The three-dimensional nature of ‘symmetric’ instability

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SUMMARY

The importance of three-dimensional effects in a flow with negative potential vorticity is considered here; this is the three-dimensional counterpart of symmetric instability. The term symmetric instability refers to a two-dimensional flow with negative potential vorticity which develops roll circulations aligned along the thermal wind. The requirement for exact two-dimensionality is here relaxed by using three-dimensional numerical simulations. Two classes of simulation are described; both take a two-dimensional basic flow.

In the first part we explore the response of a flow with uniform negative potential vorticity to localized initial perturbations. It is found that circulations become established which elongate at an angle to the thermal wind. This angle, it is shown, is determined by viscous effects and is in accord with the structure of linear viscous tilted modes for this flow. If, as happens on occasions, such an angle formed by rainbands to the thermal wind is observed in nature, it gives an indication of the importance of viscous effects. In the cases presented here the orientation is such that the rolls tilt towards the warm air as viewed in the direction of the thermal wind; i.e. they are rotated anticyclonically relative to the front. It is shown that these structures, when at finite amplitude, are stable to perturbations imposed along their length.

In the second part localized regions of instability are produced which are confined along the thermal wind direction by prescribing in these zones a reduced static stability for vertical motion. This represents the effects of latent-heat release and it is only in these zones that the effective potential vorticity is negative. It is found that if the length of the zone along the thermal wind direction is smaller than about twice the roll wavelength across that direction then the growth rate is substantially reduced; otherwise it is relatively unaffected by this confinement. It therefore appears that a flow with negative potential vorticity is not only unstable in the pathological two-dimensional case.

1. INTRODUCTION

Conditional symmetric instability (CSI) has recently become the focus of much attention, owing to its hypothesized role in frontal regions. It was first proposed as a possible mechanism for the formation of frontal rainbands by Bennett and Hoskins (1979). Further indications of the importance of CSI in frontal regions come from observational studies by Emanuel (1988). These showed that frontal regions are typically in a state of neutrality to CSI. The instability may also be a contributing factor in explosive cyclogenesis. Studies of several explosively deepening cyclones conducted with the Meteorological Office’s fine-mesh model (Shutts 1990) revealed substantial amounts of available potential energy for CSI prior to cases of explosive development.

All theoretical studies of symmetric instability to date have been two-dimensional. Although the horizontal aspect ratio of frontal rainbands makes this a reasonable first approximation when attempting to account for their formation, these rainbands also have three-dimensional characteristics. Such rainbands frequently exhibit structure in the along-front direction, and have a finite length in this direction. When considering the possible role of symmetric instability in explosive cyclogenesis, three-dimensional effects must also be considered, since such cyclones frequently have small horizontal scale and hence strongly curved flow. For these reasons this paper considers the extension of the theory of symmetric instability to three dimensions.

The criterion for symmetric instability to occur (Helmholtz 1888; Solberg 1936; Sawyer 1949) is that the absolute vorticity on a potential-temperature surface is negative. An equivalent condition is that the potential vorticity (PV) is negative (Hoskins 1974).

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Ooyama (1966) showed that for two-dimensional inviscid flow this is both a necessary and sufficient criterion for symmetric instability. In other words, any region of negative potential vorticity in a two-dimensional inviscid flow is symmetrically unstable. The derivation of this instability criterion does not apply to three-dimensional flow. Indeed, owing to the analytically intractable nature of the three-dimensional problem, it is difficult to see how such an instability criterion might be derived. Thus it is open to question whether any region of negative potential vorticity, be it two- or three-dimensional, is necessarily unstable to three-dimensional disturbances.

Since it is difficult to tackle the three-dimensional problem other than numerically, most of the work described here uses a three-dimensional nonlinear hydrostatic numerical model, which is described in section 2. The aim of this work is to investigate the basic dynamics of three-dimensional motion in a region of negative potential vorticity. Hence consideration of processes such as convective instability and the viscous properties of atmospheric turbulence is left for future work, and the model contains only dry dynamics and a second-order diffusion scheme.

All the simulations described here use as their basis a two-dimensional flow with a constant horizontal temperature gradient balanced by constant vertical shear. For large positive potential vorticity this is the Eady basic state. When the potential vorticity is negative, this state is symmetrically unstable. For clarity of exposition the direction along the basic-state isentropes will be referred to as the along-front direction and the perpendicular direction as across-front; although strictly speaking we are considering a broad baroclinic zone and not a front.

One of the observed properties of frontal rainbands is their finite length. It is conceivable that such finite-length rainbands might occur in a two-dimensional symmetrically unstable flow if the instability is only released locally. In the first part of this paper, to explore this possibility we consider the response of a two-dimensional flow with negative potential vorticity to perturbations which have a localized and therefore three-dimensional structure. This represents the realistic possibility that the initiation of CSI in the atmosphere is likely to be localized via, for example, synoptic forcing or humidity anomalies, which are locally enhanced. These simulations are described in section 3. A feature common to all the simulations is the development of roll circulations about a horizontal axis, which exhibit banded structure in horizontal cross-section. These bands elongate in the along-front direction with time and have a tendency to tilt in the horizontal with an angle of several degrees to the thermal-wind direction.

Throughout this paper the term 'tilt' is used when describing a horizontal orientation. A given banded structure is said to be tilted when it is oriented at an angle to the thermal wind. This tilt is referred to as cyclonic if the band is rotated in an anticlockwise direction relative to the thermal wind (when viewed from above). Similarly the tilt is described as anticyclonic for a clockwise rotation.

To explain the tilted nature of the simulations, we explore in section 4 the dynamics of infinitely long bands, which may be tilted in the horizontal relative to the front. In the linear regime both inviscid and viscous modes are considered, which have an arbitrary tilt, $\alpha$, relative to the mean-flow direction. The growth rates and structures of these tilted bands can be found from a normal-mode analysis. This was first considered for inviscid flow by Stone (1966, 1970, 1972) and Tokioka (1970, 1971), and for a weakly unstable viscous flow by Miller and Antar (1986). In section 4 we enlarge on the results given by these authors. We then describe the nonlinear development of such tilted structures. This is compared with the nonlinear development of the symmetric mode, which has been described by Thorpe and Rotunno (1989), referred to here as TR. The effects of perturbing such a band along its length are also considered.
In the second part of this paper a second possible explanation for the finite length of rainbands is explored; namely, that the instability is itself confined in the along-front direction. As has already been mentioned, it is not at present known whether a region of negative potential vorticity which is confined in the along-front direction is symmetrically unstable. In considering atmospheric flow, we might realistically expect regions of negative potential vorticity to have a finite length, rather than to be infinitely long. Hence the question of whether such a finite region can support unstable motions is crucial to the understanding of three-dimensional 'symmetric' instability. This question is addressed in section 5. Confined regions of instability are obtained by using a stable basic flow, but prescribing regions where the static stability is reduced in the presence of vertical motion. This represents the effects of latent-heat release, and gives confined regions where the effective potential vorticity is negative.

2. The numerical model

The numerical model used to obtain the results described here is a three-dimensional nonlinear hydrostatic limited-area model on an f-plane. Cartesian coordinates, x and y, are used in the horizontal. The vertical coordinate is a pressure-based coordinate, described by Hoskins (1971), given by

\[ z = \frac{H_s}{\kappa} \left\{ 1 - \left( \frac{p}{p_o} \right)^\kappa \right\} \]  

(1)

where \( H_s = \frac{p_o}{\rho_o g} \) and \( \kappa = \frac{R}{c_p} \), \( R \) is the universal gas constant, \( c_p \) the specific heat at constant pressure and subscript zero denotes reference state variables. In this coordinate system the density, \( \rho \), is a function of \( z \) alone. The prognostic variables are the horizontal components of the wind field, \( u \) and \( v \), and the potential temperature, \( \theta \). The vertical velocity, \( w \), and the geopotential, \( \phi \), are diagnosed. The model equations, in flux form, are

\[ \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} (uu) - \frac{\partial}{\partial y} (uv) - \frac{1}{\rho} \frac{\partial}{\partial z} (\rho uw) + f v - \frac{\partial \phi}{\partial x} + F_u \]  

(2)

\[ \frac{\partial v}{\partial t} = -\frac{\partial}{\partial x} (uv) - \frac{\partial}{\partial y} (vv) - \frac{1}{\rho} \frac{\partial}{\partial z} (\rho vw) - f u - \frac{\partial \phi}{\partial y} + F_v \]  

(3)

\[ \frac{\partial \theta}{\partial t} = -\frac{\partial}{\partial x} (u\theta) - \frac{\partial}{\partial y} (v \theta) - \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w \theta) + F_{\theta_0} + Q_{\theta_0} \]  

(4)

\[ \begin{array}{c}
\frac{\partial \phi}{\partial z} = g \theta \\
\end{array} \]  

(5)

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w) = 0 \]  

(6)
where $F_u$, $F_v$ and $F_w$ are the diffusion terms and $Q_{th}$ allows for heating to be added to the thermodynamic equation.

