Mountain wave drag over double bell-shaped orography

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SUMMARY

An integral expression is developed for the mountain wave drag (MWD) associated with linearized, hydrostatic flow over two infinitely long, bell-shaped ridges. This analysis extends previous studies of airflow over a single ridge. The MWD is expressed in canonical form with three terms: the sole effect of each ridge and a cross-term. While all three terms tend to zero with the appropriate inverse Rossby number, the cross-term also tends to zero with large ridge–ridge separation normalized by the sum of the half-widths. Asymptotic expansions of the dimensionless integral forms arising from the three terms in the MWD expression are presented.

1. INTRODUCTION

When stably stratified flow encounters irregular orography, gravity waves are generated and propagate upwards, transferring horizontal momentum vertically. The resulting pressure drag due to gravity waves is commonly referred to as wave drag and is an important part of atmospheric momentum balance. Using linearized Boussinesq equations for steady flow and a linearized lower boundary condition, the following expression for the mountain wave drag (MWD) in a hydrostatic, rotating, inviscid fluid with constant mean velocity and stability may be developed (e.g. Miles and Huppert 1969; Smith 1979a):

$$F = -2\pi \rho_0 \frac{U}{N} \int_{-\infty}^{\infty} (k^2 - k_f^2)^{1/2} \hat{h}(k) \hat{h}^*(k) \, dk$$

where $F$ is the MWD expressed as a force per unit length, $\rho_0$ is the mean density, $U$ is the mean (x-direction) velocity, $N$ is the Brunt–Väisälä frequency, $k$ is the wave number with respect to the $x$-coordinate, $k_f$ is $f_f/U$ where $f$ is a constant Coriolis parameter, $\hat{h}(k)$ is the Fourier transform of the orography and $\hat{h}^*(k)$ is its complex conjugate. This derivation assumes independence of the lateral coordinate.

Limitations of the system of equations used in this treatment are discussed by Smith (1979b) and Bannon and Chu (1988). The linearized Boussinesq approach restricts the analysis to ridge heights and wave amplitudes that are an order of magnitude less than the atmospheric scale height. Also, wave-breaking and flow-blocking is not accommodated by this linear analysis. The hydrostatic approximation is acceptable since Bretherton (1969) has shown that hydrostatic, irrotational mountain waves contribute most effectively to vertical transfer of horizontal momentum. Constant $U$ and $N$ with respect to height are mathematical simplifications; the WKB method could be used for slowly varying $U(z)$ and $N(z)$.

A canonical expression for the MWD over two parallel, bell-shaped ridges is derived as a function of appropriate inverse Rossby number and normalized ridge–ridge separation. For two ridges of heights $h_1$ and $h_2$, the term $\hat{h}(k) \hat{h}^*(k)$ in (1) produces a cross-term of order about $h_1 h_2$, i.e. $O(h_1 h_2^2)$, in addition to the known quadratic single-ridge terms, $O(h_1^2)$ and $O(h_2^2)$, analogous to $(h_1 + h_2)^2$.

2. WAVE DRAG OVER DOUBLE BELL-SHAPED OROGRAPHY

Figure 1 shows an idealized, two-dimensional orography as the superposition of two independent bell-shaped (Witch-of-Angnesi) ridges. Each ridge is defined by two parameters: ridge height $h$ and half-width $a$. Use of the linearized Boussinesq equations requires that the ridges have dimensions $h \ll a$. From Fig. 1, a suitable orographic function may be defined as

$$h(x) = \frac{h_1 a_1^2}{x^2 + a_1^2} + \frac{h_2 a_2^2}{(x - x_2)^2 + a_2^2}$$

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The Fourier transform of (2) is found to be
\[
\hat{h}(k) = \frac{1}{2} \left[ h_1 a_1 \exp(-ka_1) + h_2 a_2 \exp(-ka_2 + ikx_2) \right] \tag{3a}
\]
and
\[
\hat{h}(k)\hat{h}^*(k) = \frac{1}{2} \left[ h_1^2 a_1^2 \exp(-2ka_1) + h_2^2 a_2^2 \exp(-2ka_2) + 2h_1 h_2 a_1 a_2 \cos(kx_2) \right] \exp[-k(a_1 + a_2)] \tag{3b}
\]
where \(i\) is the imaginary unit, the Fourier normalization factor \((2\pi)^{-1}\) is in the forward transform \((x \rightarrow k)\), and the wavenumber \(k\) is physically restricted to be real and positive. Since \(\hat{h}(k)\hat{h}^*(k)\) is real and the drag force is given by the real part of (1) only, the limits of integration in (1) are redefined to be \(k \approx k_f\) and twice the value of the integral is used.

