/gh + \text{the Zilitinkevich correction}\). The terms \(0.2(w'\theta')_s\) and \(2.5 u_{*s}^2 T/gh\) account for convective and mechanical turbulence, respectively, and the Zilitinkevich correction is a turbulent kinetic-energy storage term that ensures finite growth rate of the internal boundary layer near the ground. The relative importance of mechanical and convective turbulence as well as the Zilitinkevich correction is discussed. The Zilitinkevich correction dominates the growth process of the internal boundary layer when it is lower than roughly 50 m. As the layer grows, the importance of the Zilitinkevich correction diminishes. Then mechanical turbulence dominates the growth process until the internal boundary layer has reached a height of approximately \(1-4L\). Further growth is controlled mainly by convective turbulence. Conditions of high wind speed and large values of the potential temperature gradient over water result in a deep zone where the growth of the internal boundary layer is controlled by mechanical turbulence. With zero-zero potential temperature gradient over water, the zone becomes shallow and may vanish.

1. INTRODUCTION

Owing to the difference in surface temperature and roughness between land and water, an internal boundary layer develops downwind from a coastline.

There are several semi-empirical expressions for the growth of the internal boundary layer based on physical considerations, dimensional analysis and experimental results; for a summary see Sunder and SethuRaman (1985). In 1961 Prophet proposed a phenomenological model that relates the height of the internal boundary layer to overwater stability, wind speed and downwind distance (see Collins 1971). Raynor et al. (1974) proposed a modified expression, later theoretically substantiated by Venkatram (1977). Batchvarova and Yordanov (1987b) compared several semi-empirical expressions with independent data and found agreement within a factor of 2–3.

Some of the early numerical simulations of the internal boundary layer were carried out with two-dimensional, steady-state Eulerian models (Onishi and Estoque 1968; Peterson 1969). The growth rate under neutral conditions was reported to be roughly one to ten (Peterson 1969). Taylor (1969) used a mixing-length model relating turbulent shear stress to mean velocity under neutral stability. The theory was later extended to include thermal conditions (Taylor 1970). Rao et al. (1974) investigated numerically the distributions of mean wind, shear stress and turbulent energy by a higher-order closure model. Recently, Andrén (1987) simulated the internal boundary layer over Copenhagen on one of the days of the Øresund experiment (Gryning 1985) with a higher-order closure model. Agreement between simulation and experiment was good.

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A quite different approach uses modelling principles of the daytime convective layer over flat terrain (Tennekes (1973) and Carson (1973)) in a Lagrangian manner. Venkatram (1977) and Steyn and Oke (1982) studied the dynamics of the internal boundary layer with numerical models formulated in this manner. By simplifying the governing equations, Venkatram (1977) obtained an analytical expression that was almost identical to the simple model of Raynor et al. (1974). Batchvarova and Yordanov (1987a) and Yordanov and Batchvarova (1988) derived an analytical expression for the height of the thermally induced boundary layer, based on the equation for heat conservation and continuity. In this paper we present an analytical model formulated along these lines for the height of the internal boundary layer for atmospheric neutral and convective conditions.

2. INTERNAL BOUNDARY LAYER

Before formulating the model, let us briefly look at the properties of turbulence and temperature in an internal boundary layer. Only conditions of stably stratified air over cold water are dealt with here. Then, the internal boundary layer forming downwind of the coastline is characterized by vigorous turbulence, whereas turbulence in the air outside is limited. The interface between the two turbulence regimes can be very sharp, as noted in Fig. 1(a) (see also Batchvarova and Gryning 1989), and it fluctuates considerably in time, with a size related to the penetration depth of the larger eddies.

The entrainment zone is the outermost portion of the internal boundary layer. It is confined between the height reached by only the most vigorous turbulent eddies and the height below which the internal boundary layer is almost always present. The thickness of the zone is considerable. For convectively mixed layers, Deardorff et al. (1980) found it to be 0.2 to 0.4 of their height. The potential temperature is nearly constant from the ground up to the entrainment zone, where it increases to conform with the temperature aloft (Smedman and Högsström 1983). The internal boundary layer height will be taken here as the height of an infinitesimally thin temperature jump, introduced in the entrainment zone in such a way that conservation is obtained of the total heat deficit of the internal boundary layer with respect to the air aloft, Fig. 1(b). In this paper we consider a model for the internal boundary layer, the height of which will be denoted \( h \).

