A concise derivation of the semi-geostrophic equations

By S. CHYNOWETH* and M. J. SEWELL

Department of Mathematics, University of Reading, Whiteknights, P. O. Box 220, Reading RG6 2AX

(Received 23 August 1990; revised 26 April 1991)

SUMMARY

A self-contained account of the semi-geostrophic equations is given. This contains formulae not previously available in the literature. The equations are presented in terms of four alternative sets of independent variables, and the Legendre mappings between these variables are specified. The approach is facilitated by the introduction of a general theorem in vector analysis, which is proved, concerning the total or convected derivative of a Jacobian following the particle. An expression for the curvature of a front is included.

1. INTRODUCTION

The purpose of this paper is to give a self-contained derivation of the semi-geostrophic equations that emphasizes their structure, and so offers insight into their general physical properties and aspects that will be involved in their solution. To facilitate our derivation we introduce a general theorem in vector analysis. The theorem is given in an Appendix, which can be read independently of the main body of the paper.

The semi-geostrophic equations have become an accepted system of approximate equations for the study of mid-latitude motions of the atmosphere on a synoptic scale, while allowing for the presence of mesoscale phenomena such as fronts. In our studies (e.g. Chynoweth and Sewell 1989, 1990) we have felt the need for a unified account of their derivation. The foregoing theorem allows this to be given in succinct style, marry ing the variable changes to the balance laws and physical approximations introduced sequentially. In section 7 we arrive at four alternative concise statements of the governing equations which have not been given in this form in the literature. The prominent role in this theory of the potential temperature, which is commonly used, is shown to be a peculiarity of the ideal gas, and is straightforwardly replaced by another function of entropy in the account given here.

The vector analysis theorem concerns the total derivative of a Jacobian, first of all in a formal sense in Appendix A, and subsequently in the true sense of the material or convected derivative following a particle which is familiar in fluid dynamics and in continuum mechanics generally. (The mathematical adjective ‘convected’ here includes both horizontal advection and vertical convection in the familiar physical usage.) The flexibility of the approach is indicated in Appendix B by making explicit some of the several variable changes which are needed in the fluid dynamics.

* Present address: Shell Research Ltd, Thornton Research Centre, P.O. Box 1, Chester, CH1 3SH.
2. PSEUDO-HEIGHT OF A PARCEL OF AIR

We consider an infinitesimal parcel of air (otherwise referred to as an air particle). The enthalpy, \( H \), per unit mass of a given parcel is the same given function \( H(\eta, p) \) of entrophy, \( \eta \), and pressure \( p \), for all time in any motion, according to standard thermodynamics (for example see Truesdell and Toupin 1960). The absolute temperature, \( \tau \), and density, \( \rho \), of the parcel always satisfy \( \tau = \partial H/\partial \eta \) and \( 1/\rho = \partial H/\partial p \), thus defining functions \( \tau(\eta, p) \) and \( \rho(\eta, p) \).

Let \( \eta_0 \) and \( p_0 \) denote assigned reference values of \( \eta \) and \( p \), implying associated reference values \( H_0 = H(\eta_0, p_0) \) and \( \rho_0 = \rho(\eta_0, p_0) \) in particular. We define the pseudo-height of the parcel as

\[
\left(1 - \frac{H(\eta, p)}{H(\eta, p_0)}\right) \frac{H_0}{g} = z \text{ (say)}
\]

(1)

where \( g \) is the combined acceleration due to gravity and the vertical effect of the earth's rotation. Then \( z = 0 \) when \( p = p_0 \) and, if \( H(\eta, 0) = 0 \), the pseudo-height of the atmosphere at \( p = 0 \) is \( H_0/g \) for that parcel.

A light assumption is to suppose that

\[
H(\eta, p) = \sigma(\eta) \zeta(p)
\]

(2)

where \( \sigma(\eta) \) is a strictly convex positive function of \( \eta \), and \( \zeta(p) \) is a strictly concave function of \( p \), which is positive for \( p > 0 \). Then

\[
z = (\zeta_0 - \zeta(p)) \frac{\sigma_0}{g}
\]

(3)

is a strictly convex function of \( p \) which is independent of \( \eta \), where \( \zeta_0 = \zeta(p_0) \) and \( \sigma_0 = \sigma(\eta_0) \). This applies to a parcel of the ideal gas in particular, for which (1) becomes

\[
z = \left[1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma} \right] \frac{\gamma p_0}{(\gamma - 1)p_0 g}
\]

(4)

(the definition of pseudo-height introduced by Hoskins (1971)), where \( \gamma \) is the constant ratio of specific heats. The ideal gas has \( p = k \rho^\gamma \) where \( k \) is an exponential function of \( \eta \) alone, \( \sigma = k^{1/\gamma} \) and

\[
\zeta = \frac{\gamma}{\gamma - 1} p^{(\gamma-1)/\gamma}
\]

in (2).

We now assume that the values \( \eta_0, p_0 \) and the form of the function \( H(\eta, p) \) are the same for every parcel. In any motion the dependence in (3) of \( z \) on time \( t \) and on the three Eulerian position coordinates \( x_i \) for \( i = 1, 2, 3 \) will be controlled entirely by the known function \( g(x_i) \), and by the function \( p(x_i, t) \) which is unknown in advance. Then (3), and (4) in particular, lead to a function \( z(x_1, x_2, x_3, t) \), unknown in advance but with the property

\[
\frac{\partial z}{\partial x_3} = - \frac{\sigma_0}{\rho g} \frac{\partial p}{\partial x_3} - \frac{z}{g} \frac{\partial g}{\partial x_3}
\]

(5)

Here \( x_3 \) is the true height. We assume \( \partial g/\partial x_3 \leq 0 \) as a very light restriction.

For motions in which the vertical momentum balance equation is approximated by the hydrostatic equation
\[
\frac{\partial p}{\partial x_3} + \rho g = 0
\]  

(6)

it follows that \(\partial z/\partial x_3 > 0\). Then there is a non-singular transformation

\[
z = z(x_1, x_2, x_3, t) \Leftrightarrow x_3 = x_3(x_1, x_2, z, t)
\]

(7)

between true height \(x_3\) and pseudo-height \(z\).

This provides an example of (B5) in which \(\xi_1 = x_1\), \(\xi_2 = x_2\) and \(\xi_3 = z\). There is a vertical pseudo-velocity \(\dot{z}\) related to the true velocity by

\[
\dot{z} = \frac{\partial z}{\partial t} + \dot{x}_1 \frac{\partial z}{\partial x_1} + \dot{x}_2 \frac{\partial z}{\partial x_2} + \dot{x}_3 \frac{\partial z}{\partial x_3}.
\]

(8)

The Jacobian

\[
\alpha = \frac{\partial z}{\partial x_3} = \frac{\sigma_0}{\sigma} \frac{z}{g} \frac{\partial g}{\partial x_3}
\]

(9)

has the total derivative property

\[
\frac{\dot{\alpha}}{\alpha} = \frac{\partial \dot{z}}{\partial z} = \frac{\partial \dot{x}_3}{\partial x_3}
\]

(10)

by choosing \(a = x_1\), \(b = x_2\), \(c = z\) in (B13). We use (10) in section 4.

