The response of a turbulent boundary layer to arbitrarily distributed
two-dimensional roughness changes

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SUMMARY

We investigate small changes to a two-dimensional turbulent boundary layer on a rough surface caused by arbitrary variations, in the streamwise direction, of the surface roughness length, $z_i(x)$. The linear changes to the flow are calculated as asymptotic sequences in the limit $\varepsilon = u_e / U_0 \to 0$ ($u_e$ is the upwind friction velocity, and $U_0$ is a typical value of the approach wind speed). The flow is divided into two regions, each of which is subdivided into two layers; the previous analyses may then be extended by calculating the second-order effects. Throughout the bulk of the inner region, of thickness $l$, the Reynolds shear-stress gradient balances the acceleration, but very close to the surface (in the inner surface layer), up to a height $O(\sqrt{U_0})$, the shear-stress gradient is constant and matches with its surface value. The outer region (where the leading-order perturbations are inviscid) also divides into two layers: the middle layer extending to a height $h_{int}$, where the shear in the upwind profile determines the perturbations; and the upper layer, in which the perturbations decay in potential flow, and a weak perturbation pressure develops which produces second-order changes in the inner region. At second order the normal Reynolds stresses also accelerate the mean-flow perturbations, but their effect is numerically small. The solutions in the inner surface layer satisfy the full non-linear equations, and by matching prudently, the layers above also contain a non-linear correction. We show that this means that the theory is valid when $M u_e / U_0 \ll 1$ (so that it may be applied to atmospheric problems even when $M = \ln(z_o/z_i)$ is of order one). The theory agrees well with experiments and numerical simulations (which use complex models for the turbulence closure) and improves on the previous analytic theories, especially in the prediction of the surface shear stress.

1. INTRODUCTION

In order to quantify the changes to the atmospheric boundary layer as it flows over complex terrain, one of the fundamental problems to consider is how a turbulent boundary layer changes as it passes over a level surface with varying roughness length. The presence of a roughness-length change leads to the development of an internal boundary layer (which we refer to as the inner region) through which the significant changes to the flow occur (see Fig. 1(a)). For an abrupt change in roughness length, the height of this inner region increases downstream of the change as the boundary layer adjusts to the new equilibrium structure. The aim of this work is to gain a physical understanding of, and to quantify the changes within, this inner boundary layer and the secondary motions which it drives.

The central importance of this problem is reflected in the large number of theoretical, numerical and experimental studies it has recently received. Important theoretical contributions for small roughness changes have been made by Elliot (1958), Panofsky and Townsend (1964), Townsend (1965, 1966), Blom and Wartena (1969), Mulhearn (1977), Walmsley et al. (1986) and Beljaars et al. (1987). Numerical works on the full non-linear equations, using various approximations for the Reynolds stresses, have been carried out by Taylor (1969a, b), Peterson (1969), Shir (1972), Rao, Wyngaard and Cote (1974) and Wood (1978). The best experiments have been performed by Bradley (1968). Finally, reviews of the field can be found in Panofsky (1974), Hunt and Simpson (1982) and Smits and Wood (1985).

In the present study, we analyse the linear changes to the flow caused by two-dimensional variations in surface roughness, so that the perturbations to the mean velocity and Reynolds stresses are assumed small compared with their upwind values. If
$z_0$ is a typical upwind value of the roughness length and $z_1$ typical in the changing region, then, following Townsend (1965), a sufficient condition for the linear analysis to be valid is that $M = \ln(z_1/z_0) \ll 1$, but this is unnecessarily restrictive and the necessary condition is studied in section 7. Even the linearized perturbation equations cannot be solved exactly, so approximate solutions are found asymptotically in the limit $u_\infty/U_B(l) \to 0$.
where $u_*$ is the upwind friction velocity and $U_0(l)$ is the upwind mean-flow velocity. (In the atmosphere $u_* / U_0(l)$ is typically 0.01–0.03.)

There have been a number of theoretical studies of this problem based on linear analysis, but they have three main limitations which reduce their applicability.

Hitherto, most of the analyses have been based on the work by Townsend (1965); they have considered a step change in surface roughness at $x = x_1$, and are based on the assumptions that (i) the change in the streamwise velocity is small compared with the approach flow velocity (so that the governing equations may be linearized), (ii) throughout the inner boundary layer of the perturbed flow, the advection velocity is the upwind value at the top of the inner region, $U_0(l(x))$, and (iii) the variation in the roughness length is slow enough that $(x/M)(dM/dx) \ll 1$; the flow is then self-preserving (Townsend 1965). The assumption (ii) provides the leading order approximation over most of the depth of the internal boundary layer, but results in a significant error very close to the surface when $\ln(z/l)$ is of the same order as $\ln(l/z_0)$. The mean velocity is adequately modelled by such deficient theories, but the shear stress and its gradient are not; these latter quantities are of special importance to calculations of the heat and mass flux at the surface (Weng et al. 1989).

In previous analytic studies the second-order effects on the flow in and above the inner region have not been considered, and, as a consequence, nor have the weak pressure gradients generated by the roughness change or the effects of the normal Reynolds stress on the mean flow in the inner region. Although small, the perturbation pressure plays a central role in the change to the mean drag force on an undulating surface (Belcher, Newley and Hunt, 1990, referred to as BNH). Using the solutions derived here, together with the perturbations due to an undulating surface, it is possible to construct a theory for the flow over a moving wave (see Belcher and Hunt 1990). It is found that the pressure perturbations caused by roughness changes along the wave are important in determining the amplitude growth rate of the wave (the same result, qualitatively, as Gent and Taylor (1976), but quantitatively different). Additionally, for three-dimensional changes in roughness length (considered by Xu et al. 1990), the second-order lateral pressure gradient strongly affects the divergence of the streamlines, and hence the turbulent dispersion of pollutants.

Finally, previous theories, and of course the numerical and experimental studies, have tended to deal with specific geometries, such as step changes in roughness: no general solutions have been developed. An exception is a recent analysis of the leading order perturbations for arbitrary changes in roughness by Walmsley et al. (1986).

In order to overcome these weaknesses, we develop a general analysis which retains the pressure gradient and normal stress terms in the momentum equations so that the second-order perturbations may be calculated. This, in turn, provides insight into how the method may be applied to three-dimensional variations. To develop a solution for arbitrary roughness changes, we take Fourier transforms in the $x$-direction, so that the fundamental problem is to calculate the flow over a sinusoidal variation in roughness length (Fig. 2(b)), rather than the step change (Fig. 2(a)) studied by Townsend (1965). By solving the problem in terms of Fourier transforms, we do not need to use the idea of self preservation, and the analysis does not require Townsend's (1965) condition that $(x/M)(dM/dx) \ll 1$. Using Fourier transforms has the further advantage that the solutions can be used in combination with other surface disturbances such as elevation and temperature.

The approach in this paper is similar to the perturbation analysis developed for turbulent flow over hills by Hunt, Leibovich and Richards (1988) (referred to as HLR) and extended to consider the turbulence structure and to calculate the mean drag force
on the surface by BNH. This means that in the inner region, the shear stress is approximated by a mixing-length model, and the changes to the normal stresses are assumed to be proportional to the change in the shear stress. Furthermore, the height of the inner region is defined so that the leading order perturbations in the outer region are inviscid. These assumptions have been carefully studied by BNH, who found that they are significantly more accurate than using a mixing-length model for the shear stress through all heights. Hence, the perturbed flow is divided into two regions, see Fig. 1(a). Furthermore, the outer region is composed of two layers: an upper layer, and a middle layer of height \( h_m \). The inner region, of thickness \( l \), is also made up of two layers: a shear-stress layer (referred to as the SSL), and a very thin inner surface layer (referred to as the ISL). By subdividing the inner region into two distinct layers, we obtain consistent solutions which describe the shear stress and its gradient more accurately than the previous theories, and the rational description of the outer region means that the second-order terms emerge naturally from the analysis.