![Figure 1](image-url) (a) Turbulence (dissipation rate) over Øresund and surroundings at 310 m on 5 June 1984. The abrupt change in turbulence about 5 km inland over Copenhagen marks the interface of the turbulent internal boundary layer. (b) Illustration of the height, \( h \), of the internal boundary layer. Full line is the measured potential temperature. The areas of the two hatched regions are equal.
3. Model

Figure 2 shows schematically the physical basis of our model. Above the internal boundary layer, air is stably stratified with a potential temperature gradient independent of height and time. Inside the internal boundary layer, turbulence is assumed to be sufficient to maintain a uniform vertical distribution of potential temperature. The finite inversion, which caps the layer, is assumed to be infinitesimally thin.

\[
\frac{\partial \theta}{\partial z} = \frac{\partial \Omega}{\partial z}
\]

Figure 2. Schematic illustration of the physical system that forms the basis of the model.

(a) Inversion strength

Here we obtain an expression for the inversion strength, \( \Delta \), as a function of \( h \) and atmospheric stability. It is derived following the modelling principles of Tennekes (1973) for boundary layer height over horizontally homogeneous areas, and we assume the results to be applicable to internal boundary layers as well.

Because warm air is entrained in the cooler boundary layer, the heat flux at the top of the boundary layer is downward. Invoking the entrainment rate \( dh/dt \), we have (Lilly 1968)

\[
-(w' \theta')_h = \Delta \frac{dh}{dt} \tag{1}
\]

where \( (w' \theta')_h \) is the vertical kinematic heat flux at the top of the boundary layer and \( t \) is time. The inversion strength, \( \Delta \), will tend to increase, caused by entrainment of the internal boundary into stable air above. It will tend to decrease as the boundary layer is heated, because of entrainment of warm air from above and heating of the ground. Therefore

\[
d\Delta/dt = \gamma dh/dt - (\partial \theta/\partial t)_b \tag{2}
\]

where \( \gamma \) is the potential temperature gradient above the layer, \( \theta \) the potential temperature, and \( (\partial \theta/\partial t)_b \) the heating rate of the air in the boundary layer. The heating rate is

\[
(\partial \theta/\partial t)_b = (w' \theta')_s/h - (w' \theta')_h/h \tag{3}
\]

where \( (w' \theta')_s \) is the vertical kinematic heat flux at the surface. As the heat flux at the
inversion is negative, it contributes to the heating of the layer and
\[ -(w'\theta')_h = A(w'\theta')_s + Bu^3_*/T/gh \]  
(4)
is applicable. Here, $u_*$ is the friction velocity, $g/T$ the buoyancy parameter, and $A$ and $B$ are constants. This was derived from the turbulent energy budget in the entrainment zone (Tennekes and Driedons 1981). The terms $A(w'\theta')_s$ and $Bu^3_*/T/gh$ account for convective and mechanical turbulence, respectively. Tennekes (1973) estimated $A$ to be 0.2, a value now generally accepted. Presumably $B$ is of order unity, but its precise value is still uncertain. Tennekes (1973) and van Dop et al. (1982) suggest $B = 2.5$. Driedons (1982) points out that the value of $B$ is uncertain, but recommends $B = 5$.

An expression for the temperature difference across the inversion is derived by substituting Eqs. (1) and (3) into (2):
\[ (h d\Delta/dt + \Delta - \gamma h)dh/dt = -(w'\theta')_s \]  
(5)
and then from Eq. (1):
\[ h d\Delta/dh + \Delta - \gamma h = -\Delta \frac{(w'\theta')_s}{A(w'\theta')_s + Bu^3_*/T/gh} \]  
(6)
By introducing the Obukhov length
\[ L = -u^3_*/T/\kappa \gamma (w'\theta')_s \]  
(7)
where $\kappa$ is the von Kármán constant, Eq. (6) can be rewritten as
\[ \frac{d\Delta}{dh} + \Delta \left( \frac{1}{h} + \frac{1}{Ah - B\kappa L} \right) = \gamma. \]  
(8)
This is an ordinary linear first-order differential equation; the solution is
\[ \frac{\Delta}{\gamma h} = \left\{ \frac{1}{(1 + 2A)(Ah - B\kappa L)^{(1+2A)/A}} + \frac{B\kappa L}{1 + A} \frac{(Ah - B\kappa L)^{(1+A)/A}}{1 + 2A} \right\}^{-1} \]  
(9)
where the integration constant was chosen so that Eq. (9) always remains finite:
\[ \Delta/\gamma h = \begin{cases} 1/2 & \text{for } h/L \rightarrow -0 \\ A/(1 + 2A) & \text{for } h/L \rightarrow -\infty. \end{cases} \]  
(10)
Figure 3 shows the dimensionless inversion strength, $\Delta/\gamma h$, plotted as a function of the stability parameter $h/L$ for $A = 0.2$ and $B = 2.5$. A simple approximation with correct asymptotic limits for neutral and convective conditions
\[ \frac{\Delta}{\gamma h} \approx \frac{Ah - B\kappa L}{(1 + 2A)h - 2B\kappa L} \]  
(11)
is also shown. By using a somewhat larger value of $B$ in Eq. (11), we obtain a better fit to Eq. (9), but the form suggested here results in major simplifications in the equations at a later stage.
Figure 3. Dimensionless temperature difference across the inversion, $\Delta / y h$, as a function of atmospheric stability $h/L$. The solid line shows the analytic solution, Eq. (9); the dashed line its approximation, Eq. (11).