The resulting total MWD in (1) can be expressed in canonical form:
\[
F = F_1 + F_2 + F_{12} 
\]
\[
= F_{1, f=0} R_1 + F_{2, f=0} R_2 + F_{12, f=0, x_2=0} Q. \tag{4b}
\]

\(F_{1, j=1, 2}\) represents the MWD for the sole effects of the \(j\)-th ridge. \(F_{12}\) is the MWD resulting from the linear superposition of the single ridge profiles and is called the cross-term. \(F_{1, f=0}\) represents the MWD for a single ridge when the Coriolis parameter is zero \((k_f = 0)\). \(F_{12, f=0, x_2=0}\) is the cross-term when the separation and the Coriolis parameter are zero. \(R_i\) and \(Q\) are dimensionless integrals involving the Coriolis parameter. \(F_{12}\) is expressed such that \(Q\) also contains the ridge–ridge separation dependency.

(a) MWD when the Coriolis parameter is zero

For irrotational flow and using (1) and (3b), each term in (4) becomes
\[
F_{1, f=0} = -2\pi \rho_0 U N \cdot 2 \int_0^\infty \frac{k}{4} (h_1 a_1)^2 \exp(-2ka_1) \, dk = \frac{-\pi \rho_0 U Nh_1^2}{4} \tag{5}
\]
\[
F_{2, f=0} = -2\pi \rho_0 U N \cdot 2 \int_0^\infty \frac{k}{4} (h_2 a_2)^2 \exp(-2ka_2) \, dk = \frac{-\pi \rho_0 U Nh_2^2}{4} \tag{6}
\]
\[ F_{12,f=0} = -2 \pi \rho_0 \, UN \cdot 2 \int_0^\infty k \, h_1 \, h_2 \, a_1 \, a_2 \, \exp(-k(a_1 + a_2)) \, \cos(kx_2) \, dk \]

\[ = -2 \pi \rho_0 \, UN h_1 \, h_2 \, a_1 \, a_2 \, \frac{1}{(a_1 + a_2)^2} \frac{(1 - y^2)}{(1 + y^2)^2} \]  

where the normalized separation

\[ y = \frac{x_2}{a_1 + a_2}. \]

Substituting (5), (6) and (7) into (4), \( F_{f=0} \) can be written as

\[ F_{f=0} = -\frac{\pi \rho_0 \, UN}{4} \left\{ h_1^2 + h_2^2 + 8 h_1 h_2 \frac{a_1 a_2}{(a_1 + a_2)^2} \frac{1 - y^2}{(1 + y^2)^2} \right\}. \]  

Figure 2 shows total MWD and the cross-term plotted against \( y \) for \( h_1 = h_2 \) and \( a_1 = a_2 \). Three subdomains may be inferred:

(i) \( y \approx 0 \). The orography approaches a single, composite ridge of height \( h_1 + h_2 \). The MWD attains its largest values owing to the quadratic dependence on height.

(ii) \( y \approx 3^{1/2} \). \( F_{12,f=0} \) has a minimum at \( y = 3^{1/2} \). The MWD is reduced over the subdomain (i), because the overall orographic height is decreased while the overall width is increased.

(iii) \( y \gg 1 \). \( F_{12,f=0} \) tends to zero as expected.

Figure 2. Total drag \( F \) and the cross-term \( F_{12} \) normalized by \( F_1 + F_2 \) as functions of the non-dimensional normalized separation \( y = x_2/(a_1 + a_2) \) when the Coriolis parameter is zero. The figure is plotted for the specific orography \( h_1 = h_2 \) and \( a_1 = a_2 \). Note that when \( y > 1 \) the net total MWD is less than that which would result from two non-interactive ridges.

For normalized separation \( y > 1 \), the cross-term reduces the total MWD below the sum of the MWD from individual ridges, i.e. \( F < F_1 + F_2 \). Since the second ridge is in the lee of the first, the ‘effective height’ of the second ridge is reduced owing to partial elevation of the airflow after passage over the first. As \( y \) increases further, the airflow approaches the undisturbed level before encountering the second ridge, and \( F \) approaches \( F_{1,f=0} + F_{2,f=0} \).

* A mathematical analogy of the factor \( (1 - y^2)/(1 + y^2)^2 \) appears in Hunt and Richards’ (1984) boundary-layer study of stratified flow over complex terrain.
With appropriate selection of \( a_i, h_i, \) and \( x_i \), the orography given by (2) can approach the single asymmetric ridge studied by Miller and Durrant (1991) using nonlinear numerical analysis. They found that the MWD is strongly influenced by the maximum orographic height, moderately sensitive to windspeed and stability, and weakly sensitive to terrain asymmetry. Similar behaviour is shown by (8); the MWD is quadratically dependent on height and linearly dependent on windspeed and stability. However, (8) is invariant to terrain asymmetry in the direction of the flow because asymmetry in the MWD is a nonlinear phenomenon (Drazin and Moore 1967; Miles and Huppert 1969).