It is also a light assumption to suppose that \(g\) is independent of \(x_3\), whence \(\alpha = \sigma_0/\sigma\) from (9), \(\dot{\alpha}/\alpha = -\tau\dot{\eta}/H\), so that

\[
\frac{\tau}{H} \dot{\eta} = \frac{\partial \dot{x}_3}{\partial x_3} - \frac{\partial \dot{z}}{\partial z}
\]

(11)

is a property of (3), and in particular of (4).

For adiabatic flow the energy balance equation is \(\dot{\eta} = 0\), whence

\[
\frac{\partial \dot{x}_3}{\partial x_3} = \frac{\partial \dot{z}}{\partial z}.
\]

(12)

3. GEOPOTENTIAL

For mid-latitude synoptic scale flow \(x_1\) and \(x_2\) adequately serve as the horizontal cartesian coordinates, and it is unnecessary to introduce polar coordinates. The pseudo-height (3) under the assumptions justifying (7) allows the geopotential \(gx_3 = \phi\) (say) to be regarded as a function \(\phi(x_1, x_2, z, t)\), unknown in advance of the motion. With \(g\) independent of \(x_3\) it follows from (9) that

\[
\frac{\partial \phi}{\partial z} = g \frac{\sigma}{\sigma_0}.
\]

(13)

The customary version of this formula in the literature, following Hoskins and Bretherton (1972), has \(\sigma/\sigma_0\) replaced by a quotient of the potential temperatures \(\xi_0 d\sigma/d\eta\) at \(\eta\) and \(\eta_0\), but the latter interpretation is available only for the ideal gas, being a special consequence of the exponential in the function \(\sigma(\eta) = k^{1/\gamma}\). In what follows we can sustain the more general (13) without inconvenience. Of course, we are not questioning the validity of the ideal gas approximation for the atmosphere; that is a separate point from the one we make here about the structure of the theory.
Horizontal momentum components \( M_1 \) and \( M_2 \) per unit mass are defined as

\[
M_1 = \frac{1}{f} \frac{\partial \phi}{\partial x_1} + x_1 f, \quad M_2 = \frac{1}{f} \frac{\partial \phi}{\partial x_2} + x_2 f
\]  

(14)

where this \( f \) is the Coriolis parameter. Henceforth we make the \( f \)-plane approximation in which \( f \) is assumed constant, thus ignoring its dependence on latitude and the angular velocity of the earth. The form of (14) then invites the introduction of a modified geopotential (Cullen and Purser 1984)

\[
\frac{1}{f} \phi + \frac{1}{2} (x_1^2 + x_2^2) f = P(x_1, x_2, z, t) \quad \text{(say)}
\]  

(15)

such that (14) and (15) can be rewritten as

\[
M_1 = \frac{\partial P}{\partial x_1}, \quad M_2 = \frac{\partial P}{\partial x_2}, \quad M_3 = \frac{\partial P}{\partial z}
\]  

(16)

where we define \( M_3 = g \sigma/f \sigma_0 \), a given function of \( \eta \) alone.

Equations (16) comprise a gradient mapping example of the transformation \( M_i = M_i(\xi_i, t) \) in (B6) with \( \xi_1 = x_1, \xi_2 = x_2, \xi_3 = z \). With \( t \) passive, there are at least three ways of inverting it, as follows.

(i) The inverse \( \xi_i = \xi_i(M_i, t) \) is expressible in terms of a Legendre transformation as

\[
x_1 = \frac{\partial R}{\partial M_1}, \quad x_2 = \frac{\partial R}{\partial M_2}, \quad z = \frac{\partial R}{\partial M_3}
\]  

(17)

in terms of a function \( R(M_i, t) \) such that

\[
P + R = x_1 M_1 + x_2 M_2 + z M_3
\]  

(18)

as observed by Sewell (1987, p. 157), Purser and Cullen (1987) and Chynoweth et al. (1988) in the more particular case when \( \sigma \) is replaceable by potential temperature. Chynoweth and Sewell (1989) give explicit examples showing how domains can be chosen over which \( P \) and \( R \) are single-valued, differentiable and convex functions at each fixed \( t \), and such that \( P \) contains a curved line of gradient discontinuity representing an atmospheric front.

If we now choose the \( a_i = M_i \) in (B14) we find that the total derivative of

\[
\begin{vmatrix}
\frac{\partial M_1}{\partial x_1} & \frac{\partial M_1}{\partial x_2} & \frac{\partial M_1}{\partial z} \\
\frac{\partial M_2}{\partial x_1} & \frac{\partial M_2}{\partial x_2} & \frac{\partial M_2}{\partial z} \\
\frac{\partial M_3}{\partial x_1} & \frac{\partial M_3}{\partial x_2} & \frac{\partial M_3}{\partial z}
\end{vmatrix} = q \quad \text{(say)}
\]  

(19)

satisfies

\[
\frac{\dot{q}}{q} = \frac{\partial \dot{M}_i}{\partial M_i} - \left( \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{z}}{\partial z} \right).
\]  

(20)

We use the conventional symbol \( q \) for this particular Jacobian, instead of our generic \( s \) in the Appendix. Since \( q \) is also the determinant of the Hessian matrix formed by the second derivatives of \( P(x_1, x_2, z, t) \) with respect to the spatial coordinates, it is necessary
for the strict convexity of $P$ that $q > 0$ almost everywhere. Such convexity has been justified in terms of a physical stability argument by Shufts and Cullen (1987), at least for the ideal gas in which $\sigma$ is replaceable by potential temperature in (16). The quantity $q_{T_0}/g$ can be identified with the geostrophic potential vorticity defined by Hoskins (1975), where $T_0 = \tau(\eta_0, p_0)$. An application of (20) is indicated in section 6.

(ii) Chynoweth and Sewell (1989) show that (16) can be used as the starting point for two more Legendre transformations, with either $z$ or $x_1$ and $x_2$ having only a passive role. In the first case a partial inverse of (16) is expressible via a function $S(M_1, M_2, z, t)$ of the so-called geostrophic coordinates $M_1, M_2, z$ and $t$ such that

$$x_1 = \frac{\partial S}{\partial M_1}, \quad x_2 = \frac{\partial S}{\partial M_2}, \quad M_3 = -\frac{\partial S}{\partial z},$$

$$P + S = x_1 M_1 + x_2 M_2.$$  \hfill (21)

Convexity of $P$ implies a domain over which $S$ is saddle-shaped at each fixed $t$, in the sense of being convex in $M_1$ and $M_2$, and concave in $z$.