When the surface heat flux is zero, it follows from Eq. (9) that $\Delta = \frac{1}{2} y h$, irrespective of the value of $B$, as can be seen in Fig. 4(a). Originally stratified air has been mixed into the boundary layer without net addition of heat, and therefore the two hatched regions have equal areas. Figures 4(b) and (c) illustrate conditions when the internal boundary layer is convective and it is permissible to disregard the effect of mechanical turbulence. The encroachment situation, Fig. 4(b), where the heat flux at the inversion is completely disregarded, corresponds to $A = 0$. Then $\Delta = 0$ and heat supplied from the surface is simply used to fill the original temperature profile. The physically more realistic case where $A = 0.2$, as in Fig. 4(c), results in $\Delta = \frac{1}{2} y h$. Then the boundary layer receives heat both through the interface and from the surface. Let us leave for a moment the boundary layer case and consider an internal boundary layer. Equations (9) and (11) give the temperature difference across the inversion as a function of $h$ when the production of mechanical and convective turbulence varies inland in such a way that $L$ is constant.

Figure 4. The heat budget in the case of (a) mechanical entrainment, (b) convective entrainment with $A = 0$ (encroachment), and (c) convective entrainment with $A = 0.2$. 
(b) Height of the internal boundary layer

Derivation of a prediction equation for $h$ as a function of downwind distance from the coastline involves the basic equation of heat conservation (steady-state, no internal sources or sinks)

$$\partial(u\theta)/\partial x + \partial(w\theta)/\partial z = - \partial(w'\theta')/\partial z$$  \hspace{1cm} (12)

and the continuity equation

$$\partial u/\partial x + \partial w/\partial z = 0$$  \hspace{1cm} (13)

where $u$ and $w$ are the mean wind velocities in the $x$ and $z$ directions, respectively. A detailed description of the derivation is given in appendix A. Briefly, Eq. (12) is integrated from $z = 0$ to $h(x)$, obtaining the boundary condition for $w$ at the top of the internal boundary layer, $w_h$, from Eq. (13). The result is

$$\frac{d\theta_h}{dx} \int_0^{h(x)} u \, dz = (w'\theta'),_s - (w'\theta')_h$$  \hspace{1cm} (14)

where $\theta_h$ is the potential temperature at height $h(x)$. From Fig. 2 it can be seen that

$$\theta_h = \gamma h - \Delta + \text{constant.}$$  \hspace{1cm} (15)

Differentiation of Eq. (15) with respect to $x$, taking $d\Delta/dh$ from Eq. (8) and $\Delta$ from the approximation Eq. (11), and then substituting these into Eq. (14) leads to

$$\left\{ \gamma \left( \frac{(1 + A)h - B\kappa L}{(1 + 2A)h - 2B\kappa L} \right) \int_0^{h(x)} u \, dz \right\} \frac{dh}{dx} = (w'\theta'),_s - (w'\theta')_h.$$  \hspace{1cm} (16)

To calculate the integral in Eq. (16), the wind profile within the internal boundary layer must be known. Little information is available on this profile. However, describing the wind velocity profile by Obukhov similarity theory (strictly, valid only under homogeneous, stationary conditions) yields

$$\int_0^{h(x)} u \, dz = \{u_h - R(h/L)u_0\}h$$  \hspace{1cm} (17)

where $u_h$ is the wind velocity in the upper part of the internal boundary layer, and

$$R(h/L) = \frac{1 - (1 - 16h/L)^{3/4}}{12\kappa(h/L)}.$$  \hspace{1cm} (18)

Figure 5 shows $R$ as function of the stability parameter $h/L$. Under very convective conditions $R = 0$, and in a completely neutral atmosphere $R = 1/\kappa$.