For all of the simulations described here the Boussinesq approximation has been made, where $\rho(z) = \rho_o$. The parameters used in the model are $g = 9.81 \, m \, s^{-2}$, $f = 1 \times 10^{-4} \, s^{-1}$, $\theta_o = 300 \, K$, $\rho_o = 1.16 \, kg \, m^{-3}$, and $p_o = 1 \times 10^5 \, N \, m^{-2}$. The model has a flat lower boundary and a rigid lid at $z = H$, giving the boundary conditions $w = 0$ at $z = 0$ and $z = H$. The vertical velocity is specified to be zero at $z = 0$ and the continuity equation integrated upward. The vertical velocity at $z = H$ is thus not constrained to be zero. If the model solution is well behaved, $w$ at $z = H$ should not become significantly larger than the model truncation error. This is monitored during any model run as a check on the behaviour of the model. An equation for the surface geopotential, $\phi_s$, is obtained as described by Dalu (1978):

$$\nabla^2 \phi_s \int_0^H \rho \, dz = \int_0^H \rho \left\{ \frac{\partial A_u}{\partial x} + \frac{\partial A_v}{\partial y} + f \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial F_u}{\partial x} + \frac{\partial F_v}{\partial y} - \nabla^2 \int_{z=0}^Z g \frac{\theta}{\theta_o} \, dh \right\} \, dz \tag{7}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$}

The lateral boundary conditions used are that the perturbations are periodic in both horizontal directions. Hence Eq. (7) can be solved using discrete Fourier transforms.

The equations described above are represented by finite differences on the Arakawa 'C'-grid (Arakawa and Lamb 1977). The variables are also staggered in the vertical, with $w$ stored at an intermediate level to all the other fields. The first time-step of the model is forward; subsequently the model uses a leapfrog scheme. This scheme has a computational mode associated with it, the effects of which can be reduced by using the time smoother proposed by Robert (1966) and described by Asselin (1972). All the simulations described here use a coefficient of 0.01 for the time smoother.

The diffusion terms are given by

$$F_x = \nu_x \frac{\partial^2 \chi}{\partial x^2} + \nu_y \frac{\partial^2 \chi}{\partial y^2} + \nu_z \frac{\partial^2 \chi}{\partial z^2} \tag{8}$$

where $\chi$ is $u$, $v$ or $\theta$ and $\nu_x$, $\nu_y$, and $\nu_z$ are constants. The use of different diffusion coefficients in each of the $x$, $y$ and $z$ directions has little physical justification, but it is necessary for numerical reasons. The vertical diffusion coefficient, $\nu_z$, is crucial for controlling the development of the instability; if it is too large symmetric circulations will not grow. The difference between the horizontal and vertical grid lengths necessitates that the horizontal diffusion coefficients be larger than the vertical ones, so as to control small-scale noise in the model. All the work described below has used the same diffusion coefficients for the mixing of heat and momentum i.e. a Prandtl number equal to unity. Inclusion of these diffusion terms requires extra boundary conditions at the top and bottom boundaries. The effects of different boundary conditions have been discussed by Emanuel (1979, 1985). The work described below uses stress-free boundary conditions for perturbation quantities, i.e. $\partial \chi / \partial z = 0$ at $z = 0$, $z = H$ where $\chi$ is $u$, $v$, $\theta$. An important feature of the diffusion boundary conditions is their effect on the domain-averaged potential vorticity. This has been discussed by TR.
As described below, a linear version of the model has been used to obtain normal modes. This version of the model simply replaces the advection terms by their linear counterparts.

Various tests of the model have been performed during its development. Calculation of the energy budget for a symmetrically unstable flow showed the discrepancy between the source terms and the rate of change of eddy kinetic energy to be less than 5% of the rate of change of eddy kinetic energy. The model has been used to obtain linear normal modes. It reproduces the Eady mode of baroclinic instability, in terms of both growth rate and structure. The nonlinear behaviour of the model has been tested by performing nonlinear simulations of two-dimensional symmetric instability, and comparing them with those described by TR.

3. THREE-DIMENSIONAL STRUCTURES IN A TWO-DIMENSIONAL FLOW

If finite-length frontal rainbands are a manifestation of symmetric instability, the unstable circulations must have a finite length. This might occur in a two-dimensional symmetrically unstable flow if the instability is only released locally. Alternatively, the instability itself may be localized in the along-front direction. The first of these possibilities is considered in this section, the second in section 5.

The localized release of instability is studied by introducing a three-dimensional perturbation into a two-dimensional symmetrically unstable flow. Since, at the present time, symmetric instability theory is applicable only to two-dimensional flow, there is no guarantee that such a perturbation will release the instability. The form of the initial flow is given by Eq. 9, with $a = 0$ and $V_o = -\frac{1}{4}V_e H$. Hence the basic-state flow is zero at the mid level of the layer. The values of $V_e$ and $N^2$ are $5 \times 10^{-3}$ s$^{-1}$ and $1 \times 10^{-5}$ s$^{-2}$ respectively, giving a Richardson number of 0.4 and a PV of $-0.04$ pv units ($10^{-6}$ kg$^{-1}$ m$^2$ s$^{-1}$ K). The effects of three different initial perturbations are described here. The first is a streamfunction perturbation with a constant wavelength in the across-front direction. The second has the same wavellike structure but is a thermal perturbation. The third is a completely localized thermal perturbation. Parameters used in the various simulations are detailed in Table 1.

The form of the first perturbation is given by

$$\psi' = \delta(y) \psi_0 \sin \left(\frac{\pi z}{H}\right) \text{Re}[\exp(i(kx + \mu z))],$$

where the streamfunction, $\psi'$, is defined by $u' = \partial \psi'/\partial z$. This has the form of the fastest-growing symmetric normal mode in the $x - z$ plane, but its $y$-variation is described by $\delta(y)$. The value of $k$ was chosen so that the horizontal wavelength is 132 km (half the domain length in the $x$-direction). For the value of the vertical diffusion coefficient, $v_z$, used here (20 m$^2$ s$^{-1}$), this is close to the wavelength of maximum growth for the viscous symmetric normal mode, as will be discussed in section 4. The amplitude $\psi_0$ was chosen so that the maximum amplitude of $u'$ was 0.5 m s$^{-1}$. The $y$-variation of this perturbation is specified by the function $\delta(y)$. In this case $\delta(y) = 1$ from $y = 410$ km to $y = 550$ km and vanishes elsewhere. Some concern might be felt over the sharp gradients this involves at each end of the perturbation. To investigate this effect an identical experiment has been performed where the amplitude was modulated by a sine wave in the $y$-direction, thus reducing the sharp cut-off at the edge of the perturbation. This did not significantly affect the features described below.

The initial perturbation is in the $u$ and $w$ components of the wind field, and in horizontal cross-section is aligned parallel to the basic-state isotherms at all levels. As
TABLE 1.

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<th>Figures</th>
<th>Domain length in x-direction (km)</th>
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the circulations develop, this horizontal alignment alters. During the first five hours the geopotential, potential temperature and along-front wind perturbations all develop with an orientation rotated cyclonically from the thermal-wind direction. The largest horizontal tilt is seen in the geopotential perturbation and the smallest in the potential temperature. These tilts vary with height. The along-front wind has a tilt of 10 degrees at z = ½H, decreasing to 7 degrees at the horizontal boundaries. The across-front wind has its largest tilt of 15 degrees at the horizontal boundaries, but tilts by only 2 degrees at z = ½H (Fig. 1).