If we choose $a_1 = M_1, a_2 = M_2, a_3 = z$ and $\xi_1 = x_1, \xi_2 = x_2, \xi_3 = z$ in (B14) we find that the total derivative of

$$\beta = \begin{vmatrix} \quad \frac{\partial M_1}{\partial x_1} & \frac{\partial M_1}{\partial x_2} \\ \frac{\partial M_2}{\partial x_1} & \frac{\partial M_2}{\partial x_2} \\
\frac{\partial M_3}{\partial x_1} & \frac{\partial M_3}{\partial x_2} \end{vmatrix}$$  \hfill (23)

satisfies

$$\frac{\dot{\beta}}{\beta} = \frac{\partial M_1}{\partial M_1} + \frac{\partial M_2}{\partial M_2} - \left( \frac{\partial x_1}{\partial M_1} + \frac{\partial x_2}{\partial M_2} \right).$$  \hfill (24)

This $\beta/f$ is the $z$ component of the geostrophic absolute vorticity, as distinct from the potential vorticity. Chynoweth and Sewell (1989) show that the selection of saddle functions $S(M_1, M_2, z, t)$ at each fixed $t$ is a flexible starting point for the construction of models of atmospheric fronts.

(iii) Another partial inverse

$$M_3 = M_3(x_1, x_2, z, t) \iff z = z(x_1, x_2, M_3, t)$$  \hfill (25)

of (16) is expressible via a function $T(x_1, x_2, M_3, t)$ such that (Chynoweth and Sewell 1989)

$$M_1 = \frac{\partial T}{\partial x_1}, \quad M_2 = \frac{\partial T}{\partial x_2}, \quad z = -\frac{\partial T}{\partial M_3},$$

$$P - T = z M_3.$$  \hfill (26)

Convexity of $P$ implies a domain over which $T$ is also saddle-shaped at each fixed $t$, in this case in the sense of being convex in $x_1$ and $x_2$ and concave in $M_3$. Since $M_3 = g\sigma/f\sigma_0$ depends only on $\eta$, this set of independent variables is referred to as isentropic coordinates.

If we choose $a_1 = \xi_1 = x_1, a_2 = \xi_2 = x_2, a_3 = M_3$ and $\xi_3 = z$ in (B14) we find that the total derivative of

$$\gamma = \frac{\partial M_3}{\partial z} = \frac{g}{f\sigma_0} \frac{\partial \sigma}{\partial z}$$  \hfill (28)
satisfies
\[ \frac{\dot{\gamma}}{\gamma} = \frac{\partial M_3}{\partial M_3} - \frac{\partial \dot{z}}{\partial z}. \]  
(29)

We can combine (7) and (25) to obtain a mapping
\[ M_3 = M_3(x_1, x_2, x_3, t) \Leftrightarrow x_3 = x_3(x_1, x_2, M_3, t) \]  
(30)
with Jacobian
\[ \delta = \frac{\partial M_3}{\partial x_3} = \frac{g}{f \sigma_0} \frac{\partial \sigma}{\partial x_3}, \]  
(31)
such that
\[ \frac{\dot{\delta}}{\delta} = \frac{\partial M_3}{\partial x_3} - \frac{\partial \dot{x}_3}{\partial x_3}. \]  
(32)
Since
\[ \dot{M}_3 = \frac{g}{f \sigma_0} \frac{d \sigma}{d \eta} \dot{\eta}, \]  
(33)
we have \( \dot{M}_3 = 0 \) in adiabatic flow, with consequent simplifications of (29) and (32).

As remarked after (13), the potential temperature is a function
\[ \zeta_0 \frac{d \sigma}{d \eta} = \theta(\eta) \quad \text{(say)} \]  
(34)
of entropy alone, with \( \theta_0 = \theta(\eta_0) \). For the ideal gas \( d \sigma/d \eta = \sigma/\gamma c_v \) where \( c_v \) is the specific heat at constant volume, so that
\[ \frac{\theta}{\theta_0} = \frac{\sigma}{\sigma_0} \]  
(35)
in that particular case. Then in (31) we have \( \partial \theta/\partial x_3 \), which is the square of the Brunt-Väisälä frequency for vertical oscillations in the atmosphere, and (32) gives a formula for its rate of change.

4. TRANSFORMED CONTINUITY EQUATION AND PSEUDO-DENSITY

The change of variable in (7) allows the equation of continuity (C1) to be rewritten in the forms (37) and (41) below, using the pseudo-height (3) as the independent vertical coordinate instead of the true height, \( x_3 \), by the following sequence of steps. Write (C1) as \( \dot{\rho} + \rho \ddot{x}_1/\alpha = 0 \) and insert \( \ddot{x}_3/\alpha = \dot{z}/\partial z - \dot{\alpha}/\alpha \) from (10) to give
\[ \frac{1}{\alpha} \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1} (\rho \ddot{x}_1) + \frac{\partial}{\partial x_2} (\rho \ddot{x}_2) + \frac{\partial}{\partial z} (\rho \dot{z}) \right\} - \rho \frac{\dot{\alpha}}{\alpha^2} = 0 \]  
(36)
after multiplying by \( 1/\alpha \). Hence
\[ \frac{\partial}{\partial t} \left( \frac{\rho}{\alpha} \right) + \frac{\partial}{\partial x_1} \left( \frac{\rho \ddot{x}_1}{\alpha} \right) + \frac{\partial}{\partial x_2} \left( \frac{\rho \ddot{x}_2}{\alpha} \right) + \frac{\partial}{\partial z} \left( \frac{\rho \dot{z}}{\alpha} \right) = 0 \]  
(37)
where \( 1/\alpha = \partial x_3/\partial z \) from (9).
THE SEMI-GEOSTROPHIC EQUATIONS

We can define a pseudo-density in terms of enthalpy $H(\eta, p)$ as

$$\rho \frac{H(\eta, p)}{H(\eta_0, p_0)} = r \text{ (say)}$$

(38)

such that $r = \rho$ when $\eta = \eta_0$. In the separable case (2) this $r$ becomes

$$\rho \frac{\sigma}{\sigma_0} = \frac{1}{\sigma_0} \frac{dp}{d\zeta} = r(z)$$

(39)

i.e. a function of $z$ alone in any motion. These definitions of $r$ include that of Hoskins and Bretherton (1972) for the ideal gas. For $g$ independent of $x_3$ and with the hydrostatic assumption, (9) and (39) show that

$$\frac{\rho}{\alpha} = \rho \frac{\partial x_3}{\partial z} = r(z)$$

(40)

so that the equation of continuity (37) becomes

$$\frac{\partial}{\partial x_1} \left( r\dot{x}_1 \right) + \frac{\partial}{\partial x_2} \left( r\dot{x}_2 \right) + \frac{\partial}{\partial z} \left( r\dot{z} \right) = 0$$

(41)

i.e. as if for a medium whose density can only depend on pseudo-height. We can also write (41) as

$$\frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{z}}{\partial z} = -\frac{\dot{z}}{r} \frac{dr}{dz}.$$

(42)

5. HORIZONTAL MOMENTUM BALANCE

The $x_2$ and $x_1$ equations of horizontal momentum balance, after the hydrostatic approximation (6) has been incorporated, may be written

$$-\frac{1}{f} \frac{\partial \phi}{\partial x_2} = \dot{x}_1 + \frac{1}{f} \ddot{x}_2, \quad \frac{1}{f} \frac{\partial \phi}{\partial x_1} = \dot{x}_2 - \frac{1}{f} \ddot{x}_1$$

(43)

respectively. The terms on the left of these equations are the $x_1$ and $x_2$ components, respectively, of the geostrophic wind.