In section 2 the physical model for the analysis is discussed, and in section 3 the basic equations are solved. The structure of the asymptotic solutions is discussed in section 4, and the present solutions are compared with the perturbation induced by a hill. The theory is applied to a step change in roughness and two successive abrupt changes and the results are presented in sections 5 and 6 respectively. The solutions compare well with the numerical results of Rao et al. (1974), the experiments of Bradley (1968) and the computations of Blom and Wartena (1969).

2. MATHEMATICAL MODEL

Consider a fully developed turbulent boundary layer on a flat surface with roughness length \( z_0 \). The mean flow is in the \( x \)-direction and the unperturbed velocity profile is assumed to be logarithmic (for a neutral atmosphere), so that

\[
U_B = \frac{\mu_*}{\kappa} \ln(z/z_0),
\]

(2.1)

where \( \mu_* \) is the upwind friction velocity, \( \kappa \) the von Karman constant taken as 0.4 and subscript B will denote dimensional upwind quantities. Downstream, there is an arbitrarily distributed roughness length change, and the local roughness length is \( z_I(x) \).

The variation in the roughness length causes a perturbation to the basic flow which
arises, mathematically, from the new boundary condition at the lower surface. We expand
the perturbed flow variables as $u^* = U_B + \Delta u$, etc. (the superscript * will denote total
dimensional quantities), and then the surface boundary condition is

$$\begin{align*}
U_B + \Delta u &= 0 & \text{on } z = z_1. \\
\Delta w &= 0
\end{align*}$$  \hspace{1cm} (2.2)

Using (2.1), the boundary condition on the velocity perturbation is

$$\Delta u = \frac{u_*}{K} M,$$  \hspace{1cm} (2.3)

where the roughness parameter $M = \ln(z_1/z_0)$. This expression shows that a roughness change is equivalent to a surface of constant roughness with a varying tangential surface velocity.

In order to solve the equations for the linear perturbations caused by an arbitrary change in roughness length, Fourier transforms are taken in the $x$-direction, so that for example, the $u$ perturbation transforms like

$$u(s, z) = \int_{-\infty}^{\infty} u(x, z)e^{-ist} \, dx.$$  

Note that no special symbols are used for the transforms, and in the following we shall be mainly concerned with transformed variables. When confusion may arise, the two quantities are distinguished by writing out their arguments. The Fourier transform of the roughness parameter also has to be calculated,

$$\tilde{M}(s) = \int_{-\infty}^{\infty} M(x)e^{-ist} \, dx.$$  

To make the equations non-dimensional, a value of the approach wind speed must be selected. There are two choices: the upwind friction velocity, $u_*$, or a value within the flow, $U_0$. It is convenient to make the velocity perturbations non-dimensional using $U_0 = U_B(h_m)$, the wind speed at a height $h_m$, the top of the middle layer. The height $h_m$ is where the pressure perturbation is determined and so it is natural to make the pressure perturbation non-dimensional using $\rho U_0^2$. Furthermore, the Reynolds stresses are non-dimensionalized using $\rho u_*^2$. This scheme has three advantages over any other: first, the non-dimensional equations immediately show that the perturbation is singular (see Eq. (2.5) below); second, and most importantly, it is the same scheme used by HLR and BNH for the flow over a hill, so their solutions may be used in conjunction with those derived here to describe the flow over 'complex terrain'; and finally, this scaling immediately shows that the pressure perturbation is second order.

The boundary condition (2.3) suggests that the perturbations scale on $\frac{\varepsilon}{K} \tilde{M}$ (where $\varepsilon = u_*/U_0$) and so they are formally expanded in power series:

$$\begin{align*}
\Delta u(s, z) &= \left[\frac{\varepsilon}{K} \tilde{M}\right] u^{(1)} + \left[\frac{\varepsilon}{K} \tilde{M}\right]^2 u^{(2)} + \ldots \\
\Delta w(s, z) &= \left[\frac{\varepsilon}{K} \tilde{M}\right] w^{(1)} + \left[\frac{\varepsilon}{K} \tilde{M}\right]^2 w^{(2)} + \ldots \\
\Delta p(s, z) &= \left[\frac{\varepsilon}{K} \tilde{M}\right] p^{(1)} + \left[\frac{\varepsilon}{K} \tilde{M}\right]^2 p^{(2)} + \ldots
\end{align*}$$  \hspace{1cm} (2.4)
We calculate only the linear terms in these expansions, but even they cannot be evaluated exactly, so approximate asymptotic solutions are found in the limit $\varepsilon \rightarrow 0$.

Making the flow variables non-dimensional as described above, the Reynolds averaged momentum equations may be linearized so that the perturbations are described by

$$
\frac{U}{\partial x} + w \frac{dU}{dz} = -\frac{\partial p}{\partial x} - \varepsilon^2 \left( \frac{\partial u'w'}{\partial z} + \frac{\partial u'^2}{\partial x} \right)
$$

$$
\frac{U}{\partial x} = -\frac{\partial p}{\partial x} - \varepsilon^2 \left( \frac{\partial u'w'}{\partial z} + \frac{\partial w'^2}{\partial z} \right)
$$

$$
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
$$

where $\varepsilon$ is the basic small parameter of the problem. Small letters denote the non-dimensional linear perturbations so that for example,

$$
\Delta u = U_0 \frac{\varepsilon}{k} \tilde{M} u + O\left(\frac{\varepsilon}{k} \tilde{M}^2\right)
$$

$$
\Delta \tau = \rho u^2_\ast \frac{\varepsilon}{k} \tilde{M} \tau + O\left(\frac{\varepsilon}{k} \tilde{M}^2\right).
$$

The boundary conditions are

$$
u = 1 \quad w = 0 \quad \text{on } z = z_1
$$

$$
u, w, p, \tau \rightarrow 0 \quad \text{as } \frac{z}{L} \rightarrow 0
$$

where $L$ is the largest scale of the roughness change.

Finally, the Reynolds stress terms in Eq. (2.5) must be modelled. BNH carried out a detailed examination of the structure of the turbulence in a perturbed boundary layer using scaling arguments and the results of numerical computations, with a second-order turbulence model, of boundary-layer flow over sinusoidal topography. The results show that the structure of the perturbations to the turbulence separate into two regions which are broadly the same as the inner and outer regions of the mean-flow perturbations.

In the inner region $z$ is normalised on $l$, and $x$ on $s^{-1}$; $u'w'$ and $u'^2$ are of the same order. The shear-stress term in Eq. (2.5) has a dynamical effect on the mean flow at $O(\delta)$ and, because of the length scalings, the $u'^2$ term in (2.5) has an effect at $O(\delta l/s^{-1}) = O(\delta^2)$. Hence it is most important to construct an accurate model for the Reynolds shear stress. Towards the surface, the turbulence tends to a local equilibrium (BNH) so that the shear stress can be accurately determined by a mixing-length formula. At the top of the inner region, the advection terms in the turbulent kinetic energy equation are of the same order as the production and dissipation terms, so that the mixing-length model ceases to be the correct approximation. Following HLR, however, we continue to use mixing-length theory throughout the inner region. This leads to inaccurate predictions of the shear stress at the top of the inner region, but the errors in the predictions of the mean flow are small, since it is only in the lower half of the inner region that the shear stress is dynamically significant. This means that throughout the inner region the Reynolds shear-stress term is approximated by

$$
\tau^* = \left( \kappa z \frac{\partial u_\ast}{\partial z} \right)^2
$$
so that the linear perturbation becomes
\[ -\bar{u}'\bar{w}' = \tau = \frac{2\kappa z}{\varepsilon} \frac{\partial u}{\partial z}. \quad (2.9) \]

Following BNH, the changes to the normal stresses are assumed to be proportional to the changes to the shear stress,
\[ \bar{u}'^2 = \alpha \tau \quad \bar{w}'^2 = \beta \tau \quad (2.10) \]
where \( \alpha \) and \( \beta \) are constants determined from an equilibrium boundary layer. We use \( \alpha = 6.3 \) and \( \beta = 1.7 \), derived from measurements of the atmospheric boundary layer.