For the term $(w'\theta')_h$ in Eq. (16), we use a slightly modified version of Eq. (4). When $h$ is small, the internal boundary layer entrains rapidly. To maintain a finite growth rate, the effect of the time derivative of the turbulent kinetic energy, the so-called Zilitinkevich correction (Zilitinkevich 1975) is included as a third term on the right-hand side of Eq. (4). The expression for $(w'\theta')_h$ then reads

$$-(w'\theta')_h = A(w'\theta'),_s + Bu_0^2T/gh - (Bu_0^2T/Cgh) \, dh/dt.$$  \hspace{1cm} (19)

In the original formulation, both convective and mechanical turbulence are considered in the Zilitinkevich correction. For simplicity, we include only mechanical turbulence.
Inserting Eqs. (19) and (17) into Eq. (16) and setting $t = x/u_*$ in the Zilitinkevich correction, leads to the differential equation for $h$:

$$\left\{ \frac{h^2}{(1+2A)h-2BkL} \right\} \left( u_h - R(h/L)u_* \right) + \frac{Bu^2 u_0 T}{\gamma \sigma((1+A)h-BkL)} \int \frac{dh}{dx} = \left( \frac{w' \theta'}{\gamma} \right).$$

(20)

Close to the ground is follows that

$$dh/dt = C u_*.$$

(21)

When $C = 1.3$, Eq. (21) conforms to Miyake's theory for the growth of an internal boundary layer in the neutral surface layer (after Panofsky and Dutton 1984). The analytic solution to Eq. (20) is rather unattractive (appendix B). A simple approximation to the analytic solution is obtained by neglecting the rather weak dependence of the $R$ function on $h$. Taking $R$ as constant and with the boundary conditions $h = 0$ at $x = 0$, we obtain

$$\left( u_h - R u_* \right) \left[ \frac{h^2}{2(1+2A)} + \frac{2BkL}{1+2A} h + \left( \frac{2BkL}{1+2A} \right)^2 \ln \left( -\frac{1+2A}{2BkL} h + 1 \right) \right] +$$

$$+ \frac{Bu^2 u_0 T}{\gamma \sigma(1+A)} \ln \left( -\frac{1+A}{BkL} h + 1 \right) = \left( \frac{w' \theta'}{\gamma} \right).$$

(22)

A characteristic value of $R$ is determined from Eq. (18) by inserting the actual Obukhov length and a typical value of the internal boundary layer height. Only a rough estimate is needed because Eq. (22) varies little as a function of the characteristic height; a value in the range $h = 0.05x$ to $0.1x$ where $x$ is downwind distance will be suitable in most cases.

The relative contribution to the growth of the internal boundary layer from the mechanical and convective turbulence and the Zilitinkevich correction can be deduced from Eq. (20). On the left-hand side, the first term stems from the combined effect of
mechanical and convective turbulence (hereafter denoted \(M + C\)), the second term from the Zilitinkevich correction (denoted \(Z\)). Figure 6 shows the ratio

\[
\frac{Z}{M + C} = D \frac{(1 + 2A)h - 2\kappa L}{(1 + A)h - B\kappa L}h^2
\]

(23)

where

\[
D = Bu^2 T/\gamma C g.
\]

(24)

The term \(u_h/(u_h - Ru_g)\) has been omitted in Eq. (24), where \(u_h\) in the numerator and \(u_h - Ru_g\) in the denominator both represent the mean wind speed inside the internal boundary layer. Their inequality is due to the simplified way the time–distance relationship is modelled in the Zilitinkevich correction, Eq. (19). Use of the same relationship as in the other part of the model makes the ratio equal to one. It can be seen that the contributions from \(Z\) and \(M + C\) are equal at a height of about 50 m. Below this height, the growth is controlled mainly by the Zilitinkevich correction, above by mechanical and convective turbulence. Figure 6 illustrates that the height at which \(Z = M + C\) is very sensitive to the value of \(D\), but not to that of \(L\). The height increases when \(u_g\) increases and \(\gamma\) approaches zero, and vice versa. In the limit \(\gamma = 0\), the growth of the internal boundary layer is controlled entirely by the Zilitinkevich correction, Eq. (21). The present formulation, however, is inapplicable in this limit, because it disregards the decrease with height of the friction velocity, which becomes essential when the internal boundary layer grows beyond the surface layer (Larsen et al. 1982).