The geostrophic momentum approximation (Eliassen 1948) replaces the derivatives $\ddot{x}_1$ and $\ddot{x}_2$ of the true wind in (43) by the time derivatives of the geostrophic wind components. In terms of the momentum variables (14) the resulting equations are

$$\frac{1}{f} \dot{M}_1 + M_2 - x_2 f = 0$$

(44)

and

$$\frac{1}{f} \dot{M}_2 - M_1 + x_1 f = 0.$$

(45)

These complete the system of physical balance laws, in association with the balance of mass (42), energy $\eta = 0$, and vertical momentum (6).
6. POTENTIAL VORTICITY DERIVATIVE

The properties (17) allow (44) and (45) to be rewritten in a form which has the same structure as Hamilton's equations of motion for mass-point mechanics, namely

\[
\dot{M}_1 = \frac{\partial \dot{R}}{\partial M_2}, \quad \dot{M}_2 = - \frac{\partial \dot{R}}{\partial M_1}
\]

(46)

but where

\[
\dot{R} = R f^2 - \frac{1}{2} (M_1^2 + M_2^2) f
\]

(47)

is a function of \(M_1, M_2, M_3\) and \(t\) which is not known in advance of the motion. Therefore momentum balance implies

\[
\frac{\partial \dot{M}_1}{\partial M_1} + \frac{\partial \dot{M}_2}{\partial M_2} = 0
\]

(48)

in both (20) and (24). This is a version of Liouville's theorem, which in mass-point mechanics states that the particles in phase space move as an incompressible fluid.

Energy balance \(\eta = 0\) for adiabatic flow implies \(\dot{M}_3 = 0\) from (33), whence from (20)

\[
\frac{\dot{q}}{q} = - \left( \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} \right) \text{ by (48)}
\]

\[
= \frac{\dot{z}}{r} \frac{dr}{dz} = \frac{\dot{r}}{r} \text{ by (41).}
\]

(49)

In other words, all the balance equations taken together imply

\[
\begin{pmatrix} \dot{q} \\ \dot{r} \end{pmatrix} = 0,
\]

(50)

as shown by Shutts and Cullen (1987) for the ideal gas.

The final approximation is the Boussinesq approximation, which has the effect of supposing that \(dr/dz = 0\) for tropospheric flow. Then (42) becomes

\[
\frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} = 0,
\]

(51)

\(\dot{r} = 0\) and

\[
\dot{q} = 0.
\]

(52)

Therefore the potential vorticity of a parcel is conserved following its motion, under the stated approximations.

7. SEMI-GEOSTROPHIC EQUATIONS

This is the name given to the system of governing equations resulting from the foregoing sequence of approximations. Here we give a résumé of the system using four different sets of independent variables. McWilliams and Gent (1980) review the advantages and disadvantages of the physical approximations which lead to these equations, but do not mention the Legendre transformation viewpoint which we highlight here, deriving from Chynoweth and Sewell (1989).
In each of the following subsections we list the appropriate versions of the time-derivative operator following the parcel, the equation of continuity, the two horizontal components of momentum balance, the vertical momentum balance, and energy balance, in the stated order.

Examples of boundary conditions are mentioned to guide the reader, but it would not be appropriate to attempt comprehensive statements of them here, because boundary conditions vary according to the particular problem being studied.

Strictly speaking, the four sets of coordinates below are physical space/pseudo-height, geostrophic/isentropic, geostrophic/pseudo-height and physical space/isentropic. However, we shall continue to use a shortened terminology which has gained acceptance in the literature.

(a) Physical space variables $x_1, x_2, z$ using pseudo-height

\[
\frac{\partial}{\partial t} + \dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + \dot{z} \frac{\partial}{\partial z} = \frac{\partial P}{\partial x_1} = x_2 f^2 - \frac{\partial P}{\partial x_2} f. \tag{53}
\]

\[
\frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{z}}{\partial z} = 0. \tag{51}
\]

\[
\frac{\partial P}{\partial x_1} = x_2 f^2 - \frac{\partial P}{\partial x_2} f. \tag{54}
\]

\[
\frac{\partial P}{\partial x_2} = -x_1 f^2 + \frac{\partial P}{\partial x_1} f. \tag{55}
\]

\[
\frac{\partial P}{\partial z} = M_3. \tag{56}
\]

\[
\dot{M}_3 = 0. \tag{57}
\]

The unknown functions of $x_1, x_2, z, t$ to be sought to satisfy these five equations are $P$, $M_3, \dot{x}_1, \dot{x}_2, \dot{z}$. Recall that $M_3 = g\sigma/\sigma_0$ by definition after (16), and is a given function of entropy alone.

The values of the functions $P(x_1, x_2, z, t)$, $\dot{x}_1(x_1, x_2, z, t)$, $\dot{x}_2(x_1, x_2, z, t)$ and $\dot{z}(x_1, x_2, z, t)$ are given everywhere at $t = 0$ (for example, see Cullen 1983). Typical lower and upper boundary conditions for all $t$ are $\dot{z} = 0$ on $z = 0$ and $z = h$, a given constant, for 'flat' horizontal boundaries, following Hoskins and Bretherton (1972); the former is an approximation to true vertical velocity at the ground. The velocities $\dot{x}_1$ and $\dot{x}_2$ are prescribed on respective lateral boundaries; alternatively periodicity could be imposed on these boundaries.