In the outer region, BNH showed that the distortion of the turbulence is ‘rapid’, i.e. determined by the history of the flow and not the local velocity gradient. Consequently, the effect of these distortions to the Reynolds stresses on the mean flow is \( O(\varepsilon^2) \). If the mixing-length model is erroneously used throughout all heights, then perturbation shear stress is over-predicted by \( O(\varepsilon) \), so that the effect of the Reynolds stresses on the mean flow is also over-predicted by \( O(\varepsilon) \). Furthermore, the ‘elliptic’ effects in the equations (i.e. the pressure effects), mean that if the Reynolds stress perturbations and their effects on the mean flow are over-predicted in the outer region, then the perturbations in the inner region (in particular at the surface) are also affected (see BNH). Hence using mixing length throughout the flow domain leads to an over-prediction of the surface shear stress as found by Beljaars et al. (1987) (see section 5).

3. LINEAR THEORY

The linearized problem posed above is now solved. The perturbations are described by the four-layer asymptotic structure (Fig. 1(a)) and each layer is defined using the horizontal length scale derived from each wavenumber, \( s \). This contrasts with HLR’s technique: they defined a single horizontal length scale for the hill and then used an asymptotic structure based on this single horizontal length scale. There are two reasons why HLR’s technique cannot be adopted here: in many roughness change geometries of practical importance (e.g. a step change) there is no single horizontal length scale to characterize the change; secondly, the forcing produced by the roughness change is rather subtle so that in order to predict accurately the changes that occur, the surface boundary condition must be satisfied at each wavenumber, and this cannot be done if a single horizontal length scale is selected. The heights of each of the layers are defined below, and in Fig. 1(b) we plot the variation of these heights with wavenumber \( s \).

(a) Inner surface layer

In the inner surface layer (ISL) the vertical-coordinate scales on the new roughness length, \( z_1 \), so \( \eta = z/z_1 \). The flow variables are scaled thus:
\[ u = \hat{u} \quad w = \hat{w} \quad \tau = \frac{2}{U(l)} \hat{\tau} \quad p = -U(l) \hat{p}. \quad (3.1) \]

The linearized equations (2.5) become:

\[ z_1 \frac{\partial \hat{p}}{\partial \eta} = z_1 \frac{\partial \hat{\tau}}{\partial \eta} = \frac{\partial \hat{\tau}}{\partial \hat{\eta}} + \frac{2\varepsilon^2}{U(l)} \frac{\partial \hat{\tau}}{\partial \hat{\eta}} - \frac{2\alpha \varepsilon^2}{U(l)} z_1 \hat{\tau} \quad (3.2a) \]

\[ U(l) \frac{\partial \hat{\phi}}{\partial \eta} = \frac{\partial \hat{\phi}}{\partial \hat{\eta}} = \frac{\partial \hat{\phi}}{\partial \hat{\eta}} + \frac{2\beta \varepsilon^2}{U(l)} \frac{\partial \hat{\phi}}{\partial \hat{\eta}} \quad (3.2b) \]
\[ \dot{w} = -z \dot{u} \int_1^{\eta} \dot{u} \, d\eta' \]  
(3.2c)

\[ \dot{t} = \frac{1}{\delta} \eta \frac{\partial \dot{u}}{\partial \eta} \]  
(3.2d)

The surface boundary conditions are

\[ \dot{u}(z_1) = 1, \quad \dot{w}(z_1) = 0. \]  
(3.2e)

The equations suggest expanding the perturbations in powers of \( z_1 |s| \), which is exponentially small compared with \( \delta = \ln(l/z_0) \), the small parameter in the layer above, the shear-stress layer (referred to as the SSL). This means that the first term in the ISL expansion must match with all the (algebraic) terms of the SSL, and so only the leading-order terms in the ISL expansion need to be calculated.

At zeroth order the \( x \)-momentum equation shows that, providing

\[ z_1 |s| \ln(z/z_1) \ll \varepsilon \]  
(3.3a)

the inertial gradient is much less than the shear-stress gradient, so that the shear stress is constant. Furthermore, given the criterion (3.3a), the coefficients of the non-linear inertial terms are much smaller than the coefficients of the shear stress even when \( \Delta u \sim U_0 \), i.e. no longer in the linear regime. Hence when (3.3a) is satisfied, at leading order, the non-linear equations have the constant shear-stress solution. Using the definition of the height, \( l \), of the inner region, we find that

\[ z_1 |s| \ln(z/z_1) \sim e^{-M \delta} e^{-l/\delta} \ln(z/z_1) \ll \delta \ln(z/z_1). \]  
(3.3b)

Hence if \( \ln(z/z_1) = O(1) \), (3.3a) is satisfied (since \( \varepsilon \sim \delta \)). Equation (3.3b) shows that when \( \delta \ll 1 \) (a condition required for the asymptotic analysis in the higher layers to be valid) the criterion (3.3a) is always satisfied for a range of heights which keep \( \ln(z/z_1) \) of the order one. It is not possible to specify the height of the ISL more accurately until the solutions to the SSL have been calculated (section 3(b) below).

At zeroth order the solution is

\[ \dot{u} = 1 + \delta \dot{t}(s, z_1) \ln \eta \]

\[ \dot{w} = 0 \]  
(3.4)

\[ \dot{t} = \dot{t}(s, z_1) \quad \text{(independent of } \eta \text{)} \]

\[ \dot{p} = \dot{p}(s, z_1) \quad \text{(independent of } \eta \text{)} \]

i.e. the ISL is the region of the perturbed flow where the 'law of the wall' holds. In section 3(b) it is shown that the height to which this law holds is very small, so that numerical simulations must be judicious in the application of this rule for their first grid point.

The solution (3.4) has two free constants, \( \dot{t}(z_1) \) and \( \dot{p}(z_1) \), which must be determined by matching with the (SSL) above. Anticipating the form of the (SSL), (3.4) can be prepared for matching. First, the constants \( \dot{t}(z_1) \) and \( \dot{p}(z_1) \) are expanded in powers of \( \delta \),

\[ \dot{t}(z_1) = \dot{t}^{(0)}(z_1) + \delta \dot{t}^{(1)}(z_1) + O(\delta^2) \]

\[ \dot{p}(z_1) = \dot{p}^{(0)}(z_1) + \delta \dot{p}^{(1)}(z_1) + O(\delta^2) \]  
(3.5)

The vertical coordinate in the SSL is \( \zeta = z/l \) so using the scalings (3.1) gives
\[ \begin{align*}
u & \sim \left[ 1 + \delta (\hat{v}^{(0)} + \delta \hat{v}^{(1)} + \mathcal{O}(\delta^2)) \right] \ln \left( \frac{l}{z_0} \right) + \mathcal{O}(\delta^{-1/\delta})] \\
w & \sim \mathcal{O}(\delta^{-1/\delta})] \\
p & \sim -U(l) [\delta^{(0)} + \delta \delta^{(1)} + \mathcal{O}(\delta^2)] \\
t & \sim \frac{2}{U(l)} [\hat{v}^{(0)}(z_0) + \delta \hat{v}^{(1)}(z_0) + \mathcal{O}(\delta^2)]
\end{align*} \tag{3.6} \]

to match with the SSL.

The expression for \( u \) contains \( \ln(l/z_1) = \ln(l/z_0) + \ln(z_0/z_1) = 1/\delta + M \), and so leads to an \( \mathcal{O}(M^2) \) correction to the velocity. In a strictly linear analysis this \( \mathcal{O}(M) \) part of (3.6a) should therefore be discarded; there is, however, no difficulty in retaining it and then, on matching, the solutions in the SSL contain a non-linear correction. Similarly, this expression also contains a \( \delta \) term, so to produce a rational expansion in \( \delta \) the two brackets in the \( u \) expression should be multiplied out. Suppose, however, that \( \ln(l/z_1) \) is formally \( \mathcal{O}(1) \) so that it matches into the leading-order solution in the SSL, which will then contain corrections of \( \mathcal{O}(\delta^2) \) and \( \mathcal{O}(M^2) \) and in consequence will be rather accurate.