(b) MWD when the Coriolis parameter is non-zero

Angular momentum conservation requires inclusion of the Coriolis parameter in analysis of airflow over barriers of characteristic horizontal width \( L \geq 100 \text{ km} \) for perpendicular windspeed \( 10 \text{ m s}^{-1} \) and Coriolis parameter \( 10^{-4} \text{ s}^{-1} \). The first term in (4) becomes

\[
F_1 = -\pi \rho_0 \frac{U N}{k_f} \int_{k_f}^{\infty} (k^2 - k_f^2)^{1/2} h_i^2 a_i^2 \exp(-2ka) \, dk
\]

\[
= \frac{-\pi \rho_0 UN}{4} h_i^2 \int_{k_f}^{\infty} \left((ka_1)^2 - (k_a a_1)^2\right)^{1/2} \exp(-2k) \, d(k)
\]

\[
= F_{1,f=0} R_1
\]

where

\[
R_1 = R(p_{f_1}) = \frac{F_1}{F_{1,f=0}} = 4 \int_{p_{f_1}}^{\infty} (p^2 - p_{f_1}^2)^{1/2} \exp(-2p) \, dp,
\]

(9)

\[p = ka_1 \text{ and } p_{f_1} = k_f a_1 = \frac{fa_1}{U}, \text{ i.e. the inverse Rossby number.}\]

Similarly \( F_2 \) may be developed using \( R_2 \). A plot of \( R \) obtained using numerical integration is given in Fig. 3 and is equivalent to the one by Smith (1979a, Fig. 1). Also shown in Fig. 3, and derived in the Appendix as (A1), is an asymptotic expression for \( R \), namely

\[R_{\text{asymp}} = \exp(-2p) \cdot (1 + p_f).\]

![Graph](image)

Figure 3. \( R \) and \( R_{\text{asymp}} \) versus inverse Rossby number, \( p_f \). \( R \) is obtained from numerical integration and defined to be the wave drag on a single ridge normalized by the drag on the same ridge when the Coriolis effect is zero. The abscissa is given by \( p_f = af U = 1/R_a \), where \( R_a \) is the Rossby number. \( R \) corresponds to Fig. 1 from Smith (1979a).
This clearly shows exponential decay with the Coriolis parameter and is within 10% of \( R \) for \( p_f \approx 2 \).

The cross-term is a function of both \( y \) and \( k_f \). Substituting \( p_{f12} = (a_1 + a_2)k_f \) into (7), \( F_{12} \) becomes

\[
F_{12} = -2\pi p_0 UNh_1 h_2 \frac{a_1 a_2}{(a_1 + a_2)^2} \int_{p_{f12}}^{\infty} (p^2 - p_{f12}^2)^{1/2} \exp(-p) \cos(py) \, dp
\]

\[= F_{12, f=0, x=0} \cdot Q(p_{f12}, y) \tag{10} \]

where

\[
F_{12, f=0, x=0} = -2\pi p_0 UNh_1 h_2 \frac{a_1 a_2}{(a_1 + a_2)^2}
\]

\[= \int_{p_{f12}}^{\infty} (p^2 - p_{f12}^2)^{1/2} \exp(-p) \cos(py) \, dp. \tag{11} \]

and

\[
Q(p_{f12}, y) = \frac{F_{12}}{F_{12, f=0, x=0}} = \int_{p_{f12}}^{\infty} (p^2 - p_{f12}^2)^{1/2} \exp(-p) \cos(py) \, dp.
\]

Inspection of (7) with \( y = 0 \) gives the definition of \( F_{12, f=0, x=0} \) which is the maximum value of \( F_{12} \) and the chosen normalization factor for \( Q \). It can be shown that

\[
\int_{0}^{\infty} Q(p_{f12}, y) \, dy = 0
\]

precluding any anomalous source or sink of drag. Figure 4 shows \( Q \) obtained by numerical integration. With increasing \( y \), \( Q \) decreases rapidly and, with non-zero \( p_{f12} \), oscillates about zero with decreasing amplitude. Because the cosine term in the integral includes both independent variables, \( Q \) also oscillates with \( p_{f12} \) for non-zero \( y \). An asymptotic expansion for \( Q \), \( Q_{\text{asy}} \), is given in the Appendix. The decrease of \( Q \), and hence of the cross-term, with \( y \) and \( p_{f12} \) is clearly shown.