The relative contribution from mechanical and convective turbulence to the growth of the internal boundary layer can be deduced from the first term on the left-hand side of Eq. (20). It is readily seen that the contributions are equal when

\[
(1 + 2A)h = -2\kappa L.
\]

(25)

With \(A = 0.2\) and \(B = 2.5\) this corresponds to

\[
h \approx -1.4L.
\]

(26)

Figure 6. Ratio of the contribution to the growth of the internal boundary layer from the Zilitinkevich correction, \(Z\), to that of the mechanical and convective type turbulence, \(M + C\), as function of internal boundary layer height.
The growth of the internal boundary layer is mainly controlled by convective turbulence when its height is larger than $-1.4L$, and by mechanical turbulence when it is smaller. It should be noted that for $|L|$ small and $\gamma$ near zero, the layer in which mechanical turbulence controls the growth process becomes shallow and may even vanish. Under these conditions, the growth is controlled by the Zilitinkevich correction and convective turbulence alone. However, conditions of large values of $|L|$ and $\gamma$ result in a deep intermediate layer situated above the layer with Zilitinkevich correction domination and below the layer where the convective turbulence is essential. Such conditions exist at high wind velocities in coastal areas when the water is cold, and they were found to be characteristic for the Øresund experiment.

4. COMPARISON WITH DATA

The model for the growth of the internal boundary layer is evaluated using independent data from the Øresund experiment in Denmark and Sweden, the Nanticoke experiment in Canada, and experiments carried out over Long Island in New York.

The Øresund experiment (Gryning 1985) included extensive meteorological measurements over a cross-section of the Øresund, the 20 km-wide strait between Denmark and Sweden. East of Øresund, the land is undulating farmland. Copenhagen with its suburbs lies on the western side of Øresund. This area is flat but has a high surface roughness due to its urban character. The height of the internal boundary layer over Copenhagen was determined from aircraft turbulence measurements. Turbulence, measured at 115 m, 5 km inland over Copenhagen, provided the friction velocity and wind speed. Heat flux was measured at 10 m over the Swedish coast east of Copenhagen, in an area somewhat similar to suburban Copenhagen. Radio-sonde soundings carried out in the middle of the Øresund Strait show that the air over the water surface is (strongly) stably stratified up to a height of 50–100 m. Above, the air is also stable but the temperature gradient is less. The growth of the internal boundary layer over Copenhagen was simulated on 29 May and 4–5 June 1984. On those days the wind blew over the water towards Copenhagen. To take into account the height dependence of the temperature gradient over the water, the model simulations were carried out in two height intervals. Up to a height of 100 m, the characteristic temperature gradient in the layer of very stable air near the water surface was used; above 100 m, the temperature gradient in the upper layer was used.

The Nanticoke experiment (Portelli 1982) was conducted in the spring of 1978 on the northern shore of Lake Erie. The meteorological part of the experiment consisted of deployment of acoustic sounders, and profile and flux measurements made using free rising and tethered balloons and meteorological towers. Kerman et al. (1982) report data on the height of the internal boundary layer, including the meteorological conditions for two different days. The temperature gradient over the water is given indirectly by the so-called effective Brunt–Väisälä frequency. This reflects the mean temperature gradient of the onshore flow over the height of the internal boundary layer. The model simulations are restricted to June 1 between 1200 and 1500 LST. During this period, the effective Brunt–Väisälä frequency changed only slightly as the internal boundary layer grew inland. The temperature gradient was derived from the effective Brunt–Väisälä frequency, pertinent up to a height of approximately 400 m. The data set from Kerman et al. does not include the friction velocity. It is assumed here to be 10 per cent of the wind velocity.