A consequence of these varying horizontal tilts is that the correlation between the perturbation $v$ and $w$ is small along the length of the circulations. As this is the main contributor to the rate of change of eddy-kinetic energy, it leads to slow growth in the early stages of development. As the circulations develop further they begin to rotate in an anticyclonic manner. The difference between the tilts observed in the various perturbation quantities becomes progressively smaller. Hence the eddy correlations increase in magnitude and in spatial extent (Fig. 2). This in turn leads to faster growth.

After 25 hours the eddies consist of tilted bands which fill the domain. The maximum eddy amplitude occurs in the region in which the initial perturbation was applied, as seen in the vertical velocity shown in Fig. 3(a). The horizontal structure of the horizontal components of the wind field is very similar to that of the vertical velocity. Despite the three-dimensional nature of the circulations, their development strongly resembles that of two-dimensional symmetric circulations described by TR. In the region of maximum amplitude the circulations are distinctly nonlinear, with strong anticlockwise circulations (Fig. 3(b)). Away from this region clockwise and anticlockwise circulations still coexist (Fig. 3(c)).
Figure 1. x–y cross-sections 5 hours after the application of the streamfunction perturbation. Only a limited portion of the domain is shown in the y-direction. (a) $u'$ at $z = 2.52$ km. Contour interval is 0.1 m s$^{-1}$; (b) $u'$ at $z = 2.52$ km. Contour interval is 0.13 m s$^{-1}$; (c) $u'$ at $z = 120$ m. Contour interval is 0.13 m s$^{-1}$; (d) $u'$ at $z = 120$ m. Contour interval is 0.11 m s$^{-1}$.

An interesting consequence of the differing stages of nonlinear development along the bands can be seen in the modification of the initial flow. As in two-dimensional symmetric instability, the circulations result in enhanced horizontal temperature gradients and distortion of the along-front wind. For the three-dimensional case these effects vary in the along-front direction. This is of interest in the light of observations from FRONTS 87 (Thorpe and Clough 1991), where distortion of absolute momentum ($m = V + fx$) surfaces was observed in varying degrees in the along-front direction.
The second localized perturbation has the form of the potential temperature perturbation of the two-dimensional normal mode in the x–z plane. This consists of alternating warm and cold anomalies with maximum amplitude at the horizontal boundaries. These maxima and minima occur where the symmetric circulations cross the basic-state isentropes, and so the potential temperature perturbation tilts in the vertical. It is applied to the same localized area as the previous perturbation, with the same wavelength. The amplitude used was that which would be associated with the first perturbation.

The growth of circulations from this perturbation follows the pattern described above. In the early stages cyclonic tilts developed and the growth is slow. The circulations then rotate in a clockwise direction (when viewed from above) and grow more rapidly. The initial cyclonic tilts are larger with this perturbation, and have larger variations with height; hence the unstable growth is weaker than in the previous case. There is one marked difference between the two simulations: in the case of the thermal perturbation two regions of maximum amplitude develop at each end of the initial perturbation (Fig. 4). In the y–z plane the circulations consist of localized pools of either warm or cold air.
Figure 3. Cross-sections 25 hours after the application of the streamfunction perturbation: (a) $x$–$y$ cross-section of the vertical velocity at $z = 2.52$ km. Contour interval is $3\,\text{cm}\,\text{s}^{-1}$; (b) $x$–$z$ cross-section of $(u', w')$ at $y = 145$ km; (c) as (b) but $y = 485$ km.

Hence circulations develop in the $y$–$z$ plane, with strong ascent or descent at each end of the thermal perturbation. These circulations have larger amplitude in the early stages of development than the circulations in the $x$–$z$ plane, giving rise to the two maxima seen in Fig 4.

The third perturbation used is localized in the $x$–$z$ plane as well as in the $x$–$y$ plane. It mimics the effects of moving a parcel of air along an absolute-momentum surface by applying a thermal perturbation to a confined volume of fluid. The geometry of the perturbation is shown in Fig. 5. The perturbation was 375 km long and was applied at the centre of the model domain; the amplitude of the perturbation was 0.07 K. Such a small-amplitude was used in order that the initial conditions would be statically stable.

The initial growth from this perturbation consists of a clockwise and anticyclonic tilt, which is oriented in the along-front direction with a length similar to that of the initial perturbation. In contrast to the previous two simulations no initial cyclonic tilts are observed. As the simulation proceeds the eddies elongate in the along-front direction and rotate anticyclonically (Fig. 6). Further circulations develop in the $x$–$z$ plane adjacent to the original pair, but the initial anticyclonic circulation retains the
Figure 4. $x$-$y$ cross-sections at $z = 2.52$ km of $\omega'w'$ for the potential temperature perturbation: (a) $t = 5$ hours. Contour interval is $1 \times 10^{-3}$ m$^2$s$^{-2}$; (b) $t = 15$ hours. Contour interval is $2 \times 10^{-3}$ m$^2$s$^{-2}$; (c) $t = 25$ hours. Contour interval is $2.5 \times 10^{-3}$ m$^2$s$^{-2}$.

Figure 5. The geometry of the parcel perturbation in the $x$-$z$ cross-section. A thermal perturbation of 0.07 K was applied to the shaded region between $y = 1850$ km and $y = 2225$ km.
Figure 6. $x$-$y$ cross-sections 24 hours after the application of the parcel perturbation. The entire domain is not shown in the $y$-direction. (a) $u'$ at $z = 2.52$ km. Contour interval is 0.5 m s$^{-1}$; (b) $v'$ at $z = 2.52$ km. Contour interval is 0.6 m s$^{-1}$; (c) $w$ at $z = 3.96$ km. Contour interval is 1.5 cm s$^{-1}$.

largest amplitude (Fig. 7(a)). This gives rise to significant local perturbation of the along-front wind (Fig. 7(b)). Such a circulation could give rise to the perturbation of the absolute-momentum surfaces observed in FRONTS 87 (Thorpe and Clough 1990), as these were localized, rather than having a wavelike structure.
Figure 7. x–z cross-sections 24 hours after the application of the parcel perturbation: (a) \((u', w')\) at \(y = 2062.5\) km; (b) the along-front wind, \(v\) at \(y = 2075.0\) km. Contour interval is 2.5 m s\(^{-1}\).

The effects of varying the length of the initial perturbation were investigated. Perturbations were used varying from those of length 40 km to one of two-dimensions. All the perturbations resulted in growing circulations, but the initial growth rate of the shorter perturbations was very small. The 40 km-long perturbation had negligible growth in 24 hours, although growing circulations did appear on a longer timescale.

4. TILTED STRUCTURES IN A SYMMETRICALLY UNSTABLE FLOW

The predominant feature of the simulations described in the previous section is the tendency to produce tilted structures. In an attempt to explain this characteristic, we explore in this section the dynamics of two-dimensional tilted structures in both the linear and nonlinear regimes.

Linear normal-mode analyses have long been used in the study of unstable flows. It is now widely recognized that the fastest-growing normal mode alone will not necessarily account for all the structures observed in such unstable flows. When some arbitrary perturbation is applied to an unstable flow, it may be possible to represent it as a sum of the growing modes. In this case the subsequent growth of the perturbation would depend on the structures and growth rates of all the normal modes present, not just on
those of the fastest-growing mode. It is more likely, however, that neutral modes or non-modal structures will also be present in such an initial perturbation. This can lead to transient growth in excess of the growth rate of the fastest-growing mode. (Farrell 1984). However, when considering the development of such flow over a period several times the e-folding time of the normal mode, the structures which develop have much in common with those of the normal mode. In addition, study of the normal modes shows the ways in which energy can be extracted from an unstable flow. To obtain this information it is not sufficient to consider only the growth rates of the normal modes. Rather, the structures and energetics of all modes with significant growth rates should be studied.