Although the solutions for the perturbations have been calculated, they have not led to an estimate of the height for the ISL. To do this the SSL solutions must be calculated.

\( (b) \) Shear-stress layer

The four-layer structure is constructed so that the height of the shear-stress layer, referred to as the SSL, occurs when there is a balance at first order in the local parameter \( \delta = \ln(l/z_0) \) between the shear-stress gradient and the inertial gradient. This occurs at a height \( l \), where (HLR; BNH)

\[ l \ln(l/z_0) = 2k^2 |s|^{-1}. \tag{3.7} \]

The perturbations are scaled in the same way as in the ISL (Eq. (3.1)) and the vertical coordinate is normalized on \( l \), so that \( \zeta = z/l \). The linearized equations then become

\[ \begin{align*}
(1 + \delta \ln \zeta) i \operatorname{sign}(s) \hat{u} + \frac{\hat{\psi}}{2k^2 \zeta} & = i \operatorname{sign}(s) \hat{\rho} + \frac{\partial}{\partial \zeta} \left( \zeta \frac{\partial \hat{u}}{\partial \zeta} \right) - \delta 2 \operatorname{sign}(s) k^2 \alpha \zeta \frac{\partial \hat{u}}{\partial \zeta} \\
2k^2 \delta (1 + \delta \ln \zeta) i \operatorname{sign}(s) \hat{\psi} & = \frac{\partial \hat{\rho}}{\partial \zeta} + 4k^4 \operatorname{sign}(s) \delta^2 \zeta \frac{\partial \hat{u}}{\partial \zeta} - \delta 2 k^2 \beta \frac{\partial}{\partial \zeta} \left( \zeta \frac{\partial \hat{u}}{\partial \zeta} \right) \\
\hat{w} & = -i \operatorname{sign}(s) 2k^2 \delta \int_{\zeta}^{\zeta'} \hat{u}(\zeta') \, d\zeta' \\
t & = \frac{1}{\delta} \frac{\partial \hat{u}}{\partial \zeta}
\end{align*} \tag{3.8} \]

and the small parameter in this region is \( \delta = \ln^{-1}(l/z_0) \). The flow variables are expanded in powers of \( \delta \) and solutions found iteratively.

There is nothing in the forcing of the problem to produce a zeroth order pressure perturbation, so \( \delta^{(0)} = 0 \): a pressure perturbation is only produced in association with a vertical-velocity perturbation. Continuity shows that \( \hat{w}^{(0)} = 0 \); and so at \( \mathcal{O}(\delta^0) \), the \( x \) momentum reduces to

\[ \frac{\partial}{\partial \zeta} \left( \zeta \frac{\partial \hat{u}^{(0)}}{\partial \zeta} \right) - i \operatorname{sign}(s) \hat{u}^{(0)} = 0. \tag{3.9} \]
The solution which remains bounded at large $\zeta$ is

$$\hat{u}^{(0)} = A_0 K_0[2 \sqrt{i \text{sign}(s) \xi}]$$  \hspace{1cm} (3.10)

where $K_0$ is the modified Bessel function of zeroth order. Note that throughout when $K_0$ or its derivative is written, it is a function of $2 \sqrt{i \text{sign}(s) \xi}$. $A_0$ is a constant which must be determined by matching with the ISL. Recalling the scaling arguments of BNH, the shear stress does not affect the mean flow till $O(\delta)$ in the SSL, hence we expect $A_0$ to be $O(\delta)$. An important feature of the solution (3.10) is that $K_0$ has real and imaginary parts, so that the velocity perturbation is out of phase with the surface velocity. Using the asymptotic form of $K_0$ (see e.g. Abramowitz and Stegun 1972), gives

$$u^{(0)} \sim -\frac{A_0}{4} (i \pi \text{sign}(s) + 2 \ln \zeta + 4\gamma) \quad \text{as} \quad \zeta \to 0$$  \hspace{1cm} (3.11)

where $\gamma$ is Euler’s constant. Matching the coefficient of the logarithmic term with (3.6) shows that

$$-\frac{1}{2} A_0 = \delta \xi^{(0)}(z_1)$$  \hspace{1cm} (3.12)

and matching constants gives

$$-\frac{1}{4} A_0 (i \pi \text{sign}(s) + 4\gamma) = 1 + \delta \xi^{(0)}(z_1) \ln(l/z_1)$$  \hspace{1cm} (3.13)

so that

$$A_0 = \frac{4}{2 \ln(l/z_1) - 4\gamma - i \pi \text{sign}(s)} = \frac{2\delta}{1 + \delta(M - 2\gamma - i\pi/2 \text{sign}(s))}$$  \hspace{1cm} (3.14)

which is indeed $O(\delta)$. The leading-order surface shear stress is then calculated using (3.12).

At $O(\delta)$, the continuity equation gives,

$$\hat{\omega}^{(1)} = -2i \text{sign}(s) \kappa^2 A_0 \int_{\zeta_0}^{\zeta} \xi K_0 \, d\xi'$$  \hspace{1cm} (3.15)

and $\zeta_0$ is chosen so that the solution matches with the ISL. From the ISL solution

$$w \sim O(\delta e^{-i\delta}) \quad \text{as} \quad \zeta \to 0$$

and so the required solution is

$$\hat{\omega}^{(1)} = -2\kappa^2 A_0 \left[ \xi \frac{\partial K_0}{\partial \xi} + \frac{1}{2} \right].$$  \hspace{1cm} (3.16)

So the asymmetry of $u$ at $O(\delta)$ leads, via continuity, to an asymmetric $O(\delta^2)$ vertical velocity perturbation, which must in turn be driven by an asymmetric pressure perturbation.

At $O(\delta)$ the $z$-momentum equation becomes

$$\frac{\partial \hat{p}^{(1)}}{\partial \xi} = -2\kappa^2 \frac{\partial \xi}{\partial \xi} = 2\kappa^2 \beta \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \hat{u}^{(0)}}{\partial \xi} \right).$$  \hspace{1cm} (3.17)

This shows that the first-order perturbations are affected by the gradients of the normal Reynolds stresses, and because $\hat{u}^{(0)} = O(\delta)$, the resulting perturbations are $O(\delta^2)$. At
leading order we only needed to provide a detailed model for the Reynolds shear stress, $-\rho u w$ (the only assumption about the normal stresses was that they scaled on $\rho u_f^2$) but at next order we require a model for the normal Reynolds stress perturbations and assume that they are proportional to the change to the shear stress (see section 2).

Using the expression for $u^{(0)}$ and integrating gives

$$\tilde{\rho}^{(1)} = \tilde{\sigma}^{(1)}(s, z_1) + 2k^2 \beta A_0 \left[ \frac{\xi}{\delta} \frac{\partial K_0}{\partial \xi} + \frac{1}{2} \right].$$

(3.18)

Where $\tilde{\sigma}^{(1)}$ is the leading-order surface-pressure perturbation, which is shown below to be complex, i.e. at $O(\delta^2)$ there is a pressure perturbation which has a component out of phase with the surface velocity.