(b) Momentum/entropy space variables $M_1, M_2, M_3$

\[
\frac{\partial}{\partial t} + \dot{M}_1 \frac{\partial}{\partial M_1} + \dot{M}_2 \frac{\partial}{\partial M_2} + \dot{M}_3 \frac{\partial}{\partial M_3}. \tag{58}
\]

\[
\dot{q} = 0 \quad \text{where} \quad q = \left| \frac{\partial^2 R}{\partial M_i \partial M_j} \right|^{-1}. \tag{59}
\]

\[
\dot{M}_1 = \frac{\partial R}{\partial M_2} f^2 - M_2 f. \tag{60}
\]
\[ \dot{M}_2 = -\frac{\partial R}{\partial M_1} f^2 + M_1 f. \]  
\[ z = \frac{\partial R}{\partial M_3}. \]  
\[ \ddot{M}_3 = 0. \]

The unknown functions of \( M_1, M_2, M_3, t \) to be sought to satisfy the first three of these equations are \( R, \dot{M}_1 \) and \( \dot{M}_2 \). The solutions of (62) and (57) for \( z \) and \( \dot{M}_3 \) are immediate, and the latter simplifies (58). The dependence of \( R \) on \( M_3 \) is evidently restricted only by (59) with (57), apart from the convexity required earlier.

The function \( R(M_1, M_2, M_3, t) \) is assigned at \( t = 0 \). Typical lower and upper boundary conditions in systems using \( M_3 \) as a vertical coordinate prescribe \( z(M_1, M_2, M_3, t) \) (for example, see Davies and Horn 1988). On respective lateral boundaries one can prescribe \( \partial R/\partial M_1 \) and \( \partial R/\partial M_2 \), or impose some form of periodicity.

The operator (58) with (57) allows (59) to be rewritten in the expanded form
\[ \frac{\partial q}{\partial t} + \frac{\partial}{\partial M_1} (q \dot{M}_1) + \frac{\partial}{\partial M_2} (q \dot{M}_2) = 0 \]  
after also using (48). If \( q \neq 0 \) we find that (57)–(61) similarly imply
\[ \frac{\partial}{\partial t} \left( \frac{1}{q} \right) + \frac{\partial}{\partial M_1} \left( \frac{\dot{M}_1}{q} \right) + \frac{\partial}{\partial M_2} \left( \frac{\dot{M}_2}{q} \right) = 0. \]

This equation, with (60) and (61), can be recognised as the starting point of an investigation by Cullen and Purser (1989), based on the Monge–Ampere equation (59) as an equation to find \( R \) from given \( 1/q \).

(c) Geostrophic coordinates \( M_1, M_2, z \)
\[ (\dot{M}) = \frac{\partial}{\partial t} + \dot{M}_1 \frac{\partial}{\partial M_1} + \dot{M}_2 \frac{\partial}{\partial M_2} + \frac{\partial z}{\partial z}. \]
\[ \dot{q} = 0 \quad \text{where} \quad q = -\frac{\partial^2 S}{\partial z^2} \begin{vmatrix} \frac{\partial^2 S}{\partial M_1^2} & \frac{\partial^2 S}{\partial M_1 \partial M_2} \\ \frac{\partial^2 S}{\partial M_2 \partial M_1} & \frac{\partial^2 S}{\partial M_2^2} \end{vmatrix}^{-1} \]
\[ \dot{M}_1 = \frac{\partial S}{\partial M_2} f^2 - M_2 f. \]
\[ \dot{M}_2 = -\frac{\partial S}{\partial M_1} f^2 + M_1 f. \]
\[ \frac{\partial S}{\partial z} = -M_3. \]
\[ \ddot{M}_3 = 0. \]

Here \( S \) denotes the saddle function introduced in (21). The formula (66) for \( q \) was given by Chynoweth and Sewell (1989). A special case in two dimensions with constant \( q = 1 \) would be harmonic \( S(M_1, z) \) at each fixed \( t \), but other saddle functions are allowed in
THE SEMI-GEOSTROPHIC EQUATIONS

The unknown functions of \( M_1, M_2, z, t \) to be sought to satisfy these five equations are \( S, M_3, \ddot{M}_1, \ddot{M}_2, \dot{z} \). If \( S \) were known it is evident from (67)–(69) that \( M_3, \dot{M}_1 \) and \( \ddot{M}_2 \) would be available immediately. Another form of the equations with this choice of independent variables was given by Hoskins (1975).

The functions \( S(M_1, M_2, z, t) \) and \( \dot{z}(M_1, M_2, z, t) \) are assigned at \( t = 0 \). Lower and upper boundary conditions are chosen as in \((a)\) above. On respective lateral boundaries one can prescribe \( \partial S / \partial M_1 \) and \( \partial S / \partial M_2 \), or again impose some form of periodicity.

\[
\frac{\partial}{\partial t} \left( \begin{array}{c}
\dot{S} \\
\dot{x}
\end{array} \right) = \frac{\partial}{\partial t} \dot{x}_1 + \frac{\partial}{\partial x_1} \dot{x}_2 + \frac{\partial}{\partial M_3} \dot{M}_3,
\]

\[
\frac{\partial}{\partial t} \left( \begin{array}{c}
\ddot{S} \\
\ddot{x}
\end{array} \right) = \frac{\partial^2 T}{\partial x_1^2} \frac{\partial^2 T}{\partial x_2} \left( \frac{\partial^2 T}{\partial M_3^2} \right)^{-1}.
\]

\[
\dot{q} = 0 \quad \text{where} \quad q = -\left| \begin{array}{cc}
\frac{\partial^2 T}{\partial x_1^2} & \frac{\partial^2 T}{\partial x_1 \partial x_2} \\
\frac{\partial^2 T}{\partial x_2 \partial x_1} & \frac{\partial^2 T}{\partial x_2^2}
\end{array} \right|.
\]

\[
\frac{\partial T}{\partial x_1} = x_2 f^2 - \frac{\partial T}{\partial x_2} f.
\]

\[
\frac{\partial T}{\partial x_2} = -x_1 f^2 + \frac{\partial T}{\partial x_1} f.
\]

\[
z = -\frac{\partial T}{\partial M_3}.
\]

\[
\dot{M}_3 = 0.
\]

Here \( T \) denotes the saddle function introduced in (26). The formula (71), was given by Chynoweth and Sewell (1989). A special case in two dimensions with constant \( q = 1 \) would be harmonic \( T(x_1, M_3) \) at each fixed \( t \), but other saddle functions are allowed in (71). The unknown functions of \( x_1, x_2, M_3, t \) to be sought to satisfy the first three of these equations are \( T, \dot{x}_1 \) and \( \dot{x}_2 \). The solutions of (74) and (57) for \( z \) and \( \dot{M}_3 \) are immediate, and the latter simplifies (70). The dependence of \( T \) on \( M_3 \) is evidently constrained only by (71) with (57), apart from its saddle property. Another form of the equations with this choice of independent variables was given by Hoskins and Dragnechi (1977).

The function \( T(x_1, x_2, M_3, t) \) is assigned at \( t = 0 \). Lateral boundary conditions can be chosen as described in \((a)\) above, and upper and lower boundary conditions as described in \((b)\) above.