The normal modes that are relevant to the three-dimensional problem are two-dimensional tilted bands whose phase lines may be oriented at an angle to the basic-state flow. (The classical symmetric instability problem considers only those perturbations with zero tilt. If the perturbations are tilted by 90 degrees to the front they have the configuration associated with the Eady mode of baroclinic instability.) Note that two-dimensional perturbations with a constant tilt, which is neither zero nor 90 degrees, can exist only in the absence of horizontal shear in the basic flow. In the presence of horizontal shear the bands will either rotate in the horizontal with time or else develop structure along their length. (This can be demonstrated mathematically by considering the rate of change of the gradient of perturbation quantities in the along-band direction.) Although the tilted modes are themselves two-dimensional they can be combined to form fully three-dimensional structures. In section 4(a) the linear normal modes are discussed, in section 4(b) the viscous problem is considered, and in section 4(c) the nonlinear development of the tilted structures is described.

(a) The inviscid normal modes

The tilted normal modes of an inviscid flow have been described by Stone (1966, 1970, 1972), who showed that modes that are tilted in the horizontal relative to the basic flow are unstable, but that the highest growth rates are associated with the symmetric mode. The growth rates of the ‘tilted’ modes depend only on the amplitude of the tilt, and not on the direction. The structure of the modes does, however, depend on the direction of the tilt relative to the front (Tokioka 1970, 1971).

In order to consider the relevance of the tilted modes to three-dimensional flow we present here the results of an eigenvalue analysis. The growth rates and phase speeds obtained from this analysis are analogous to those discussed by Stone (1970). They are presented, however, in a different manner, which we feel helps to clarify the tilted nature of these modes. We then discuss the detailed structure and energetics of the various modes. These will later be related to those of the viscous normal modes.

The eigenvalue problem is formulated using a coordinate rotation as described by Nehrkorn (1986), which consists of defining the horizontal coordinate such that the y-axis lies parallel to the phase lines of the normal mode. Hence the perturbations have no variations in the y-direction and are fully described by an x–z cross-section. The basic state now varies in both the x and y directions and is given by

\[
\vec{u}(z) = (V_0 + V_z z) \sin \alpha \quad \vec{v}(z) = (V_0 + V_z z) \cos \alpha \quad \vec{w} = 0
\]

\[
\vec{\theta}(x, y, z) = \theta_o + \frac{\theta_o}{g} fV_z x \cos \alpha - \frac{\theta_o}{g} fV_z y \sin \alpha + \frac{\theta_o}{g} N^2 z, \tag{9}
\]

where \( \alpha \) is the angle between the y-axis and the direction of the basic-state shear.
From the above it can be seen that there is an asymmetry between modes with positive tilts and those with negative tilts. For positive $\alpha$ the basic-state temperature gradient in the $y$-direction is negative, whilst for $\alpha < 0$ it is positive. Because of the balanced nature of the basic flow, this results in a vertical shear across the mode, of opposite sign for the two different angles (Fig. 8).

Consideration will be given to the energetics of the various modes obtained from the stability analysis. The eddy-kinetic-energy equation used is

$$\frac{d}{dt} \left( \frac{\mathbf{\bar{u}}^2 + \mathbf{\bar{v}}^2}{2} \right) = \frac{\mathbf{\bar{u}} \cdot \mathbf{w}' \mathbf{V}_z \sin \alpha - \mathbf{\bar{v}} \cdot \mathbf{w}' \mathbf{V}_z \cos \alpha}{E_{ku}} + \frac{g}{E_{ku}} \cdot \frac{\mathbf{w}' \theta'}{E_{ke}} \cdot \frac{\mathbf{\bar{u}} \cdot \mathbf{\bar{v}}}{E_p}$$

where eddy quantities are defined as deviations from the stationary initial state and are denoted by a prime. The overbar denotes an average across one wavelength in the horizontal from $z = 0$ to $z = H$. The conversion from basic-state kinetic energy to eddy kinetic energy has a component, $E_{ku}$, due to the across-band shear, and a component, $E_{ke}$, due to the along-band shear. $E_p$ is the conversion from basic-state potential energy to eddy kinetic energy.

To obtain the normal modes, Eqs. (2)-(6) are linearized about this basic state. (Since we are considering inviscid adiabatic flow, $F_u = F_v = F_{th} = Q_{th} = 0$.) The equations are non-dimensionalized using

$$(x, y) = \frac{NH}{f} (\bar{x}, \bar{y}) \quad z = H \bar{z} \quad t = \frac{N}{fV_z} \bar{t} \quad (u, v) = V_z H (\bar{u}, \bar{v})$$

$$w = \frac{fV_z H}{N} \bar{w} \quad \theta = \frac{\theta_o}{g} NH \bar{z}$$

and $\phi = NH^2 V_z \bar{\phi}$, where tildes denote non-dimensional quantities. In contrast to the previous section $V_o$ is taken to be zero. Hence the basic flow is zero at the ground rather than at the mid level of the layer. Perturbations of the form $\chi' = \chi(z) \exp(i(kz - \sigma t))$ are substituted into these equations, which are then combined to give the eigenvalue equation (with tildes omitted):

$$[(\sigma - k z \sin \alpha)^2 - Ri] \psi_{zz} + 2k \left\{ -\frac{Ri \sin \alpha}{(\sigma - k z \sin \alpha)} + i \frac{Ri \cos \alpha}{Ri + \frac{2i}{(\sigma - k z \sin \alpha)}} \right\} \psi_z +$$

$$+ k^2 \left\{ Ri + \frac{2i}{(\sigma - k z \sin \alpha)} \right\} \psi = 0. \quad (10)$$

This cubic eigenvalue equation can be represented by finite differences in the vertical and the complex eigenvalues and eigenfunctions obtained using a standard eigensolver package.

The growth rates and phase speeds as a function of wavelength for a variety of tilts are shown in Fig. 9 for a Richardson number of 0.4. A striking feature of these graphs is the contrast between the growth-rate curve of the symmetric mode and that of all the other modes. The symmetric mode has maximum growth rate at zero horizontal wavelength. All the tilted modes have a growth-rate maximum at finite wavelength. This maximum becomes more pronounced as the angle between the phase lines of the perturbation and the basic-state flow increases. For all the tilted modes the maximum-growing mode is the gravest mode in the vertical, which propagates with the speed of the mid-level flow, resolved perpendicular to its phase lines. At shorter wavelengths there are two modes with equal growth rates. One of these has its maximum amplitude
near the lower horizontal boundary, the other exhibits maximum amplitude near the upper boundary. As the wavelength decreases, the maximum amplitude of the disturbance becomes more confined to the boundary. This is reflected in the phase-speed curves. As the wavelength of the modes decreases, the phase speed of the modes approaches the speed of the basic flow at the boundary. The extent to which the modes at high wavenumber are confined to the boundaries also varies with the magnitude of the tilt. For small values of $\alpha$, although the position of maximum amplitude is shifted away from $z = \frac{1}{4}H$, these modes almost fill the domain. As $\alpha$ increases, the confinement to the boundaries is much more noticeable.

When the Richardson number is increased (i.e. the flow becomes more stable) the broad features outlined above are still observed. The range of wavenumbers over which
the symmetric mode has the highest growth rate decreases as \( Ri \) increases, since the long wavelength cut-off in the growth-rate curve for \( \alpha = 0 \) moves to higher wavenumber. At \( Ri = 0.95 \) the baroclinic mode and symmetric mode have equal growth rates (this is in agreement with Stone’s results).

By using this coordinate system to display the growth rate and phase-speed curves we can see the relationship between the tilted modes and the baroclinic and symmetric modes. In fact we could regard both the baroclinic and symmetric modes as special cases of the tilted modes. Because the baroclinic mode is oriented at right angles to the thermal wind it cannot extract energy from the basic flow in a ‘symmetric’ manner. Similarly, the symmetric mode cannot take advantage of the fact that the basic flow is baroclinically unstable, since it is constrained to be parallel to the basic flow. The tilted modes, however, display features of both baroclinic and symmetric instability. The proportional influence of the two instabilities depends on the magnitude of the tilt. For small \( \alpha \) the tilted modes are distinctly symmetric in character. As \( \alpha \) increases, the influence of baroclinic instability is seen in the more pronounced maximum in the growth-rate curve and the boundary confinement of the high-wavenumber modes. The influence of symmetric instability is still present, however, as the boundary modes, which are neutral in the Eady problem, retain significant growth rates.