![Figure 4](image)

Figure 4. Three-dimensional representation of \( Q \) as a function of inverse Rossby number \( p_{f12} \) and normalized separation \( y \). \( Q \) is defined to be the normalized cross-term \( (Q = F_{12}/F_{12, f=0, x=0}) \). The qualitative behaviour of \( Q \) is emphasized here. Although this figure is obtained from numerical integration, the asymptotic behaviour of \( Q \) is given by (A3).
by $Q_{\text{asy}}$ since the controlling behaviour in (A3) is given by

$$Q \sim \frac{\exp(-p_{f12})}{1 + y^2}.$$  

Using $R$ and $Q$, the total MWD can be written as

$$F = \frac{-\pi \rho_0 U N}{4} \left\{ h_1^2 R(p_{f1}) + h_2^2 R(p_{f2}) + \frac{8 h_1 h_2 a_1 a_2}{(a_1 + a_2)^2} Q(p_{f12}, y) \right\}. \quad (13)$$

As in (8), $F$ in (13) is linearly dependent on $N$; however, unlike (8), it is nonlinearly dependent on the mean windspeed, $U$, through $R$ and $Q$ and does not decrease monotonically.

Figure 5 shows $F$ normalized by the constant $F_{1,f=0} + F_{2,f=0}$ for the simplified case $h_1 = h_2$ and $a_1 = a_2$ (i.e. $p_{f12} = 2p_{f1}$). Thus

$$\frac{F}{F_{1,f=0} + F_{2,f=0}} = R(p_{f1}) + Q(2p_{f1}, y).$$

For a non-zero Coriolis parameter, constructive and destructive interference between the imposed wavelengths of the perturbed airflow causes the total MWD to oscillate with varying $y$ and $p_{f12}$. These oscillations appear only in the cross-term. When $f = 0$, as in Fig. 2, the wavelengths which contribute most significantly to the MWD approach infinity; hence oscillations are not observed for finite $y$. When $f \neq 0$, as in Fig. 5, there is a finite upper bound on allowed wavelengths, $\lambda$, namely $U/N \leq \lambda/(2\pi) \leq U/f$ (Queney 1948). This permits interference to occur for finite values

![Figure 5](image_url)

Figure 5. Total drag $F$ normalized by $F_{1,f=0} + F_{2,f=0}$ as a function of inverse Rossby number, $p_{f1}$, and normalized separation $y$. The figure is plotted for the specific orography $h_1 = h_2$ and $a_1 = a_2$ (Note: $p_{f1} = p_{f2} = p_{f12}/2$). The $p_{f1} = 0$ axis corresponds to Fig. 2.
of $y$. As $f$ increases, the interference pattern shifts towards smaller values of $y$. Oscillatory behaviour of $F_{12}$ can be inferred from (10) implicitly, and is explicitly shown by (A3).

As expected, normalized $F$ decreases with inverse Rossby number, and the system approaches quasi-geostrophy. The quasi-geostrophic assumption is generally considered valid when $R_0 = O(0)$ and invalid when $R_0 \gg O(1)$. However, MWD is not always negligible in the intervening range. For a single ridge and $p_f = 1$ ($R_0 = 1$), the integral $R_1 \approx 0.24$; when $p_f = 2$ ($R_0 = 0.5$), $R_1 \approx 0.05$. Also, for the double-ridge orography with normalized separation $y = 2$, normalized MWD is about 0.33 at $p_{f_1} = 1$ and about 0.14 at $p_{f_1} = 1.5$. Therefore, appreciable wave drag remains for Rossby numbers less than 1 ($p_{f_1} > 1$).

(e) An example

For $h_1 = 1.0$ km, $h_2 = 1.5$ km, $a_1 = 35$ km, $a_2 = 50$ km, $x_2 = 100$ km, $U = 10$ m s$^{-1}$, $N = 10^{-2}$ s$^{-1}$, $\rho_0 = 1$ kg m$^{-3}$, and $f = 10^{-4}$ s$^{-1}$, it follows that $R(p_{f_1} = 0.35) = 0.735$, $R(p_{f_2} = 0.50) = 0.602$, and $Q(p_{f_2} = 0.85, y = 1.176) = -0.235$. Substitution into (13) yields $F = -1.1 \times 10^5$ N m$^{-1}$. Using $R_{\text{asy}}$ and $Q_{\text{asy}}$ with the given values, (13) yields $F = -9.7 \times 10^4$ N m$^{-1}$, which is about 12% less. Using $R_{\text{asy}}$ from (A2) and $Q_{\text{asy}}$, $F$ is about 4% more than the numerical result.