As part of an extensive coastal meteorological programme, Brookhaven National Laboratory studied the development of the internal boundary layer over Long Island,
New York. Characteristics of the coastal boundary layer were primarily investigated by aircraft measurements of turbulence and temperature. Flights were made at selected altitudes across the width of Long Island during periods of air flow from water to land, usually at midday. Vertical cross-sections of turbulence and temperature were compiled by combining measurements from various heights. These were used to determine the development of the internal boundary layer and associated temperature gradient. Wind speed was measured at the meteorological tower at Brookhaven National Laboratory. A complete description of the experimental equipment has been given by Raynor et al. (1979). Stunder and SethuRaman (1985) have given data for the experiment on 16 June 1979, 1330 to 1500 EST, which we have simulated with our model.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Date</th>
<th>Time</th>
<th>(u_h) (m s(^{-1}))</th>
<th>(u) (m s(^{-1}))</th>
<th>(\langle w'\theta' \rangle) ((\text{K m s}^{-1}))</th>
<th>(L) (m)</th>
<th>(\gamma) (K m(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Øresund</td>
<td>29 May 84</td>
<td>1140 CET</td>
<td>7.55</td>
<td>0.78</td>
<td>0.058</td>
<td>-630</td>
<td>0.039(\dagger) 0.0041(\ddagger)</td>
</tr>
<tr>
<td>Øresund</td>
<td>4 June 84</td>
<td>1150 CET</td>
<td>10.7</td>
<td>0.84</td>
<td>0.107</td>
<td>-420</td>
<td>0.040(\ddagger) 0.0042(\ddagger)</td>
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<tr>
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<td>5 June 84</td>
<td>1110 CET</td>
<td>9.57</td>
<td>0.86</td>
<td>0.041</td>
<td>-1190</td>
<td>0.045(\ddagger) 0.0051(\ddagger)</td>
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<tr>
<td>Nanticoke</td>
<td>1 June 78</td>
<td>1200 LST</td>
<td>4.8</td>
<td>0.48</td>
<td>0.190</td>
<td>-45</td>
<td>0.0064</td>
</tr>
<tr>
<td>Nanticoke</td>
<td>1 June 78</td>
<td>1300 LST</td>
<td>5.0</td>
<td>0.50</td>
<td>0.215</td>
<td>-45</td>
<td>0.0088</td>
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<tr>
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<td>1400 LST</td>
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<tr>
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<td>-53</td>
<td>0.0151</td>
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<td>Brookhaven</td>
<td>16 June 78</td>
<td>1330 EST</td>
<td>4.5</td>
<td>0.50</td>
<td>0.125</td>
<td>-76</td>
<td>0.0150</td>
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</tbody>
</table>

\(\dagger\) Potential temperature gradient in the air below 100 m
\(\ddagger\) Potential temperature gradient in the air above 100 m

A summary of the meteorological conditions during all the experiments (Table 1) shows atmospheric stability falling in the range of near-neutral to unstable conditions. Near-neutral atmospheric conditions, characterized by high wind speed and low surface heat fluxes, are typical for the Øresund experiment, whereas the Nanticoke and Brookhaven experiments were carried out under unstable atmospheric conditions.

Table 2 shows the height of the internal boundary layers predicted by Eq. (B1) and Eq. (22), and measured values. For the individual experiments, the data are illustrated in Fig. 7 out to a distance of 10 km. In the Øresund experiment, the reported heights of the internal boundary layer represent instantaneous measurements of the turbulent internal boundary layer. The deviation between the internal boundary layer defined in section 2 and the turbulent one can be large, depending on the amplitude of the undulation in the interface. In the Nanticoke experiment, the layer was estimated from temperature profile measurements in some cases, in other cases from acoustic sounders yielding facsimile records of the temperature structure several hundred metres above the ground. Kaimal et al. (1982) and Coulter (1979) report good agreement between boundary layer heights deduced from the temperature structure detected by acoustic sounders and temperature profiles, especially when the inversion is strong and well defined. In the Brookhaven experiment, aircraft measurements of vertical temperature profiles and turbulence were used to estimate the internal boundary layer height. The exact criteria used to identify the height are not clear.
## Table 2. Observed and Predicted Internal Boundary Layer Heights

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Date</th>
<th>Time</th>
<th>Distance (m)</th>
<th>Internal boundary layer height</th>
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<tr>
<td></td>
<td></td>
<td></td>
<td>Observed</td>
<td>Predicted</td>
</tr>
<tr>
<td>Øresund</td>
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<td>1140 CET</td>
<td>6400</td>
<td>310</td>
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<tr>
<td>Øresund</td>
<td>4 June 84</td>
<td>1150 CET</td>
<td>6300</td>
<td>270</td>
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<tr>
<td>Øresund</td>
<td>5 June 84</td>
<td>1110 CET</td>
<td>4900</td>
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<td>Nanticoke</td>
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## 5. Discussion