The growth-rate curves presented here do not at first sight resemble Stone’s (1970). This is a result of the coordinate rotation used. The features of the growth-rate curves described above can also be deduced from the results of Stone by plotting the growth rates as a function of the total wavenumber whilst keeping \( \lambda/k \) constant.

It is of interest to consider the behaviour of the baroclinic mode (\( \alpha = 90^\circ \)) when the potential vorticity is negative. The effects of non-geostrophy on the Eady problem have been reported by Nakamura (1988) and Garner (1991), for positive but small potential vorticity. As the potential vorticity is decreased and becomes negative, these effects become more pronounced. It is interesting to note that there is no fundamental difference between the modes with positive PV and those with negative PV. This might seem surprising, since the Eady mode is usually understood in terms of some concept of balance and invertibility (Hoskins et al. 1985). The insensitivity of the baroclinic mode to the transition between positive and negative PV suggests that the strong distinction which is often made between flows with positive and negative potential vorticity is an artefact of the use of balanced equations, rather than a physical property of atmospheric flow.

The most intriguing aspect of the inviscid eigenvalue problem is the behaviour of modes with the same magnitude but opposite sign of \( \alpha \). The growth-rate curves shown in Fig. 9 are all for \( \alpha > 0 \), but could equally well be for \( \alpha < 0 \). The growth rates of modes with opposite sign of \( \alpha \) are almost identical, differing by less than 1%; although this is not a formal symmetry of the eigenvalue equation (10). There are, however, significant differences between the structures of two such modes, as was first noted by Tokioka (1970). This is illustrated in Fig. 10 for modes with \( \alpha = +5^\circ \) and \( \alpha = -5^\circ \). The slope of the circulations in the vertical, and the location of the maximum amplitude show substantial differences. For \( \alpha = +5^\circ \) the circulations are more vertical than the basic-state isentropes, and all of the perturbation quantities have significant amplitude at the horizontal boundaries. The dominant energy source in this case is the basic-state kinetic energy. For \( \alpha = -5^\circ \) the circulations are more horizontal than the basic-state isentropes and all the fields have their maximum amplitude in the centre of the domain. The dominant energy source in this case is the buoyancy term, although the shear term is still significant.

These structures can be understood by considering the difference in the basic flow for the two cases. For \( \alpha > 0 \) the basic-state shear in the \( x-z \) plane is positive, and the
Figure 9. (a) Growth rates (s\(^{-1}\)) and (b) phase speeds in m s\(^{-1}\) of the inviscid normal modes plotted as a function of the horizontal wavenumber for various angles and Ri = 0.4. For \(\alpha = 0\) only the fastest-growing mode is shown and the phase speed is zero. For all other values of \(\alpha\) the two fastest-growing modes are shown. The dimensionless quantities shown were obtained using \(N^2 = 1 \times 10^{-3} \text{s}^{-2}\), \(V_i = 5 \times 10^{-3} \text{s}^{-1}\), \(H = 5 \text{ km}\) and \(f = 1 \times 10^{-4} \text{s}^{-1}\). If the second growing mode for \(\alpha = 0\), and the third and fourth growing mode for other values of \(\alpha\) had been plotted the graph would look the same but with lower growth rates; the modes shown are the gravest modes in the vertical, the next highest growing modes have multiple vertical structure.
Figure 10. The structures of two of the inviscid normal modes for a Richardson number of 0.4: (a) $\psi'$ for $\alpha = -5^\circ$. Contour interval is 600 m$^2$s$^{-1}$; (b) $\theta'$ for $\alpha = -5^\circ$. Contour interval is 0.1 K; (c) $\phi'$ for $\alpha = -5^\circ$. Contour interval is 1.3 m$^2$s$^{-1}$; (d) $\psi'$ for $\alpha = +5^\circ$. Contour interval is 500 m$^2$s$^{-1}$; (e) $\theta'$ for $\alpha = +5^\circ$. Contour interval is 0.13 K; (f) $\phi'$ for $\alpha = +5^\circ$. Contour interval is 3 m$^2$s$^{-1}$. 
corresponding along-front temperature gradient is negative. The positive shear tends to rotate the circulations in a clockwise direction about the y-axis, giving them a more vertical orientation. This increases the contribution to the energetics of the shear term and decreases that of the buoyancy term. Since the modes extract energy from the basic-state kinetic energy there is a negative correlation between $v'$ and $w'$. This, combined with the negative along-front temperature gradient, leads to motion in the y–z plane directed across the basic-state isentropes. This further reduces the contribution of the buoyancy term to the growth of the mode, especially in the centre of the domain. In contrast, for $\alpha < 0$ the basic-state shear is negative, so the circulations are rotated in an anticlockwise direction about the y-axis. This increases the contribution of the buoyancy term and decreases that of the shear term. For this case a negative $v'w'$ correlation can result in positive $w'\theta'$ owing to the motions in the y–z plane. Hence for $\alpha < 0$ these two sources of eddy kinetic energy are in competition, whilst for $\alpha < 0$ they can cooperate to promote growth.

The asymmetry of the modes about $\alpha = 0$ is further emphasized by the energetics plotted as a function of $\alpha$ in Fig. 11. All three contributions to the rate of change of kinetic energy show this asymmetry. The asymmetry in the $E_{ku}$-term is due to the tilt of the symmetric circulations, which results in $u'w'$ being negative except for large negative $\alpha$. Since $E_{ku}$ is proportional to $\sin \alpha$ it therefore changes sign with $\alpha$. Around $\alpha = 0$, the inertial contribution to the energetics, $E_{ku}$, is large, whilst the buoyancy term is small. This shows that the modes with small $\alpha$ are strongly related to the symmetric mode. The inertial contribution is large when the ascent is more vertical than the absolute momentum surfaces, in which case it will be close to or more vertical than the $\theta$-surfaces, leading to small or negative $E_{p}$. Hence the maximum in $E_{ku}$ coincides with the minimum in $E_{p}$. This feature is also observed for higher values of the Richardson number. For $Ri = 0.8$ the

![Figure 11](image.png)

Figure 11. The source terms in the eddy kinetic energy equation for the fastest-growing inviscid mode, plotted as a function of $\alpha$. The amplitudes are normalized so that all the modes have the same (arbitrary) domain-averaged kinetic energy.
maximum in $E_k$ still occurs at positive $\alpha$. For $Ri = 0.9$, however, it occurs at negative $\alpha$. As the horizontal tilt increases the buoyancy term becomes the dominant energetic source. This may be attributed to baroclinic instability having a greater influence on the modes. As mentioned above, the buoyancy and inertial terms are in competition for $\alpha > 0$. Hence for $\alpha < 0$ the inertial contribution to the energetics is significant over a wider range of angles than for $\alpha > 0$.

This study of the inviscid normal modes shows us that there are a variety of ways in which tilted structures can extract energy from a symmetrically unstable flow. There exists a marked asymmetry about the symmetric axis (in fact if a mode with a given vertical structure is growing for $\alpha < 0$ it will decay when $\alpha > 0$). This asymmetry is a fundamental property of the basic flow. It is a rather surprising fact that, despite this asymmetry, the total rate of change of eddy kinetic energy (and hence the growth rate) is symmetric about $\alpha = 0$.

(b) The viscous normal modes

The importance of viscous effects in symmetric instability has been demonstrated by McIntyre (1970) and Emanuel (1979). McIntyre showed that a symmetrically stable flow could be destabilized by the unequal diffusion of heat and momentum. Busse and Chen (1981) studied this double diffusive instability for modes tilted at small angles to the direction of the basic flow. They found that a mode at a small angle to the symmetric mode had a higher growth rate than the modes discussed by McIntyre.

The tilted modes were studied for a weakly unstable viscous flow by Miller and Antar (1986), referred to here as MA. For a Prandtl number (the ratio of the coefficients of viscous and thermal diffusion) of unity they consider flows with $Ri = 0.92$ and $Ri = 0.8$ (In the first case the symmetric and baroclinic modes have equal growth rates for the values of viscosity used, in the second the symmetric mode has the highest growth rate.) For $Ri = 0.92$ they report a minimum (negative) growth rate at small positive tilt. The growth-rate curve, however, still exhibits strong symmetry about $\alpha = 0$. The energetics for this case display strong asymmetries similar to those seen in the inviscid problem. For $Ri = 0.8$ the maximum growth rate occurred with a small negative tilt: the tilt at which the maximum growth varied with the Prandtl number.