These solutions are substituted into the $O(\delta^2)$ $x$-momentum equation, and we find

$$\tilde{u}^{(1)} = A_1 K_0 + \tilde{\sigma}^{(1)} + A_0 \kappa^2 \beta$$

$$+ A_0 \left[ \kappa^2 (\alpha - \beta) \left[ \text{sign}(s) \xi K_0 - \frac{\xi}{\delta} \frac{\partial K_0}{\partial \xi} \right] \right]$$

$$+ \left[ \ln \xi - 2 \right] \xi \frac{\partial K_0}{\partial \xi} - \frac{1}{2} K_0 \ln \xi - \frac{1}{2} \mathcal{F} \left( \frac{1}{\xi} \right).$$

(3.19)

The constant $A_1$ is determined by matching with the ISL, and $\tilde{\sigma}^{(1)}$ matching with the middle layer. $\mathcal{F}(1/\xi)$ is the particular integral of $1/\xi$, which cannot be found exactly, but approximate solutions may be found for large and small $\xi$, viz.

$$\mathcal{F} \sim \frac{1}{2} \ln^2 \xi - 2 \ln \xi - 1 + O(\xi) \quad \text{as} \quad \xi \to 0$$

(3.20a)

and

$$\mathcal{F} \sim \frac{1}{i \text{sign}(s) \xi} + O \left( \frac{1}{\xi^2} \right) \quad \text{as} \quad \xi \to \infty.$$ 

(3.20b)

Then for the SSL, where $\xi < 1$ the solution for small $\xi$ is appropriate. Matching the coefficients of the logarithmic terms and the free coefficients with the ISL solution (3.6), gives

$$\frac{1}{4} A_1 \{2 \ln(l/z_1) - 4\gamma - i \pi \text{sign}(s)\} = -\tilde{\sigma}^{(1)} - A_0 \left( \frac{3}{2} - \frac{1}{2} \kappa^2 (\alpha - \beta) \right)$$

$$- \frac{1}{4} (4 + 4\gamma + i \pi \text{sign}(s))$$

(3.21)

and the surface shear stress is given by

$$\delta \tilde{\tau}^{(1)} = \frac{1}{4} A_0 (i \pi \text{sign}(s) + 4\gamma + 4) - \frac{1}{4} A_1.$$ 

(3.22)

Then $\tilde{u}^{(0)}$ and $\tilde{u}^{(1)}$ can be differentiated to give the shear-stress perturbations

$$\tilde{\tau}^{(0)} = a_0 \xi \frac{\partial K_0}{\partial \xi}$$

(3.23a)

$$\tilde{\tau}^{(1)} = \xi \left\{ A_1 \frac{\partial K_0}{\partial \xi} + a_0 \left( \kappa^2 (\alpha - \beta) i \text{sign}(s) \xi \frac{\partial K_0}{\partial \xi} + \frac{\partial K_0}{\partial \xi} + (\ln \xi - 2) i \text{sign}(s) K_0 - \frac{1}{2} \frac{\partial K_0}{\partial \xi} \ln \xi - \frac{1}{2} \xi K_0 - \frac{1}{2} \mathcal{F} \right) \right\}$$

(3.23b)

where
\[
\frac{a_0}{\delta} = \frac{2}{1 - \delta(M + 2\gamma + i(\pi/2) \text{sign}(s))},
\] (3.23c)

A useful check on the algebra is that as \(\xi \to 0\) this matches the value of the shear stress in the ISL at \(O(\delta)\).

The thickness of the ISL may now be estimated by examining when the SSL expansion of the shear stress breaks down. Sykes (1980) noted that the solution calculated in the SSL for the shear-stress gradient diverges logarithmically near the surface: from the SSL solution

\[
\frac{\partial \tau}{\partial \xi} \sim p_0 + q_0 \ln \xi + \delta(p_1 + q_1 \ln \xi) \quad \text{as} \quad \xi \to 0
\] (3.24)

where \(p_0, p_1, q_0\) and \(q_1\) are independent of \(\xi\). The SSL expansions cease to be asymptotic when \(q_1 \delta \ln \xi\) becomes \(O(1)\), i.e. when \(\ln \xi = r \ln(z/z_0)\), so that \(\xi = z/l = (z_0/l)^r\) for any \(0 < r < 1\). Hence, taking \(r = \frac{1}{2}\), a useful measure of the height of the ISL for practical purposes is

\[
l_s = \sqrt{(lz_0)}.
\] (3.25)

Alternatively, the height, \(l_s\), of the inner surface layer may be estimated by considering the gradient of the shear stress in the ISL. Calculating the \(O(z_0/s)\) terms in the ISL, a lengthy calculation shows that as \(z/z_1 \to \infty\), the shear-stress gradient calculated in the ISL is given by

\[
\frac{\partial \tau}{\partial \xi} \sim i\text{sign}(s)\left\{\left(\ln(z/z_0) - 1\right)\left(1 - \frac{1}{2}A_0 \ln(z/z_1)\right) - \frac{1}{2}A_0 - \delta^{(1)} - 2\delta^{2}a\delta^{(0)}(z_1)\right\}.
\] (3.26)

This expression equals \(\partial \tau/\partial \xi\) calculated in the SSL up to \(O(\delta^2)\) at a height \(z = l_s \sim \sqrt{(lz_0)}\). This value of \(l_s\) is typically of the order of \(30z_0\), so that the region of the perturbed flow governed by the law of the wall is very small. Indeed, if \(h\) is the height of a typical roughness element, then a common prescription for the roughness length is \(z_0 = h/30\) so that \(h\) may be greater than \(l_s\) and the ISL may not exist at all in the real flow.

The solution must now be prepared for matching with the middle layer, where the vertical coordinate is \(z = z/h_m = (l/h_m)\xi\). As \(\xi \to \infty\), both \(K_0\) and \(\xi \partial K_0/\partial \xi\) are exponentially small (Abramowitz and Stegun 1972), so

\[
\begin{align*}
u &\sim \left\{\delta \left(\delta^{(1)} + A_0 \kappa^2 \beta + \frac{1}{\text{sign}(s)\xi} \right) + O(\delta^3)\right\} \\
w &\sim \left\{- A_0 \kappa^2 \delta + O(\delta^3)\right\} \\
\tau &\sim \frac{2}{U(l)} \left\{O(\delta^3)\right\} \\
p &\sim - U(l) \left\{\delta^{(1)} + A_0 \kappa^2 \beta + O(\delta^3)\right\}
\end{align*}
\] (3.27)

which is to match with the middle layer.

\(c\quad\text{Middle layer}\)

In the outer region the Reynolds stresses (when correctly calculated using rapid distortion theory and not mixing length) affect the mean flow at \(O(\epsilon^2)\), so that the equations governing the leading order linear perturbations are the inviscid Euler
equations. Consequently, they can be reduced to a single equation for \( w \):
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{U''}{U} \right) w = O(\varepsilon^2). \tag{3.28}
\]

In the middle layer, the shear in the upwind velocity gradient dominates over the effects of the streamwise acceleration, i.e.,
\[
\frac{\partial^2 w}{\partial x^2} \ll \frac{U''}{U} w. \tag{3.29a}
\]

As \( z \) increases, \( U''/U \) decreases, and these two terms balance at a height, \( h_m \), defined to be the top of the middle layer. For a logarithmic boundary layer, \( h_m \) is given by the implicit relation
\[
h_m \ln \left( \frac{h_m}{z_0} \right) = \frac{1}{|s|}. \tag{3.29b}
\]

With the approximation (3.29a) the equation governing the perturbation to the vertical velocity becomes the Rayleigh equation, and can be rewritten in terms of the scaled coordinate \( \hat{z} = z/h_m \), so that
\[
\frac{\partial^2 w}{\partial \hat{z}^2} - \frac{U''}{U} w = O\left( h_m^2 |s|^2 \right). \tag{3.30}
\]

To find the asymptotic solution some care must be taken. The logarithmic form for \( U \) could be substituted and then \( z \) non-dimensionalized on \( h_m \), but this shows that the variation in \( U \) occurs only at \( O(h_m^2 |s|^2) \). The corresponding solution, which matches with the SSL, shows that this second-order variation in \( U \) becomes \( O(1) \) in the overlap region, so the second-order terms have to be calculated. Instead, the equation can be solved for a general unperturbed velocity profile and the logarithmic profile only substituted to match the solution with the inner region. Hence, equation (3.30) has the solution
\[
w^{(0)} = B_0 U + C_0 \int_{\hat{z}}^{\hat{z}} \frac{dz'}{U'(z')} \tag{3.31}
\]

where \( U = U(\hat{z} h_m) \).