8. Numerical Algorithm

In the preceding sections we have provided a self-contained account of the derivation of a set of equations known as the semi-geostrophic equations, introducing the accepted approximations only when required in order to appreciate fully the effect of those approximations. These equations contain a structure which has only recently been recognised as the Legendre dual transformation; we have used the transformation to provide a unified description of the equations in four sets of dual variables.

An example where the availability of the Legendre transformation proves useful is in the numerical algorithm for the solution of Eqs. (51) to (57) described by Chynoweth.
(1987, 1990), Chynoweth and Baines (1989) and Chynoweth and Sewell (1990), and based on an idea by Cullen and Purser (1984). Examples of its application were described by Cullen et al. (1987) and Shutts et al. (1988).

A number of numerical schemes have been used to solve the semi-geostrophic equations. The present algorithm is based on a piecewise constant representation of momentum and potential temperature over an adaptive mesh, using well-defined singularities of the Legendre transformation. While faster algorithms are available for certain purposes, for example that of Fulton (1989) (cf. Schubert et al. 1989), the advantage of the present one is that discontinuities such as atmospheric fronts can be modelled explicitly, as illustrated in Chynoweth et al. (1988).

Briefly, the method repeatedly uses the mapping between the physical space variables (using pseudo-height) and the momentum/entropy space variables, using the appropriate form of the equations in each space. The advantage of the mapping is that Eqs. (60), (61) and (57) explicitly describe the trajectories in the momentum/entropy space. Corresponding expressions (for \( \hat{x}_1, \hat{x}_2 \) and \( \hat{z} \)) are not available amongst the physical space equations.

The flow chart of the algorithm is shown in Fig. 1 where reference is made to the one-dimensional schematization shown in Fig. 2 (adapted from section 2 of Chynoweth and Sewell 1990) to aid visualization of the scheme.

9. SLOPE AND CURVATURE OF A FRONT

The trace which a frontal surface makes with a vertical plane \( x_2 = \) constant has a slope given by a formula of the type

\[
\frac{dz}{dx_1} = -\frac{M_1^+ - M_1^-}{M_3^+ - M_3^-}
\]  

(75)

which is sometimes attributed to Margules in 1906, where the + and − superscripts designate values immediately on either side of the front. Here we derive a formula for the curvature of that frontal trace. We also make the observation that (75) expresses exactly the same mathematical fact, in our different physical context, as the Clausius–Clapeyron equation for the pressure/temperature gradient along a conjunction of two phases in classical thermodynamics.

We note first that for the thermodynamics of a simple fluid, the Gibbs free enthalpy function \( G(\tau, p) \) of temperature \( \tau \) and pressure \( p \) is often swallowtail shaped (see Sewell 1987, section 5.2) of the form shown in Fig. 3, with entropy \( \eta = -\partial G/\partial \tau \) and specific volume \( \nu = \partial G/\partial p \). The conjunction of phases is along the self-intersection line, and the Maxwell stability convention is sometimes adopted to excise the tripartite cusped tail of the swallowtail and leave the single-valued convex surface in Fig. 4, having a line of gradient discontinuity or ‘crease’ terminating at the swallowtail point. If the two parts of the surface on the ‘plus’ and ‘minus’ sides of the crease are denoted by \( G^+(\tau, p) \), and \( G^-(\tau, p) \), and the projection of the crease onto the \( \tau, p \) plane is the function \( p(\tau) \), differentiation of the phase conjunction property

\[
G^+(\tau, p(\tau)) = G^-(\tau, p(\tau))
\]  

(76)

along the crease, i.e. with respect to \( \tau \), gives

\[
\frac{\partial G^+}{\partial \tau} + \frac{\partial G^+}{\partial p} \frac{dp}{d\tau} = \frac{\partial G^-}{\partial \tau} + \frac{\partial G^-}{\partial p} \frac{dp}{d\tau}
\]  

(77)
At initial $t$, calculate a tangent approximation $\tilde{P}(x_1, x_2, z, t)$ (e.g., solid line in Fig. 2(a)) to give $P(x_1, x_2, z, t)$ over a set of elements with volumes $V_\alpha(t)$ (or areas or lengths in lower dimensions).

Calculate the corresponding $M_\alpha(x_1, x_2, z, t)$ and dual $\tilde{R}(M_1, M_2, M_3, t)$ from (16) and (18) (e.g., solid lines in Figs. 2(b) and (c)). This $\tilde{R}(M_1, M_2, M_3, t)$ will be a chordal approximation to $R(M_1, M_2, M_3, t)$, coinciding with it at $N$ points $M_1^\alpha, M_2^\alpha, M_3^\alpha, R^\alpha$ for $\alpha = 1, \ldots, N$.

Change to the new time-step $t + \Delta t$, and use (60), (61) and (57) to approximate the $M_\alpha^\alpha(t + \Delta t)$, for $i = 1, 2, 3$ and $j = 1, \ldots, N$.

Guess the nodal values $R^\alpha = \tilde{R}(M_1^\alpha(t + \Delta t), M_2^\alpha(t + \Delta t), M_3^\alpha(t + \Delta t))$ at the new time-step. One option is to use the values of $\tilde{R}(M_1^\alpha(t), M_2^\alpha(t), M_3^\alpha(t))$ from the previous time-step. (Joining these guessed nodes is illustrated by the broken line in Fig. 2(c).)

Construct the dual $\tilde{P}(x_1, x_2, z, t + \Delta t)$ of this guess, and calculate the volumes $V_\alpha(t + \Delta t)$ of the associated elements.

Check to see if the equation of continuity (51) is sufficiently well satisfied, i.e., whether

$$|V_\alpha(t + \Delta t) - V_\alpha(t)| < \varepsilon$$

for some predefined tolerance $\varepsilon$, and each $\alpha$.

If Yes

If No

Refine the guesses $R^\alpha$ and return to earlier step, perhaps several times at each time step.

Either stop, or go on to next time-step and repeat et seq.

Figure 1. Flow chart for numerical algorithm.
and therefore

\[ \frac{dp}{d\tau} = \frac{\eta^+ - \eta^-}{\nu^+ - \nu^-}. \]  

(78)

This is the Clausius–Clapeyron equation.