From the above we might suppose that the preference for negative tilts, observed in the nonlinear simulations, is due to the gradual emergence of the fastest-growing normal mode as the dominant structure. The nonlinear results were obtained, however, for values of Richardson number and viscosity different from those used by MA. To ascertain whether the parameters used in section 3 also result in a tilted mode having the highest growth rate in the linear regime, we describe now the viscous normal modes for the initial state used in section 3. The viscous tilted modes are obtained from the linearized version of the numerical model referred to in section 2, using the methodology described by Simmons and Hoskins (1976). The growth rates of the fastest-growing modes as a function of wavenumber are shown in Fig. 12 for various values of $\alpha$. From these curves it can be seen that when diffusion is added to the problem the symmetric mode no longer has the highest growth rate. Instead the maximum growth, for $v = 20 \text{ m}^2\text{s}^{-1}$, occurs for a mode at around $-6$ degrees to the basic-state isotherms. When the magnitude of this tilt exceeds 6 degrees the growth rates begin to decrease again. The same happens as the mode is tilted back through $\alpha = 0$ towards the cold air. The lowest growth rates occur for small positive values of $\alpha$ (around 3 degrees), then increase again as $\alpha$ increases further.

All the curves shown in Fig. 12 reveal that the addition of diffusion to the problem gives a non-zero wavelength of maximum growth to the symmetric mode and a more
distinct maximum to the modes with small values of $\alpha$. It is noteworthy that there exists a set of modes with angles between $-12$ and $-3$ degrees, with wavelengths ranging from $100 \text{ km}$ to $200 \text{ km}$, whose e-folding times differ by less than 30 minutes. This means that it is possible for a three-dimensional structure to exist for some time if an initial perturbation consists of a combination of these modes.

The variation of growth rate with diffusion coefficient is shown in Fig. 13. The diffusion has a most dramatic effect on the symmetric mode and modes close to it, almost halving the growth rate of the symmetric mode for $\nu_z = 20 \text{ m}^2\text{s}^{-1}$. As $|\alpha|$ increases the effect of the diffusion is diminished. As the diffusion coefficient is increased the maximum growth shifts to larger angles. For $\nu_z = 10 \text{ m}^2\text{s}^{-1}$ the maximum growth occurs for $\alpha = -3$, and a minimum has not been found in the range of angles studied. It is interesting to compare these curves with those shown by MA for a diffusion coefficient of $2.5 \text{ m}^2\text{s}^{-1}$ with $Ri = 0.92$ and varying Prandtl number. (Note that the growth rates and energy conversion rates given by MA are a factor of 4 too large, owing to an error in non-dimensionalizing time—T. L. Miller, personal communication.) For a Prandtl number greater than unity the maximum growth occurred for small positive tilt. When the Prandtl number was less than unity the maximum occurred for a negative tilt, whilst a minimum accompanied a small positive angle. This shows that the behaviour of the viscous normal modes depends on the Richardson number, the Prandtl number and the magnitude of the diffusion.

The energetics of the viscous modes for the two different diffusion coefficients are shown in Fig. 14. As in the inviscid case, the maximum growth rate occurs where $E_{k\nu}$ is large and $E_p$ small. In contrast to the inviscid case, this occurs for finite negative values of $\alpha$. For the range of angles shown, $E_{k\nu}$ is the dominant energy source, in contrast to
the inviscid case (Fig. 11), for which the buoyancy term is significant for negative values of $\alpha$. A comparison of Figs. 11 and 14 shows that, with the exception of the 'total' curve, the asymmetry about $\alpha = 0$ is as pronounced in the inviscid case as it is in the viscous cases. We conclude from this that the asymmetry in the structures of the modes is a property of the unstable flow. The asymmetry in the growth-rate curves in the viscous regime may then be attributed to diffusion affecting the normal-mode structures in a different manner.

The results of the nonlinear simulations described in section 3 can be understood in terms of the gradual emergence of the fastest-growing normal mode as the dominant feature. This occurred over a timescale of several times the $e$-folding time of the fastest-growing normal mode, which is 5.5 hours for $v_z = 20 \text{ m}^2\text{s}^{-1}$. The initial perturbations used may be represented as a sum of the tilted normal modes, both growing and neutral, and perhaps also non-modal structures. The early stages of the development of the perturbations are influenced by all of these components. In the long term, the fastest-growing normal modes have the greatest influence. Note that in order for a combination of normal modes to retain coherent structure over long timescales they must have similar propagation speeds. This is the case for all the fastest-growing tilted modes, which propagate with the mid-level speed of the basic flow.

(c) The nonlinear development of the tilted modes

The linear eigenvalue problem demonstrates the significance of the tilted modes in a symmetrically unstable flow. In order to consider the implications of these tilted modes to atmospheric flow, their nonlinear behaviour must also be understood. This was investigated using the numerical model described in section 2. The model was initialized
with the balanced flow described by Eq. (9), which was then perturbed with the fastest-growing viscous normal mode of that flow. The nonlinear development of two tilted modes, one with \( \alpha = -6^\circ \) and the other with \( \alpha = +3^\circ \), was studied and compared with that of the symmetric mode.

The nonlinear development of both tilted modes resembles that of the symmetric mode in many respects. In all cases, the anticlockwise cells grow at the expense of the clockwise cells. The eddy kinetic energy reaches a maximum amplitude when the
anticlockwise cells dominate the flow, then decreases as they decay in amplitude. The most noticeable feature of the nonlinear growth is the modification of the along-front wind by the eddies. As described by TR, this results in considerable distortion of the absolute momentum surfaces. This is seen in the symmetric case, and in the development of the tilted modes.

The preferential growth of the anticlockwise (AC) cells stems from the nonlinear modification of the initial flow by the unstable eddies. In the centre of the domain the ascent and descent lies within the wedge of instability (Thorpe et al. 1989). The confined nature of the domain means the motion cannot be entirely within this wedge, leading to motion across the basic-state $\theta$-surfaces at the boundaries. Such motion does not correspond to an unstable displacement, but is driven by the horizontal geopotential gradient. A measure of the stability to this almost horizontal motion is given by the vertical component of the absolute vorticity, $f + \partial u/\partial x$ (in a similar way to the stability to slantwise motion being given by the absolute vorticity evaluated on a $\theta$-surface). In the initial conditions this is positive and constant everywhere, so the stability is the same for the clockwise and anticlockwise cells. As the eddies develop and increase in magnitude, their distinctive $u'$-component modifies the flow so as to increase the stability to horizontal motion for the clockwise cells and decrease it for the anticlockwise cells. This can be seen in Fig. 15(a) where the arrows showing the motion in the $x$–$z$ plane are superimposed on the $u'$-field. The return path of the AC cell occurs where $\partial u'/\partial x$ is negative. The converse is true for the clockwise cells. As a result of this, the growth of the clockwise cell is much more strongly inhibited than that of the anticlockwise cell.

TR discussed how the potential vorticity of the flow was modified by the unstable eddies. This is illustrated in Fig. 15(b). When the AC cells have their maximum amplitude, the strongest ascent and descent occurs in a region where the stability to slantwise motion has been increased (although the flow is still unstable). The combination of this factor with the increasing amplitude of the diffusion terms will act to inhibit the growth of the AC cell. The decrease in stability to slantwise motion means that conditions are favourable for the reformation of the clockwise cell in the region between the decaying AC cells.

Although the growth and decay of the AC cells occur in a similar manner for $\alpha = -6^\circ$, $\alpha = 0^\circ$ and $\alpha = +3^\circ$, the timing differs between the three cases. This can be seen by comparing the domain-averaged eddy kinetic energy of the three simulations. The maximum eddy kinetic energy, for $\alpha = -6^\circ$, is reached after 12 hours, compared to 18 hours for $\alpha = 0^\circ$, and 19 hours for $\alpha = +3^\circ$. The maximum eddy kinetic energy attained is almost the same magnitude for $\alpha = 0^\circ$ and $\alpha = -6^\circ$. For $\alpha = +3^\circ$ the maximum eddy kinetic energy is approximately 0.56 of that for $\alpha = -6^\circ$. In all cases the potential vorticity in the region of maximum ascent and descent has increased by similar magnitudes when the AC cells have their maximum amplitude. This supports the hypothesis that the increase in the stability of the flow is one of the factors contributing to the subsequent decay of the AC circulations.

