Energetics with an entropy flavour

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The purpose of this note is to provide an elementary motivation for, and a simple derivation of, the key results in the theory of available enthalpy presented by Marquet (1990). Our aim is to obtain a Bernoulli equation that governs the rates of change, following the motion, of energy quantities with entropy-like characteristics.

Let \(e_k = \mathbf{v} \cdot \mathbf{v}/2\) be the specific kinetic energy and \(e_g = g z\) the specific potential energy. Then as usual

\[
\frac{de_k}{dt} = -\alpha \mathbf{v} \cdot \nabla p + C(g, k) + D_k
\]

and

\[
\frac{de_g}{dt} = -C(g, k)
\]

where the energy conversion term \(C(g, k) = -g \, dz/dt\); \(D_k\) is the specific rate of working of all forces other than gravity and the pressure gradient force; \((\alpha)\) is the specific volume, \(1/\rho\), where \(\rho\) is the density; \(p\) is the pressure).

As is customary, we obtain a thermodynamic link from the enthalpy form of the first law since it contains the term \(\alpha \, dp/dt\) from which we can extract \(C(h, k) = -\alpha \mathbf{v} \cdot \nabla p\). Thus for the specific enthalpy \(h = e_h = c_p T\)

\[
\frac{de_h}{dt} = -C(h, k) + \alpha \, dp/\partial t + q.
\]

The rate of heating, \(q\), allows us to introduce entropy considerations by creating a covariance between the temperature \(T\) and the quantity \(q\):

\[
q = (T_r/T)q + (1 - T_r/T)q
\]

where \(T_r\) is a constant to be determined. For the specific entropy \(s_r\),

\[
(T_r/T)q = T_r \, ds/dt = T_r (c_p \, d\ln T/\partial t) - R \, d\ln p/\partial t
\]

and thus we can define \(\alpha(T, T) = -(RT_r/p) \, dp/\partial t\) and then combine (3), (4), and (5) to obtain

\[
\frac{d}{dt} (e_h - c_p T \, \ln T) = C(p, T) - C(h, k) + G
\]

where the generation \(G = \alpha \, dp/\partial t + (1 - T_r/T)q\). Moreover, we can define available energy forms

\[
a_T = h - c_p T_r - c_p T_r \, \ln(T/T_r)
\]

\[
a_p = RT_r \, \ln(p/p_r)
\]

that vanish when \(T = T_r = \text{const.}\) and \(p = p_r = \text{const.}\). The constants \(T_r\) and \(p_r\) can be determined as averages or specified as typical values.

We summarize these results as
\[
\frac{de_k}{dt} = C(h, k) + C(g, k) + D_k
\]
\[
\frac{de_l}{dt} = -C(g, k)
\]
\[
\frac{da_T}{dt} = C(p, T) - C(h, k) + G
\]
\[
\frac{da_p}{dt} = -C(p, T)
\]

and thus we may define the available enthalpy as the sum of the components

\[
a_h = a_T + a_p = (h - h_s) - T_r(s - s_r).
\]

Moreover, we now have a Bernoulli equation in the form

\[
\frac{d}{dt}(e_k + e_g + a_h) = G + D_k.
\]

This new Bernoulli equation, upon integration over the atmosphere, leads to the usual energy theorem. Upon multiplying by the density, integrating with the aid of the transport theorem (Dutton 1986), and recognizing that the mass-weighted integrals of \( h \) and \( T, s \), are constant, we find that

\[
\frac{d}{dt}(K + I + P) + \int_V \rho \left( R \frac{dT}{dt} - T \frac{ds}{dt} \right) dV = D + Q + \int_V \rho \left( \alpha \frac{\partial p}{\partial t} - \frac{T_r}{T} \right) q dV
\]

where \( K, I, \) and \( P \) are the usual forms of kinetic, internal, and potential energy and \( D \) and \( Q \) are the integrals of \( D_k \) and \( q \). The last terms on both sides cancel by virtue of (5). With the equation of state, direct use of the transport theorem or an application of the continuity equation and the divergence theorem shows that \( \int \rho R(dT/dt) \, dV = \int \rho \alpha \partial p/\partial t \, dV \) for the entire atmosphere. Thus (11) is the classical statement that \( d(K + I + P)/dt = D + Q \).

We have succeeded in deriving a system (8) of energy equations, following the motion, in which the thermodynamic quantities have an entropy flavour. There are, however, no implications that any conversions will actually occur or be of a known sign.

REFERENCES