In the derivation of the model, Eq. (B1), the gradient of potential temperature is assumed to be independent of height. Variations can be allowed for, however, in the differential formulation of the model, which can then be solved numerically. Venkatram (1988) discussed an integral approach, where there is no need to assume a potential temperature gradient constant with height. Kerwan et al. (1982) devised another approach based on an effective stratification parameter that accounts for vertical variations in the temperature gradient. In the present paper, the Øresund cases were modelled in two steps. First, the characteristic temperature gradient in the very stable air over the water was used, then the temperature gradient in the less stable air above. A virtual inland distance was introduced in the upper zone modelling, forcing the internal boundary layer to merge at the transition between the two zones.
Figure 7. Simulations of the growth of the internal boundary layer for the experiments in Table 1. Predictions by the analytic model, Eq. (B1), are given by the solid line; the dashed line, almost indistinguishable from the solid one, shows simulations with the approximation, Eq. (22). Also shown are the measured values.
In the limit of convective internal boundary layers, the growth rate, when omitting the Zilitinkevich correction, can be written
\[
\frac{u_h}{1 + 2A} \frac{dh}{dx} = \frac{(w'_r')_s}{\gamma}.
\] (27)

With the usual boundary condition, the solution
\[
h^2 = (1 + 2A) 2(w'_r')_s x / \gamma u_h
\] (28)
shows the well-known parabolic internal boundary layer. With \(A = 0\), the model simulates the conditions when the heat flux through the top of the internal boundary layer is zero. Equation (28) then conforms to a previously known expression for the encroachment situation (Gamo et al. 1983; Panofsky and Dutton 1984). In the limit of neutral internal boundary layers, the growth rate can be written
\[
\frac{h^2(u_h - u_s')/k) dh}{2Bu_s^* T/g} \frac{dh}{dx} = \gamma^{-1}
\] (29)
where we again have omitted the Zilitinkevich correction. The solution reads
\[
h^3 = \frac{6Bu_s^* T}{g \gamma(u_h - u_s')/k} x.
\] (30)

The model describes the growth of near-neutral and convective internal boundary layers when the upwind air is stably stratified. The model cannot be used to describe the development of stable internal boundary layers or when the upwind air is neutral or unstably stratified. Far downwind, the internal boundary layer ceases to grow and reaches a terminal height. This aspect is not discussed here. However, we consider that the addition of a model for the terminal height of the internal boundary layer would form a valuable supplement.

ACKNOWLEDGMENTS

The authors are indebted to Drs D. Yordanov and M. Nielsen for helpful discussions. Graziella Leroy and Birthe Skrumsager are acknowledged for typing various versions of the manuscript.

APPENDIX A

Some steps in deriving the prediction equation for \(h\)

Integration of Eq. (12) along \(z\) from 0 to \(h(x)\)
\[
\int_0^h \frac{\partial(u \theta)}{\partial x} \, dz + \int_0^h d(w \theta) = - \int_0^h d(w'_r')
\] (A1)
gives
\[
\frac{\partial}{\partial x} \int_0^h (u \theta) dz - u_h \theta_h \frac{\partial h}{\partial x} + w_h \theta_h = (w'_r')_s - (w'_r')_h.
\] (A2)

By use of the continuity equation
\[
w_h = - \int_0^h (\partial u / \partial x) \, dz
\] (A3)
it can be shown that
\[ \frac{\partial}{\partial x} \int_0^h (u \theta_h) dz = u_h \theta_h \frac{\partial h}{\partial x} - w_h \theta_h + \frac{\partial \theta_h}{\partial x} \int_0^h u \, dz. \] (A4)

Substituting Eq. (A4) into Eq. (A2), and rearranging, the following differential equation is obtained:
\[ \frac{\partial}{\partial x} \int_0^h u(\theta - \theta_h) dz + \frac{\partial \theta_h}{\partial x} \int_0^h u \, dz = (\overline{w'\theta'})_s - (\overline{w'\theta'})_h \] (A5)

where \( \theta - \theta_h \) represents the temperature defect. In neutral and convective internal boundary layers the temperature defect is negligible because the turbulence is sufficiently strong to maintain a near-uniform distribution of the potential temperature. However, it can be significant in stable layers. Considering only neutral and convective internal boundary layers, then
\[ (d \theta_h/dx) \int_0^h u \, dz = (\overline{w'\theta'})_s - (\overline{w'\theta'})_h. \] (A6)