The contribution to the geostrophically balanced flow made by symmetric instability has not been discussed in previous studies. We might anticipate some adjustment to balance occurring during the release of symmetric instability, since the timescale over which the unstable circulations develop is comparable to that associated with the Coriolis parameter. This is in fact observed; a significant proportion of the buckling of the absolute momentum surfaces commented on earlier can also be seen in the geostrophic absolute momentum surfaces ($m_g = \nu_g + fx$, where $\nu_g$ is the geostrophic wind). The presence of this evidence of symmetric instability in the balanced flow may have implications regarding the search for symmetric instability in nature. The modification of the along-front wind by the unstable eddies is by far the most significant feature of the instability. Since it
remains after the unstable circulations have decayed, it is more likely to be observed than the circulations themselves.

Since the viscous symmetric mode does not exhibit the fastest growth in either the linear or nonlinear regime, three-dimensional effects must be considered in a flow with negative potential vorticity. One question regarding the relevance of the two-dimensional circulations is whether they are stable to the growth of secondary three-dimensional eddies. To investigate this matter, the fastest-growing circulation ($\alpha = -6^\circ$) was perturbed along its length in a variety of manners. The structure of the perturbation in the $y$-direction ranged from long-wavelength sine waves to random noise. The amplitude of the perturbation was limited to keep the flow convectively stable through the simulation.
This restriction was adopted since a hydrostatic model does not accurately represent the dynamics of upright convection. The perturbations were applied both to a linear circulation and to a nonlinear circulation. The application of these perturbations made no significant difference to the lifecycle of the unstable eddies; however, three-dimensional structure was clearly visible throughout this life-cycle. It is interesting to note that convective instability appeared to develop more readily in the presence of a three-dimensional structure. This experiment needs to be repeated with a non-hydrostatic model, to investigate the interaction between convective and symmetric instability.

5. Localized Regions of Instability

This section considers whether a region of negative potential vorticity that is confined in the along-front direction can support unstable motions. Prescribing a three-dimensional balanced initial state which contains regions of negative PV embedded in a flow with positive PV is not a trivial matter; this has been discussed by TR for two-dimensional flow. It is much more straightforward to construct a confined region of conditional symmetric instability than one of dry symmetric instability. This can be done by specifying a symmetrically stable initial flow which contains a region in which the static stability is reduced in the presence of any vertical motion. This reduction can result in the confined region being conditionally unstable to symmetric motions.

For saturated flow, the saturated equivalent potential temperature,

$$\theta_{es} = \theta \exp \left( \frac{L q_s}{c_p T} \right),$$

is conserved. Durran and Klemp (1982) have shown that the conservation of $\theta_{es}$ can be expressed as

$$\frac{D \gamma}{Dt} + wN_m^2 = 0 \quad (11)$$

where $\gamma$ is the buoyancy term in the vertical momentum equation. The moist static stability, $N_m^2$, is

$$N_m^2 \approx g \frac{\Gamma_m}{\Gamma_d} \frac{\partial}{\partial z} (\ln \theta_{es})$$

where $\Gamma_m$ and $\Gamma_d$ are the saturated and dry adiabatic lapse rates respectively. The expression in Eq. (11) can be approximated by taking $\gamma$ to be the buoyancy, $g(\hat{\theta}/\theta_e)$. The circumflex represents a departure from a reference atmosphere in which $\theta$ varies only with height. This gives

$$\frac{D \hat{\theta}}{Dt} + \frac{\theta_e}{g} w \bar{N}^2 = 0. \quad (12)$$

$\bar{N}^2$ is the dry static stability, $N^2$, for unsaturated flow and the moist static stability, $N_m^2$, for saturated flow.

A common approach is to assume that ascending air is saturated whilst descending air is not (‘moist up, dry down’). This is considered to be a good approximation for synoptic-scale motions. Clough and Franks (1991) have shown that this is not necessarily true of mesoscale motions embedded in a cloud canopy. This is due to the fact that the evaporation of snow and crystalline ice occurs on a shorter timescale and in a shallower
depth than the evaporation of rain. For rainfall rates of around 1–10 mm h⁻¹ the evaporation of snow is sufficient to maintain the atmosphere near saturation, despite descent rates of the order of 10–30 cm s⁻¹. The sloping nature of symmetric circulations will result in precipitation falling through the descending regions. The vertical velocities in all the simulations studied in this paper have been of the order of 10 cm s⁻¹. This implies that if a region of conditional symmetric instability occurs above the freezing level the effects of moisture should be represented by ‘moist-up, moist-down’.

In order to confine the region of moist negative potential vorticity, the static stability, $\bar{N}^2$, is defined as

$$\bar{N}^2 = N^2 - \frac{g}{\theta_o} q.$$  

The symmetric stability of the flow is determined by the magnitude of $q$. Where $q = 0$ the stability depends on the dry static stability and the flow is symmetrically stable. Where $q$ is non zero the flow may contain CSI. Since the model is formulated using the dry potential temperature, $wq$ appears as a heating term on the right-hand side of the thermodynamic equation. For the simulations described in this chapter the region of instability was confined only in the along-front direction, whilst the flow in the $x$–$z$ plane was unstable everywhere. The heating term used was specified in three parts. In the region $y_1 \leq y \leq y_2$ it was ascribed a constant value, $q_o$, whose magnitude resulted in the moist PV being negative there. Rather than having an abrupt boundary between the stable and unstable region, the heating was reduced exponentially at each end of the unstable region over a distance $\Delta$. Hence for $y_1 - \Delta \leq y \leq y_1$ and $y_2 \leq y \leq y_2 + \Delta$ the heating function was given by $q_o \exp(-|y - y_{1,2}|/y_o)$ so that the heating decreased by a factor of $e$ over a distance $y_o$. Everywhere else in the model domain the heating function was zero. Specifying the heating in this manner means that the region of instability is a function of the model coordinates and its position does not change with time. If the instability is due to a confined region of moisture this might be expected to move with the flow, which would lead to there being a steering level for the unstable region. Here the steering level is supposed to be at the mid level of the unstable region. Two important questions not addressed here are what determines this steering level and how the development of unstable motions is affected when the steering level changes.

To investigate the properties of such a confined region of CSI, the model was initialized with the basic state used earlier, but with a static stability of $1 \times 10^{-4}$ s⁻², giving a Richardson number of 4.0. The heating function was applied over a region of 330 km, such that the Richardson number in this region was 0.4. Owing to the periodic boundary conditions this represents a series of unstable patches in the along-front direction. The flow was perturbed with the parcel perturbation described earlier, which was applied in the centre of the unstable region. In order that the initial motions should be confined well within the unstable region, the length of the perturbation was restricted to 60 km.

As the simulation proceeds, energy spreads outwards from the region of the initial perturbation. As in the case when the instability is unconfined, circulations develop in the $x$–$z$ plane. These circulations grow in amplitude in the ‘moist’ region but do not penetrate into the dry region. As can be seen in Fig. 16 the circulations develop with the horizontal tilt evident in the unconfined case. Hence such a region of confined conditional symmetric instability could give rise to rainbands of finite length which are tilted in the horizontal relative to the front. (The development of the unstable circulations observed in this case is slow, owing to the small perturbation used.)
Figure 16. $x$-$y$ cross sections at $z = 2.52$ km and $t = 35$ h for the confined region of negative effective potential vorticity. The boundaries between the unstable and stable regions are marked. (a) $u'$. Contour interval is 0.3 m s$^{-1}$; (b) $w'$. Contour interval is 1 cm s$^{-1}$; (c) $v'$. Contour interval is 0.4 m s$^{-1}$.