The \( z \)-momentum equation shows that the pressure is constant at leading order, and continuity shows that
\[
u^{(-1)} = -\frac{1}{ish_m} \left[ B_0 U' + \frac{C_0}{U} \left( 1 + UU' \int_{\hat{z}}^{\hat{z}} \frac{dz'}{U'(z')} \right) \right]. \tag{3.32}
\]

The unknown constants, \( B_0 \) and \( C_0 \), are determined by matching with the SSL and upper layer. To perform the matching the integral terms can be approximated by
\[
\int_{\hat{z}}^{\hat{z}} \frac{dz'}{U'(h_m Z')} \sim \frac{Z}{U^2} + O(h_m^2 |s|^2). \tag{3.33}
\]

Rewriting in the SSL coordinate, (3.31) becomes
\[
w \sim isB_0 U(l) + \frac{\delta^{1/2}}{U(l)^{1/2}} \frac{2\kappa^2}{l} isC_0. \tag{3.34}
\]

Matching the vertical velocity and the pressure with equation (3.27) from the SSL gives
\[ B_0 = -\frac{A_0 \kappa^2 \delta}{U(l)} \quad C_0 = -i \hbar m \delta U(l)(\delta^{(1)} + A_0 \kappa^2 \beta). \] (3.35)

The definitions of the layer heights show that \( i\hbar m = (\delta \kappa U(l))^{1/2} \). A detailed consideration of the \( u \) perturbation shows that it too matches.

For large values of \( \varepsilon \), the middle-layer solutions become,

\[ u \sim \delta U(l) \{ (\delta^{(1)} + A_0 \kappa^2 \beta) + O(h_m |s|) \} \]

\[ w \sim \delta \left\{ -\frac{A_0 \kappa^2}{U(l)} + O(h_m |s|) \right\} \] (3.36)

which must match with the upper layer.

\[ (d) \quad \text{Upper layer} \]

By construction, in the upper layer the shear in the unperturbed velocity profile is negligible (compared with the streamwise acceleration). The vertical coordinate scales on \( |s| \) and the perturbations are expanded in powers of \( h_m |s| \); then at zero and first orders equation (3.28) becomes

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) w = O(h_m^2 |s|^2) \] (3.37)

so that the leading-order perturbations are determined by potential flow. The solution is

\[ w^{(0)} = D_0 e^{-|s|z} \quad u^{(0)} = -i \text{sign}(s)D_0 e^{-|s|z} = -p^{(0)} \] (3.38)

i.e. the perturbations decay exponentially at each wavenumber.

These expressions are matched with the middle-layer expressions (3.36), to give

\[ D_0 = -\frac{A_0 \kappa^2 \delta}{U(l)}, \quad \text{and} \quad \delta^{(1)} = A_0 \kappa^2 \left\{ \frac{i \text{sign}(s)}{U'(l)} - \beta \right\}. \] (3.39)

Thus the outer flow is driven by the displacement caused by the non-uniform thickening of the perturbed boundary layer.

\[ (e) \quad \text{Inversion of the Fourier transforms} \]

To obtain actual profiles of the velocity and shear-stress perturbations for a given roughness-length change, it is necessary to invert the Fourier transforms of the solutions presented in the previous sections. Only one example of the calculation of these inverse transforms is presented because they are all similar.

First note the identities

\[ K_0(y e^{i\pi/4}) = \ker_0(y) + i \kei_0(y) \]

\[ K_0(y e^{-i\pi/4}) = \{ K_0(y e^{i\pi/4}) \}^* \] (3.40)

where \( \ker_0 \) and \( \kei_0 \) are the real and imaginary Kelvin functions. Then the leading order \( u \) perturbation in the SSL becomes

\[ u^{(0)}(x, z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \{ \tilde{M}(s) + \tilde{M}(-s) \} \left[ a_1 \cos(sx) - b_1 \sin(sx) \right] \\
+ i \{ \tilde{M}(s) - \tilde{M}(-s) \} \left[ a_1 \sin(sx) + b_1 \cos(sx) \right] \right] ds \] (3.41)
where

\[
a_1 = \frac{\text{ker}_0(2\sqrt{\xi}) - \pi/2 \text{kei}_0(2\sqrt{\xi})}{a^2 + \pi^2/4},
\]

\[
b_1 = \frac{\pi/2 \text{kei}_0(2\sqrt{\xi}) + a \text{kei}_0(2\sqrt{\xi})}{a^2 + \pi^2/4},
\]

\[a = \ln(l/z_1) - 2\gamma.\tag{3.42}\]

4. Structure of the solution

Before applying the results of the analysis to specific applications, we pause to examine the structure of the flow induced by a sinusoidal roughness variation,

\[z_1(x) = z_0 - z_1 e^{ix}\]  

(4.1)

so that \(M = -(z_1/z_0) e^{ix}\).

(a) Magnitudes of the perturbations

The results of the analysis show that in the inner region there is a velocity perturbation of \(O(Mu_0/\ln(l/z_0))\) which leads, via continuity, to a vertical displacement of the streamlines, \(\Delta l(x)\), of \(O(lM/\ln^2(l/z_0))\) at the top of the inner region; and thence to the flow in the outer region, which is of second order in the local parameter \(\ln^{-1}(l/z_0)\). This argument shows that the outer region flow is similar to the flow over a low hill of height \(Ml/\ln^2(l/z_0)\) and length \(L\) (the wavelength of the roughness variation). As with the flow over a hill, the vertical displacement of the streamlines induces a pressure perturbation of

\[O((l/L)(M/\ln^2(l/z_0))(U_{\infty}^2(h_m))).\]

The associated pressure gradient of

\[O((l/L^2)(M/\ln^2(l/z_0))(U_{\infty}^2(h_m)))\]

is constant across the inner region and further accelerates the flow within the inner region. Although the magnitudes of the perturbations in the outer region are the same as the flow over a low hill, they are different within the inner region. The inner region of a flow over a roughness change is affected by the induced pressure gradient only at second order; whereas for the flow over the hill, the pressure gradient from the outer region drives the inner region flow at leading order. The streamlines of the perturbed flow induced by the sinusoidal roughness variation are sketched in Fig. 3(a).

(b) Phases of the perturbations

The above arguments show that there is a similarity between the magnitudes of the perturbations due to a varying surface roughness and those induced by a hill. The relative phases of the velocity, pressure and shear stress are, however, quite different (refer to Fig. 3(b)).

When the perturbations are forced by a roughness change given by Eq. (4.1), near the surface where \(z \sim z_1, \partial \Delta \tau/\partial z \sim 0\) and the velocity and shear-stress perturbations are exactly out of phase with the surface roughness variation (Figs. 3(b)(i), 3(b)(ii)), then close to the surface the streamwise velocity is accelerated at regions of low roughness length, and decelerated at regions of high roughness. At leading order, when \(z \sim l\), there is a balance between the inertial and the shear-stress gradients, i.e. \(U(l) \sim \partial \Delta \tau/\partial z\),
and so there is a phase difference of 90° between the velocity and shear-stress perturbations. This shows that the balance of advection and Reynolds shear-stress effects in the SSL leads to a variation in the phase between the velocity and shear-stress perturbations through the depth of the inner region. Hence the streamwise velocity at $z \sim l$ has real and imaginary parts so that it is not in phase with the surface roughness variation.