To translate this into the meteorological context we have only to replace \(-G(\tau, p)\) by the modified geopotential \(P(x_1, z)\) for fixed \(x_2\), and use \((16)_1\) and \((16)_3\). Then (78) becomes (75). The result is consistent with the idea of representing a front as the line of gradient discontinuity or crease in a single-valued convex surface \(P = P(x_1, z)\), for example such as remains after convexification of a swallowtail surface, in the way illustrated by Chynoweth and Sewell (1989, Figs. 3, 7 and 12). The top of the front is the projection of a swallowtail point in the sky on this model.
A second differentiation of \( P^+(x_1, z(x_1)) = P^-(x_1, z(x_1)) \) along the trace of front, i.e. with respect to \( x_1 \), gives the curvature of the front as follows. It is common to use square brackets to denote finite jumps in a specified sense on either side of a discontinuity, and in this notation we find

\[
\frac{d^2 z}{dx_1^2} = -\frac{1}{[M_3]} \left( \frac{\partial^2 P}{\partial x_1^2} \right) + 2 \left( \frac{\partial^2 P}{\partial x_1 \partial z} \right) \frac{dz}{dx_1} + \frac{\partial^2 P}{\partial z^2} \left( \frac{dz}{dx_1} \right)^2
\]

(79)

wherein

\[
\frac{dz}{dx_1} = -\frac{[M_1]}{[M_3]}
\]

(80)

from (75). These formulae, (79) and (80), are valid, just by differentiation along the crease, regardless of whether or not one chooses to adopt the swallowtail interpretation. It will be interesting to learn in due course what observational evidence can contribute to the formula (79) for the frontal curvature, and in particular to its sign. For example, a sufficient condition to ensure a concave front \((d^2 z/dx_1^2 < 0)\) would be that \(P^+(x_1, z)\) is more convex than \(P^-(x_1, z)\), where they meet at the front, when \([M_3] > 0\) there.

**APPENDIX A**

**Vector analysis theorem**

In this section we prove a theorem in vector analysis which we have not seen in standard texts.

Let \(x_1, x_2, x_3\) be rectangular cartesian coordinates having associated fixed unit vectors \(i_1, i_2, i_3\) and gradient operator \(\nabla = i_\alpha \partial / \partial x_\alpha\), using the summation convention over \(i = 1, 2, 3\). Let \(t\) be a fourth variable, independent of the \(x_i\). Let \(v = v_i i_i\) be a vector field whose components \(v_i\) are each differentiable functions of some or all of the \(x_i\) (and perhaps also of \(t\), although we shall not require dependence of \(v\) on \(t\) in this section).
To any differentiable function of the $x_i$ and $t$ we can apply the operator defined by

$$D = \frac{\partial}{\partial t} + v \cdot \nabla.$$ (A1)

In this section we do not require $v$ to represent a velocity, and so $D$ need not be a total or convected derivative in the conventional sense. The applications in other sections will introduce such interpretations.

Let $a$, $b$, $c$ be three differentiable scalar functions of the $x_i$ and $t$, and let

$$s = [\nabla a \ \nabla b \ \nabla c]$$ (A2)

denote the scalar triple product of their gradients. This is a Jacobian determinant which can also be written

$$s = \varepsilon_{ijk} \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_j} \frac{\partial c}{\partial x_k}$$ (A3)

in terms of the alternating symbol $\varepsilon_{ijk}$ which we use in the following proof.

**Theorem 1**

$$D_s + s \nabla \cdot v = [\nabla Da \ \nabla b \ \nabla c] + [\nabla a \ \nabla Db \ \nabla c] + [\nabla a \ \nabla b \ \nabla Dc].$$ (A4)

**Proof**

$$D_s = [D \nabla a \ \nabla b \ \nabla c] + [\nabla a \ D \nabla b \ \nabla c] + [\nabla a \ \nabla b \ D \nabla c]$$

$$= \left[ \nabla \frac{\partial a}{\partial t} + \nabla (v \cdot \nabla a) - \frac{\partial v}{\partial x_i} \frac{\partial a}{\partial x_p} \ \nabla b \ \nabla c \right] + \ldots + \ldots$$

$$= [\nabla Da \ \nabla b \ \nabla c] + \ldots + \ldots - \left[ \frac{\partial v}{\partial x_i} \frac{\partial a}{\partial x_p} \ \nabla b \ \nabla c \right] + \ldots + \ldots$$

$$= [\nabla Da \ \nabla b \ \nabla c] + \ldots + \ldots$$

$$- \varepsilon_{ijk} \left( \frac{\partial v}{\partial x_i} \frac{\partial a}{\partial x_p} \frac{\partial b}{\partial x_j} \frac{\partial c}{\partial x_k} + \frac{\partial a}{\partial x_i} \frac{\partial v}{\partial x_p} \frac{\partial b}{\partial x_j} \frac{\partial c}{\partial x_k} + \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_j} \frac{\partial v}{\partial x_k} \frac{\partial c}{\partial x_p} \right)$$

$$= [\nabla Da \ \nabla b \ \nabla c] + \ldots + \ldots$$

$$- \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_j} \frac{\partial c}{\partial x_k} \left( \varepsilon_{ipk} \frac{\partial v}{\partial x_p} + \varepsilon_{ijk} \frac{\partial v}{\partial x_p} \frac{\partial v}{\partial x_p} \right) = [\nabla Da \ \nabla b \ \nabla c] + \ldots + \ldots - s \nabla \cdot v.$$  

The last step depends on the alternating symbol property that

$$\varepsilon_{ijk} a_{pi} + \varepsilon_{ijp} A_{pj} + \varepsilon_{ipj} A_{pk} = \varepsilon_{ijk} A_{pp}$$ (A5)

for any matrix $A_{ij}$, and in particular for $A_{ij} = \partial v_j / \partial x_i$.

To prove (A5) consider the following two cases. For two suffixes the same, say $i = j$, the right side is zero because $\varepsilon_{ijk} = 0$. The left side is

$$\varepsilon_{ijk} A_{pj} + \varepsilon_{ipk} A_{pj} = \varepsilon_{ijk} (A_{pj} - A_{pi}) = 0.$$  

For suffixes all different, say $i = 1, j = 2, k = 3$,

$$\varepsilon_{p12} A_{p1} + \varepsilon_{p23} A_{p2} + \varepsilon_{p13} A_{p3} = A_{11} + A_{22} + A_{33} = \varepsilon_{123} A_{pp}.$$  

Whenever $s \neq 0$ the right side of (A4) takes a particular form, as follows.
Theorem 2

If \( s \neq 0 \)

\[
\frac{Ds}{s} = \frac{\partial Da}{\partial a} + \frac{\partial Db}{\partial b} + \frac{\partial Dc}{\partial c} - \nabla \cdot \mathbf{v}. 
\]  

(A6)

Proof

The three functions \( a(x_i, t), b(x_i, t), c(x_i, t) \) have a collective local inverse

\[
x_i = x_i(a, b, c, t) \tag{A7}
\]

for \( i = 1, 2, 3 \) whenever \( s \neq 0 \). We can use (A7) to express each of \( Da, Db, Dc \) as functions of the independent variables in (A7), whence

\[
\nabla Da = \frac{\partial Da}{\partial a} \nabla a + \frac{\partial Db}{\partial b} \nabla b + \frac{\partial Dc}{\partial c} \nabla c \tag{A8}
\]

and similarly for \( \nabla Db \) and \( \nabla Dc \). Substituting these relations into (A4) gives (A6). □