Although no growing circulations are observed in the ‘stable’ region, the transport of energy into the stable region leads to the excitation of gravity waves which have the same horizontal wavelength as the symmetric circulations. The phase-shifts between the different perturbation quantities differ, since the waves are not growing with time. The waves have their maximum amplitude near the upper boundary to the north of the ‘unstable’ region and at the lower boundary to the south. This is due to the variation with height of the along-front wind.

The effect of varying the length of the unstable region is investigated by performing a set of simulations which are identical in all respects apart from the ratio of the length of the unstable region to that of the stable region. Two sets of simulations were performed; in one case the vertical diffusion coefficient was 20 m$^{2}$ s$^{-1}$, in the other 10 m$^{2}$ s$^{-1}$. The horizontal wavelength of the most unstable normal mode is approximately 140 km for $\nu_z = 20$ m$^{2}$ s$^{-1}$ and 100 km for $\nu_z = 10$ m$^{2}$ s$^{-1}$.

In Fig. 17 the eddy kinetic energy averaged over the unstable region is plotted as a function of the ratio of the length of the unstable region to the horizontal wavelength in the $x$-direction. As this ratio decreases the growth rate decreases. When the ratio is of order unity, there is negligible growth from the initial perturbation. This implies that the length of the region necessary to support unstable circulations depends on the wavelength of these circulations, and hence on the viscous properties of the flow. Experiments were also performed in which the length of the stable region was altered. It was found that the growth of the eddies was insensitive to the size of the stable region.
The reduction that takes place in the growth rate as the length of the unstable region is decreased in the along-front direction can be attributed to the loss of energy to the stable regions. In the unconfined case, as energy spreads from the initial perturbation, it excites further unstable circulations, which grow in time and so increase the overall energy of the eddies. When the unstable region is confined, however, gravity waves are excited in the unstable region, which do not contribute to the growth of the eddies, and when the rate of transport of energy away from the unstable region is equal to the rate of growth of the circulations within, there will be no overall growth. Although the initial perturbation is confined in all three directions, the transport of energy away from the perturbation in the along-front direction is by far the largest. This is due to the advection by the along-front wind.

6. DISCUSSION

This paper has investigated the extension of two-dimensional ideas of symmetric instability to three dimensions. It has been shown that three-dimensional structure is likely to occur, even when the initial state is two-dimensional. The use of a localized perturbation resulted in circulations forming which gradually became more elongated in the along-front direction. These circulations were tilted in the horizontal relative to the
front. In some cases the initial tilt was such that the bands were rotated cyclonically relative to the front; in these cases the bands rotated anticyclonically with time, and the fastest growth occurred once the bands had acquired an anticyclonic tilt. In other cases the initial tilt of the circulations was anticyclonic. Growing circulations were observed to form from perturbations as short as 40 km in the along-front direction. The initial growth was much faster, however, when a longer initial perturbation was used.

The above-mentioned preference for horizontal tilts can be understood by studying the tilted normal modes of a flow with negative potential vorticity. There exist a variety of ways in which energy can be extracted from such a flow. Growing modes exist with horizontal tilts ranging from that of the symmetric mode to that associated with baroclinic instability; the largest growth rates occurring around the symmetric axis. For a viscous flow with a Prandtl number of unity, the maximum growing mode is tilted with an orientation rotated anticyclonically relative to the front. The preference for such a tilt was also observed in the nonlinear regime. Such a tilted band can also exist with structure along its length. The magnitude and direction of the tilt depends on the Richardson number, the Prandtl number and the magnitude of viscosity used. As the nature of atmospheric viscosity is not fully understood we cannot say that a particular tilt should be observed in the atmosphere. It is likely, however, that rainbands arising because of 'symmetric' instability will be tilted relative to the thermal wind but that this tilt will not be significantly larger than 6 degrees.

When regions of negative PV are confined in the along-front direction, they are capable of supporting unstable 'symmetric' eddies. These develop with the tilt previously observed in the two-dimensional case; neither this tilt nor the wavelength of the instability in the along-front direction are affected by varying the length of the unstable region. The growing eddies are confined within the region of negative potential vorticity; inertia-gravity waves are excited outside this region. As the length of the unstable region is decreased the growth rate of the unstable circulations decreases. There is a threshold below which 'symmetric' circulations do not grow, determined by the ratio of the length of the unstable region to the horizontal wavelength of the circulations.

This result shows that the instability criterion derived by Ooyama (1966) for a two-dimensional flow does not hold in the case of a three-dimensional structure. The existence of negative PV is necessary but not sufficient for the instability to occur in three-dimensional flow. This is also true of the two-dimensional problem when viscous effects are included owing to the circulations requiring a finite scale for growth to occur. TR showed that when the region of negative PV is confined in the \(x-z\) plane it must exceed a certain size in order to support unstable motions. Thus for symmetric instability to occur the region of negative PV must be large enough in the across-front direction to support at least one unstable circulation, and longer than this in the along-front direction. It is not necessary for the dimension in the along-front direction to be larger than that in the across-front direction, so long as it is longer than the wavelength of the instability in this direction.

The three-dimensional counterpart of symmetric instability could account for the formation of rainbands of finite length if the instability is either locally released, or confined in the along-front direction. The rainbands may be expected to have a horizontal tilt relative to the front, the magnitude and direction of which depends on the viscous properties of the flow. If such a tilt can be documented in observed rainbands it may give an indication of the importance of viscous effects in frontal regions. It is noteworthy that Testud et al. (1980) observed that cold-frontal rainbands have an orientation rotated anticyclonically relative to the front. This is also the conclusion from the simulations described herein. It has also been shown that symmetric instability is a possible source
of inertia-gravity wave activity in frontal regions. This could occur at all boundaries of a
confined unstable region. When the region is also confined in the across-front direction
gravity waves are excited above the symmetric circulations, as well as the lateral bound-
daries of the unstable region.

We have considered here the three-dimensional counterpart of symmetric instability
in an idealized representation of a frontal zone. There are many features of real frontal
zones which have not been considered. Of particular interest is how the presence of
horizontal shear may affect the horizontal tilt of the unstable eddies. Future work should
also consider the interaction between symmetric instability and frontogenetical forcing.
The timescale for symmetric instability is the same as that on which frontogenesis occurs.
This implies that future work on symmetric instability should consider the stability of
time-dependent flows, such as occur at an active front. Another issue to consider is the
interaction between symmetric instability and convective instability. The numerical
studies described in this paper imply that the susceptibility of symmetric circulations to
convective instability is increased owing to three-dimensional processes, and that this
could lead to small-scale structure along the length of rainbands. It should be remem-
bered, in future studies of flows with negative potential vorticity, that three-dimensional
effects are important, and that 'symmetric' instability is in fact three-dimensional in
nature.

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REFERENCES

Arakawa, A. and Lamb, V. R. 1977 Computational design of the basic processes of the UCLA


Bennetts, D. A. and Hoskins, B. J. 1979 Conditional symmetric instability—a possible explanation for

889


Dalu, G. A. 1978 A parametrization of heat convection for a numerical sea

Atmos. Sci.*, 39, 2152-2158

Emanuel, K. A. 1979 Inertial instability and mesoscale convective systems. Part I: linear theory of inertial instability in rotating viscous
fluids. *J. Atmos. Sci.*, 36, 2645-2449

1985 Comments on 'Inertial instability and mesoscale convective
systems. Part I'. *J. Atmos. Sci.*, 42, 747-752


Meteorol. Soc.*, 97, 139-153

1974 The role of potential vorticity in symmetric stability and insta-

Hoskins, B. J., McIntyre, M. E. and Robertson, A. W.  
McIntyre, M. E.  
Miller, T. L. and Antar, B. N.  
Nakamura, N.  
Nehrkorn, T.  
Ooyama, K.  
Robert, A. J.  
Sawyer, J. S.  
Shutts, G. J.  
Simmons, A. J. and Hoskins, B. J.  
Solberg, H.  
Stone, P. H.  
Testud, J., Breger, G., Amayenc, P., Chong, M., Nutten, B. and Sauvaget, A.  
Thorpe, A. J. and Clough, S. A.  
Thorpe, A. J. and Rotunno, R.  
Thorpe, A. J., Hoskins, B. J. and Innocentini, V.  
Tokioka, T.  