Assuming an average mid-latitude skin friction of $\tau_0 = 0.2$ N m$^{-2}$ (Smith 1978), the drag produced by this orography corresponds to that produced by $\tau_0$ over a horizontal distance of 550 km ($=F/\tau$). The increased drag due to the ridges is three times the skin friction $\tau_0$ over 185 km.

3. Concluding remarks

Mountain wave drag over two bell-shaped ridges is analysed in terms of the appropriate inverse Rossby numbers, $p_f$, and normalized separation, $y$. Important simplifications are made: linear, steady-state, hydrostatic, inviscid flow over two infinitely-long ridges is assumed, and $N$ and $U$ are held constant. In addition to the single-ridge terms, $F_1$ and $F_2$, this particular orography produces a cross-term, $F_{12}$, in the total MWD expression (4). Each term in (4) depends on a dimensionless integral: $R(p_f)$ for the single-ridge terms and $Q(p_f, y)$ for the cross-term.

The canonical form in (4) allows comparison of the MWD when $f = 0$ and when $f \neq 0$. For $f = 0$, and normalized separation $y > 1$, $F_{12}$ reduces the total MWD below that for the sum of the individual ridges, i.e. $F < F_1 + F_2$. For non-zero $p_{f_{12}}$ and $y$, the wavelengths of the airflow perturbations generated by two bell-shaped ridges may interfere constructively or destructively with each other. This interference causes $Q$ (Fig. 4), and hence the total MWD (Fig. 5), to exhibit oscillatory behaviour with $p_{f_{12}}$ and $y$.

As expected, $F_{12}$ approaches zero as $y \to \infty$, and the MWD is the sum of the single-ridge terms, $F_1 + F_2$. As $f$ increases, all three MWD terms decrease, and quasi-geostrophy is approached in accord with physical intuition. Nevertheless, appreciable wave drag remains for Rossby numbers less than one, and quasi-geostrophic assumptions should be treated carefully.

MWD over $n$ ridges is indeed affected by interactions between the neighbouring wave disturbances (Gossard and Hooke 1975). The $n$-ridge case can be examined by generalization of (4). For non-adjacent ridges with $y_{ij} > 1$, cross-terms $F_{ij}$ ($i \neq j$) can only contribute significantly to the total MWD if neighbouring pairs of ridges are proximal, i.e. $y_{ij+1} \approx O(1)$; this is due to the $(1 + y^2)^{-1}$ decay of $Q$.

APPENDIX

Derivation of asymptotic forms for $R$ and $Q$

Using the substitution $t = 2p$ and a binomial expansion of the square-root term under the integral sign, $R(p_f)$ from (9) becomes

$$R(p_f) = \sum_{n=0}^{\infty} \binom{1}{n} \frac{(-1)^n p_f^n}{4^n} \int_0^\infty t^{1-2n} \exp(-t) \, dt.$$  

Expanding only the $n = 0$ and $n = 1$ terms which are non-divergent (Erdélyi 1956; Bender and Orszag 1978), integrating by parts, and retaining only the controlling behaviour, the asymptotic expression for $R$ becomes

$$R_{\text{asy}} = \exp(2p_f) \cdot (1 + p_f).$$  

(A1)

(The remaining integral from the $n = 1$ term is neglected because $R(p_f = 0)$ must equal one). A
semi-empirical relationship, $R_{se}$, employs small correction factors and gives better agreement for $p_f < 1$:

$$R_{se} = \exp(2p_f) \cdot (1 + 1.65p_f - 0.62p_f^2).$$  \hspace{1cm} (A2)

A similar binomial expansion for $Q$ yields

$$Q(p_f, y) = \sum_{n=0}^{\infty} \binom{n}{2} (-1)^n p_f y^n \int_{p_f}^{\infty} p^{1-2n} \exp(-p) \cos(py) \, dp.$$  

$Q$ may be approximated using only the first term of the infinite series. $Q_{asym}$ becomes

$$Q_{asym} = \int_{p_f}^{\infty} p \exp(-p) \cos(py) \, dp$$

$$= \frac{\exp(-p_f)}{1 + y^2} \left[ (p_f + 1) \cos(p_f y) - y(p_f + 2) \sin(p_f y) \right]$$

$$- \frac{2y^2}{1 + y^2} \left[ \cos(p_f y) - y \sin(p_f y) \right].$$  \hspace{1cm} (A3)

The difference between $Q_{asym}$ and $Q$ is negligible under the simplified conditions outlined in section 1.

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