APPENDIX B

Changes of variable

For some purposes it is more convenient to rewrite \( a, b, c \) as \( a_1, a_2, a_3 \). The Jacobian (A2) can then be rewritten

\[
s = \frac{|\partial a_i|}{\partial x_j}. \tag{B1}
\]

The transformation to (A7) is then

\[
a_i = a_i(x_j, t) \Leftrightarrow x_i = x_i(a_j, t) \tag{B2}
\]

if \( s \neq 0 \), and (A6) can then be rewritten

\[
D \ln s = \frac{\partial Da_i}{\partial a_i} - \frac{\partial v_i}{\partial x_i}. \tag{B3}
\]

In applications several different changes of variable arise which are differentiable and at least locally invertible, and we review some of them here. We denote them first by trios of functions as follows.

\[
X_i = X_i(x_j, t) \Leftrightarrow x_i = x_i(X_j, t). \tag{B4}
\]

\[
\xi_i = \xi_i(x_j, t) \Leftrightarrow x_i = x_i(\xi_j, t). \tag{B5}
\]

\[
M_i = M_i(\xi_j, t) \Leftrightarrow \xi_i = \xi_i(M_j, t). \tag{B6}
\]

A non-repeated suffix such as \( i \) and \( j \) above will be understood to range over all the values 1, 2, 3 in turn, and the variables will be regarded as cartesian components of vectors which we denote by \( \mathbf{x}, \mathbf{X}, \mathbf{\xi} \) and \( \mathbf{M} \).

A quantity with values \( y \) which depends on \( t \) and on one or these four vectors could be rewritten, using (B4)–(B6), as a different function of \( t \) and of any other one of the vectors, for example as

\[
y(\mathbf{X}, t) = \tilde{y}(\mathbf{x}, t) = \tilde{y}(\mathbf{\xi}, t) = \tilde{y}(\mathbf{M}, t) \tag{B7}
\]
when we wish to have a different symbol for each of the four functions. The partial
derivative with respect to \( t \) of the first of these functions, in which \( X \) is held constant,
will be denoted by \( \dot{y} \), i.e. using the superposed dot. The chain rule then gives
\[
\dot{y} = \frac{\partial \dot{y}}{\partial t} + \dot{x} = \frac{\partial \dot{y}}{\partial x} + \dot{\xi} = \frac{\partial \dot{y}}{\partial \xi} + \dot{\xi} = \frac{\partial \dot{y}}{\partial t} + \dot{M} = \frac{\partial \dot{y}}{\partial M}.
\]
(B8)

This displays three examples of the operator (A1) in a different but self-evident notation.
There are then three consequent examples of the formulae (A4) and (A6) or (B3), in
which \( D, v \) and \( v \) are replaced by the dot, the appropriate velocity, and the appropriate
gradient respectively.

The choice of \( a(x_i, t), b(x_i, t) \) and \( c(x_i, t) \) can be regarded as another variable change
(B2), additional to those in (B4)–(B7) if desired.

If we make the particular choices
\[
a = X_1 \quad b = X_2 \quad c = X_3
\]
so that (B2) identifies with (B4), we have
\[
\dot{X}_1 = \dot{X}_2 = \dot{X}_3 = 0
\]
by definition, and (A4) becomes
\[
\dot{s} + s \nabla \cdot \dot{x} = 0
\]
(B11)

where the Jacobian (B1) has become
\[
s = \left| \frac{\partial X_i}{\partial x_j} \right| = \left| \frac{\partial x_i}{\partial X_j} \right|^{-1}.
\]
(B12)

For any other choice of \( a, b, c \) in (A6) the example (B8) of (A1) gives
\[
\frac{\dot{s}}{s} = \frac{\partial \dot{a}}{\partial a} + \frac{\partial \dot{b}}{\partial b} + \frac{\partial \dot{c}}{\partial c} - \nabla \cdot \dot{x}
\]
\[
= \frac{\partial \dot{a}}{\partial a} - \frac{\partial \dot{x}_i}{\partial x_i} \text{ for (B1), in the other notation (cf. (B3)).}
\]
(B13)

If we had started Appendix B with \( \xi \) or \( M \) instead of \( x \) we would have arrived at formulae
with the same structure as (B13), namely
\[
\frac{\dot{s}}{s} = \frac{\partial \dot{a}}{\partial a} - \frac{\partial \dot{\xi}_i}{\partial \xi_i} \text{ for } s = \left| \frac{\partial a_i}{\partial \xi_j} \right|,
\]
(B14)

\[
\frac{\dot{s}}{s} = \frac{\partial \dot{a}}{\partial a} - \frac{\partial \dot{M}_i}{\partial M_i} \text{ for } s = \left| \frac{\partial a_i}{\partial M_j} \right|.
\]
(B15)

The invertible transformations implicit in (B14) are \( a_\xi(\xi, t) \) and \( \xi(X, t) \), and those in
(B15) are \( a_M(M, t) \) and \( M(X, t) \).

In this Appendix it has been convenient to use \( s \) as the generic symbol for a Jacobian,
even though its definition varies from case to case. In the main text different specific
Jacobians are denoted by \( a, b, c, \gamma, \delta, \psi \).

In Appendices A and B our results have been purely mathematical, without any
physical interpretations being presumed. In what follows and in the main body of the
paper we introduce physical illustrations.
APPENDIX C

Equation of continuity

To give a simple example of the foregoing formulae in a physical context, let a particle of a continuous medium have current (Eulerian) coordinates $x_i$ at time $t$, and initial (Lagrangian) coordinates $X_i$ at some fixed reference time (say $t = 0$). Then (B4) describes the motion with velocity $\mathbf{u}$.

The choice (B9) in (A2) implies $\mathbf{s} = \rho(X_i, t)/\rho(X_i, 0)$ in (B12), the quotient of the current and initial densities, and we recover from (B11) the differential equation of continuity

$$\dot{\rho} + \rho \nabla \cdot \mathbf{u} = 0. \quad \text{(C1)}$$

The physical assumption $\rho > 0$ ensures $\mathbf{s} > 0$, so that the Jacobian (B12) is non-singular and the medium cannot interpenetrate itself.

ACKNOWLEDGEMENT

S. Chynoweth is glad to acknowledge the financial support of the Science and Engineering Research Council during this research.

REFERENCES


Eliassen, A. 1948 The quasi-static equations of motion with pressure as independent variable. Geofys. Publ., 17, No. 3


Hoskins, B. J. and Draghici, I. 1977 The forcing of ageostrophic motion according to the semi-geostrophic equations and in an isentropic coordinate model. J. Atmos. Sci., 34, 1859–1867
McWilliams, J. C. and Gent, P. R. 1980 Intermediate models of planetary circulations in the atmosphere and oceans. J. Atmos. Sci., 37, 1657–1678


Truesdell, C. and Toupin, R. A. 1960 Geometric models of balanced semi-geostrophic flow. Annales Geophysicae, 6, 493–500