In the SSL, at second order, the asymmetry in the streamwise velocity perturbation induces a vertical velocity perturbation, $\Delta w^{(1)} \sim -i\Delta u^{(0)}$, and the maximum vertical velocity occurs above the region where the surface variation of $\Delta u(z \sim z_1)$ is at a minimum, i.e. where the roughness length has its maximum value. The magnitude of
this vertical velocity is proportional to $\delta \propto \ln^{-1}(s|^{-1}/z_0)$; the phase of $\Delta w^{(1)}$, however, is independent of $\ln^{-1}(s|^{-1}/z_0)$ (cf. the perturbations induced by a hill, described below). The vertical displacement of the streamlines, $\Delta l(x)$, then leads the phase of the roughness variation by $90^\circ$ (Fig. 3b(iii)). The second-order displacement of the mean streamlines drives the outer flow and so the streamwise velocity is in phase with the streamline displacement, i.e. the streamwise velocity perturbation is in phase with the slope of the roughness variation, leading the phase of the roughness variation by $90^\circ$ (Fig. 3b(iii)).

In the outer region there is a balance between the pressure and inertial gradients, i.e. $U(l)\Delta u \sim -i\Delta p$, so that the pressure perturbation develops in the outer region and is out of phase with the outer region velocity perturbation, lagging the phase of the roughness change by $90^\circ$ (Fig. 3b(iv)).

These phase relationships contrast with the solutions for the perturbations induced by a hill. The undulation induces a vertical velocity which is in phase with its slope. In the outer region, continuity then leads to a streamwise velocity perturbation which is in phase with the displacement of the surface. Furthermore, in the outer region there is a balance between the streamwise acceleration and the pressure gradient, $i\mu(h_m)\Delta u \sim -i\Delta p$, and so a leading pressure gradient develops which is out of phase with the hill surface (c.f. the phase of pressure and streamwise velocity in roughness change solutions). In the inner region the leading-order balance (except in the inner surface layer) is also between the inertial and pressure gradients, so that the streamwise velocity is still in phase with the undulation. At first order in $\ln^{-1}(l/z_0)$, however, there is a balance between the streamwise acceleration and the shear-stress gradient, so that there is an increasing phase difference between the first-order streamwise velocity, $\Delta u^{(1)}$, and the shear stress, $\Delta \tau$, up through the inner region (just as for the roughness change). The asymmetric streamwise velocity perturbation leads, via continuity, to a second-order asymmetric vertical velocity perturbation and thence there is an asymmetrical pressure perturbation also at second order in $\ln^{-1}(l/z_0)$. This means that the magnitude of the pressure perturbation induced by a hill is of $O((H/L)(\rho U_0^2))$ (where $H$ is the height of the hill, and $L$ its characteristic length scale) which is independent of $\ln^{-1}(l/z_0)$. The phase of the pressure perturbation, however, is of $O(\ln^{-2}(l/z_0))$, so that it decreases as the roughness of the surface decreases. This contrasts with the asymmetric pressure perturbation induced by a roughness change where the magnitude is proportional to $\ln^{-1}(s|^{-1}/z_0)$, but the phase is constant, independent of wavenumber.

(c) Other features of the solution

Figure 4 shows plots of the vertical profiles of the real parts of the normalized first and second-order velocity Fourier transforms, $\tilde{u}^{(0)}$ and $\tilde{u}^{(1)}$. These profiles are the basic solutions which are applicable to all roughness changes. Figure 5 shows similar plots of $\tilde{u}^{(0)}$ and $\tilde{u}^{(1)}$.

Having linearized the governing equations, approximate asymptotic solutions have been found in the limit $\varepsilon \to 0$, or equivalently $\delta = \varepsilon/kU(l) \to 0$, so we now consider what restrictions this second level of approximation places on the validity of the analysis. The expansions for the linear perturbations cease to be asymptotic when $\delta = \ln(l/z_0)$ is no longer small. The height of the inner region, $l$, has been defined (Eq. (3.7)) at each wavenumber $s$. As $s$ increases, $l(s)$ decreases, i.e. the higher wavenumber components of the roughness change influence a thinner region of the flow. Also, as $s$ increases $\delta = \ln^{-1}(l/z_0)$ increases, so that the asymptotic expansions become less accurate at the high wavenumbers. This does not matter too much, because $l/z_0$ decreases faster than $\ln^{-1}(l/z_0)$ increases, so that the depth of influence of the high wavenumber variations in
Figure 4. Normalized first- and second-order velocity perturbations in the inner region, \( \cdots \ \tilde{u}^{(0)}/A_0, \)
\( \cdots \cdots \cdots \cdots \cdots \ \tilde{u}^{(1)}/A_0. \)

Figure 5. Normalized first- and second-order shear-stress perturbations in the inner region, \( \cdots \ \tilde{\tau}^{(0)}/a_0, \)
\( \cdots \cdots \cdots \cdots \cdots \ \tilde{\tau}^{(1)}/a_0. \)
roughness decreases more rapidly than the asymptotic expansions lose accuracy. This argument shows that the present theory does not require the extra criterion of Townsend (1965), that \((x/M)(dM/dx) \ll 1\), i.e. that the roughness varies slowly.

The solutions do not show a great explicit dependence on the wavenumber \(s\), especially when compared with the solutions for the flow over a hill derived by HLR. The primary variation is due to the sign(s) terms. The absence of explicit dependence is because the four-layer asymptotic structure has been used at each wavenumber, hence the solutions show an implicit dependence on \(s\) because the height of the inner region \(l = l(s)\) (so that \(\delta = \delta(s)\)). As a specific example, consider the asymmetric part of the pressure perturbation

\[
\Delta p_{\text{asy}}(s, z_1) = -M\rho U^3_0 \delta \frac{A_0 k^2 i \text{sign}(s)}{U^2(l)}
\]

\[= -M\rho u_\infty^3 \frac{2 i \text{sign}(s)}{U^4(l)} (1 + O(\delta)).\]

Hence, for a roughness change of finite extent, because of the sign(s) term, there is a net asymmetric pressure perturbation (obtained by inverting the Fourier transform 4.2). The coefficient of the overall asymmetric pressure perturbation is strongly determined by the slow variations of \(U(l)\) and \(\delta\) with \(s\) and hence is strongly dependent on the exact shape of the roughness change.

For a specific geometry such as a step change in roughness, there is a ‘global inner region’, whose height, \(l_G\), may be defined as the height at which the shear-stress perturbation is one tenth of the value at the surface. This global inner region is then the layer where the shear-stress perturbation has a significant effect on the velocity perturbations. In the present study, the height \(l_G\) is a function of \(x\) which is automatically determined by the precise geometry of the roughness change, because an inner region has been defined at each wavenumber. Hence, by using Fourier analysis, the downstream distance required before the boundary layer achieves a new equilibrium, i.e. the value of \(x\) for which \(l_G(x) \to \infty\), is automatically determined by the geometry of the roughness change. Furthermore, the advection velocity in the inner region \(U(l_G(x))\), is also automatically calculated at each value of \(x\), and does not have to be prescribed beforehand.

5. Application to a Step Change in Roughness

We now present the results of the application of the theory to a step change in roughness. First a comparison is made between our analytic model, the numerical computations of Rao et al. (1974), and the experiments of Bradley (1968), for a step change in surface roughness with the parameters \(M = 4.83\) and \(z_0 = 0.002\) cm. Rao et al. (1974) found that their model, which used a second-order turbulence closure model, accurately predicted the velocity profile, but gave lower values of the wall shear stress than Bradley (1968) measured.

Using a mixed spectral finite difference method, Beljaars et al. (1987) have also calculated the flow over a step change in surface roughness using these parameters. They used two turbulence closure models: a mixing-length theory and a two-equation \(k-\epsilon\) model. They found that the two models predicted little difference in the mean velocity, but the mixing-length model gave higher values of the mean stress. As explained in section 2, if mixing length is used throughout the flow, the surface shear stress is overpredicted (see also BNH). The present theory is more accurate than using mixing length throughout because mixing length is only applied in the region of the flow where it is applicable.
In Figs. 6(a), (b) we plot vertical profiles of the mean velocity, $u^*(z)$, at two distances downstream of the change. There is close agreement between the results of our theory and the results of Rao et al. (1974), and the correspondence is better when the second order term is included. Figure 6(c) shows the distribution of wall shear stress. The present theory gives slightly higher values than Rao's, so that our results happen to be closer to the experimental values. Again, if mixing length is, erroneously, used throughout the flow domain, then the predicted wall shear stress is yet higher (see section 2).