The wind profile in terms of the wind velocity defect is
\[ \int_0^h u \, dz = u_h h - \int_0^h (u_h - u) \, dz \] (A7)

where \( u_h \) is the wind velocity at height \( h \). One advantage of this formulation is that the dependence on roughness length is replaced by a wind speed which often is more accurately known. From subsequent analysis, it appears to be advantageous to describe the wind velocity defect by a dimensionless function, \( R \), given by
\[ R(h/L) = \frac{1}{u_*} \int_0^1 (u_h - u) d(z/h) \] (A8)

which allows Eq. (A7) to be written
\[ \frac{1}{h} \int_0^h u \, dz = u_h - u_* R(h/L). \] (A9)

Little information is available on wind profiles inside internal boundary layers. Using the Obukhov similarity theory, which strictly speaking is valid only under homogeneous, stationary conditions, the wind profile is
\[ u(z) = \frac{u_*}{k} \{ \ln(z/z_o) - \psi(z/L) \} \] (A10)

where \( z_o \) is the roughness length, and the \( \psi \) function is an empirical function that describes dependence on atmospheric stability. The \( \psi \) function was taken as
\[ \psi \left( \frac{z}{L} \right) = 2 \ln \left( \frac{1 + x}{2} \right) + \ln \left( \frac{1 + x^2}{2} \right) - 2 \tan^{-1}(x) + \frac{\pi}{2} \] (A11)

where \( x = (1 - 16 z/L)^{1/4} \) and the corresponding value of the von Kármán constant is 0.4 (Panofsky and Dutton 1984). Then
\[ R(h/L) = \kappa^{-1} \int_0^1 \left\{ \ln \left( \frac{h}{z} \right) - \psi \left( \frac{h}{L} \right) + \psi \left( \frac{z h}{h L} \right) \right\} d \left( \frac{z}{h} \right). \] (A12)
Integration of Eq. (A12) leads to

$$R(h/L) = \frac{1 - (1 - 16h/L)^{3/4}}{12\kappa(h/L)}.$$  \hspace{1cm} (A13)

Figure 5 shows $R$ as a function of $h/L$. In the limit of neutral atmospheric stability $R = 1/\kappa$; taking $u_*$ to be approximately $0.1u_h$, then $(1/h) \int_0^h u \, dz = 0.75 \, u_h$ in near-neutral conditions. This shows that the characteristic advection velocity is 25 per cent smaller than the wind speed in the upper part of the internal boundary layer. In the convective limit $R = 0$, indicating that the wind velocity defect is zero and consequently that the wind velocity as a function of height is constant.

**APPENDIX B**

**Analytical solution to Eq. (20)**

With the boundary condition $h = 0$ at $x = 0$, the solution can be written

$$\frac{u_h}{1 + 2A} \left\{ \frac{h^2}{2} + \frac{2B\kappa L}{1 + 2A} h + \left( \frac{2B\kappa L}{1 + 2A} \right)^2 \ln \left( -\frac{1 + 2A}{2B\kappa L} h + 1 \right) \right\} +$$

$$+ \frac{u_* L}{(1 + 2A)12\kappa} \left\{ -h - \frac{2B\kappa L}{1 + 2A} \ln \left( -\frac{1 + 2A}{2B\kappa L} h + 1 \right) + f(s) - f(q) \right\} +$$

$$+ \frac{B_3^2 u_* T}{C_\gamma g (1 + A)} \ln \left( -\frac{1 + 2A}{B\kappa L} h + 1 \right) \frac{(w'\theta')^2}{\gamma}$$

where

$$f(y) = -\frac{L}{28} \left( \frac{y}{q} \right)^7 +$$

$$+ \frac{2B\kappa L}{(1 + 2A)q^3} \left[ 4y^3 - \sqrt{2} \left( 1 - \sqrt{2} \ln \left( \frac{y^2 - \sqrt{2}y + 1}{y^2 + \sqrt{2}y + 1} \right) + \arctan(\sqrt{2}y + 1) + \arctan(\sqrt{2}y - 1) \right) \right]$$

$$q = \left( \frac{1 + 2A}{32B\kappa - 1 - 2A} \right)^{1/4},$$

$$s = q(1 - 16h/L)^{1/4}.$$

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