In order to show their different vertical profiles, and their different magnitudes, the normalized first- and second-order perturbations of mean velocity downstream of the roughness change are plotted in Fig. 6(d). Figure 6(e) shows similar plots of the shear-
stress perturbations, $\tau^{(0)}$ and $\tau^{(1)}$. Clearly for most practical purposes, for two-dimensional flows (functions of $x$ and $z$), the second-order term may be neglected. This accuracy of the leading-order solution is because of the procedure we adopted when matching the ISL to the SSL: the coefficient $A_0$, of the leading-order solution is a function of $\delta$, so that some of the second-order effects have been ‘telescoped’ into the leading-order solution.

6. TWO SUCCESSIVE ABRUPT CHANGES

The analysis is now applied to the flow over two successive roughness changes. Blom and Wartena (1969) calculated the changes to a boundary layer caused by two successive step changes in surface roughness using a modified version of Townsend’s (1965) theory. Their results are now compared with the solutions obtained using the present theory. For $0 < x < L$ the local roughness length is $z_1 = 7.39$ cm, and when $x < 0$ or $x > L$, the local roughness length is $z_1 = 1$ cm; $L$ is taken to be 185 m and $u_* = 0.3775$ m s$^{-1}$.

Comparisons of the velocity profiles at different distances downstream of the second change are presented in Fig. 7(a), (b). The two theories are seen to be in close agreement. The variation of the wall shear stress downstream of the second change is plotted in Fig. 7(c), and here there is a significant difference between the two theories. The wall shear stress predicted by the theory of Blom and Wartena (1969) has a large local maximum at the second roughness change ($x = L = 185$ m) which the authors believe to be an over-prediction caused because the condition that $(x/M).dM/dx \ll 1$, is not satisfied at the position of the step change. The advective effects then cause the shear stress also to be over-predicted downwind of this roughness change. The present theory does not need the condition that the roughness length varies slowly (section 4) and does not show the wall shear-stress maximum at the second roughness change, hence also predicts lower values downwind of the change. This also means that the present theory predicts, correctly the authors believe, a larger downstream distance for the flow to adapt to the new equilibrium than Townsend (1965) or any modified versions of this theory.
Figure 7(a). Vertical profile of the mean velocity 251 m downstream of two step changes in roughness.

Figure 7(b). Vertical profile of the mean velocity 413 m downstream of two step changes in roughness.

Figure 7(c). Wall shear stress downstream of two step changes in roughness.

7. DISCUSSION

We have derived approximate solutions for the linear perturbations to a turbulent boundary layer as it passes over a surface with an arbitrary variation in surface roughness length. Although the theory also contains the linear perturbations through most of the flow, the solutions have been shown to predict accurately the velocity and shear-stress profiles for step changes of surface roughness, even when the roughness parameter $M$ is of order one. In view of this success of the theory, the fundamental approximations of the analysis are now examined in detail.
The variation in the surface roughness induces perturbations to the flow at the surface which diffuse out into the flow. The magnitudes of the perturbations to the mean velocity are as follows:

in the ISL

$$\Delta u = O(Mu_*)$$  \hspace{1cm} (7.1)

in the SSL

$$\Delta u = O(MA_0u_*) = O(M\delta u_*)$$  \hspace{1cm} (7.2)

and in the outer region,

$$\Delta u = O(Mu_*) \frac{U_B(h_m)}{U_B(l)} A_0 \delta = O\left(Mu_* \frac{U_B(h_m)}{U_B(l)}, \delta^2\right).$$  \hspace{1cm} (7.3)

This means that it is most important to calculate accurately the perturbations to the velocity, shear stress and shear-stress gradient close to the surface. As noted in section 3(a), the analysis of the ISL leads to solutions of the full non-linear equations, and hence the procedure adopted to match between the ISL and SSL leads to the solutions in the SSL having a non-linear, i.e. $O(M^2)$, correction. Hence the amplitude, $A_0$, of the leading order streamwise velocity perturbation in the SSL is

$$A_0 = \frac{2}{\ln(l/z_1) - 2\gamma - i\tau/2 \text{sign}(s)} = \frac{2\delta}{1 + \delta(M - 2\gamma - i\tau/2 \text{sign}(s))}.$$  \hspace{1cm} (7.4)

Comparing (7.1), (7.2) and (7.3), the streamwise velocity perturbation is greatest in the ISL, the layer where the solution is obtained for the full non-linear equations. In the SSL solutions the linearized equations have been obtained and so these solutions are approximate. Furthermore, the height at which the ISL and SSL solutions match is also only valid within the linearized framework. Hence in the SSL, we require that

$$u \frac{\partial u}{\partial x} \ll U \frac{\partial u}{\partial x}$$  \hspace{1cm} (7.5)

and that the velocity perturbation $\Delta u$ is small compared with the mean upwind velocity $U$ at $z \sim l$, so that the equations may be linearized. The theory is then applicable when these conditions are satisfied, so that from (7.2), we require $M\delta u_*/U_0 \ll 1$. For practical geometries, $\delta$ takes a wide range of values (because of the wide range of wavenumbers) and a practical guide for local validity of the theory is that

$$\frac{Mu_*}{U_B(l_G)} \ll 1.$$  \hspace{1cm} (7.6)

For a localized roughness change of extent $L$, a global criterion is that

$$\frac{Mu_*}{U_B(L)} \ll 1.$$  \hspace{1cm} (7.7)

The conditions (7.6), (7.7) are much less restrictive than the criterion $M \ll 1$, suggested by several authors (e.g. Townsend 1965). Similar arguments applied to the shear-stress perturbation produce the same criteria.

We have argued that it is most important to calculate accurately the perturbations very close to the surface. Within the inner region, a mixing-length formula has been used for the shear stress, and this model is most accurate where the turbulence approaches equilibrium, near the surface. Hence the turbulence model is also most accurate in the
ISL, where the perturbations are largest. It is important to note that we have not used the mixing-length model throughout the flow domain. The mixing-length theory overpredicts by $O(\varepsilon)$ the changes to the shear stress in the outer region, which then leads to errors in the pressure and shear stress in the inner region and at the surface, through the 'elliptic' terms in the equations (as discussed in section 2). This explains why the mixing-length computation of Beljaars et al. (1987) gives higher values of the wall shear stress than the two-equation $k - \varepsilon$ turbulence model (see section 5). Hence, we believe that the current shear-stress model (using mixing length in the inner region and truncating it in the outer region) is more accurate than using a mixing-length model throughout the flow (see also BNH). The normal stress gradients have been modelled by assuming that the changes to $u'_{ij}$ and $w'_{ij}$ are proportional to the changes to the shear stress. These terms have an effect on the mean flow at second order, but the resulting perturbations have numerically small values, so that we do not believe that the perturbations to the mean flow are sensitively dependent on the detailed modelling of these terms.

Having linearized the governing equations, approximate asymptotic solutions have been found in the limit $\varepsilon \to 0$, or equivalently $\delta = \varepsilon/(kU(l)) \to 0$. In section 4(c) we showed that the asymptotic expansions become less accurate (logarithmically) as the wavenumber increases, but that this does not matter because the depth of influence of these high-wavenumber components decreases linearly. Hence the present theory does not require the extra criterion of Townsend (1965), that $(x/M)(dM/dx) \ll 1$, i.e. that the roughness varies slowly. This extra criterion required by the Townsend (1965) theory leads to a spuriously large prediction of the wall shear stress at the point of a step change and then advective effects mean that the wall shear stress downwind of the change is also over-predicted (see section 6).

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