Nonlinear evolution of ordinary frontal waves induced by low-level potential vorticity anomalies

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SUMMARY

Linear, semi-geostrophic (SG) theory reveals the instability of steady fronts with low-level potential vorticity anomalies. Joly and Thorpe (1990) have shown in this context the most unstable normal modes to have sub-synoptic wavelengths. The present study uses a primitive equation (PE) model to construct, at these wavelengths and along the same fronts, the PE normal modes and extends the evolution to the nonlinear regime. It is shown that PE normal modes have a structure similar to the original SG modes at the same given wavenumber.

In the nonlinear experiments, two different kinds of behaviour are found, depending on the initial wavelength of the perturbation, the frontal baroclinicity and the width of the potential vorticity anomaly. The first kind, and main finding of this study, is characterized by the inability of a barotropically unstable mode (in the energy sense) to lead to large pressure falls in the vortex. Such a mode, with its wavelength smaller than the Rossby radius, is successful in breaking the frontal flow but saturates within two days. The other occurs when the wavelength is larger than the Rossby radius. Then, it is shown that the initially significant barotropic contribution to the growth vanishes and the wave enters a phase of classical baroclinic growth. It is only when this second phase occurs that the frontal change in structure is accompanied by significant deepening of the surface low. It saturates in a way similar to larger-scale baroclinic waves, by increasing the upper-level jet and shear.

1. INTRODUCTION

With frontal waves, forecasters are faced daily with a problem in zonally perturbed weather regimes. These waves seem to develop rather frequently but most of them remain innocuous; nevertheless, they are important whenever there is a need for accurate wind or precipitation predictions, especially since waves of this kind sometimes turn into fully developed storms.

By quantifying these assertions we hope to help the understanding of our results. With the improvements in the observing system and, in particular, the operational use of satellite imagery, a better large-scale view of frontal organization over the Atlantic and Pacific oceans is now available, but the data network is barely sufficient for subsynoptic detection (Lorenc et al. 1988). As a result, a statistical, observational study specific to frontal waves or second generation cyclones is not yet available, to our knowledge. Consider instead the one-year analysis of all cyclogenesis by Roebber (1984), as summarized by his Fig. 2. It turns out that the largest class of cyclones (36% out of a set of 1130 cases) underwent a small pressure fall, between 0 and 5 mb in 24 hours, and that 24% of them deepen between 5 and 10 mb in 24 hours. These two classes, which constitute the majority of cases, define what we refer to as 'ordinary cyclones', and among them are the 'ordinary second-generation cyclones' that grow along finite-length fronts, though only a small portion of these become severe storms or 'bombs'.

The theoretical study of frontal waves has recently been revived through building on a better knowledge of the processes of frontogenesis and frontal structure, of which the latter embodies the 'semi-geostrophic' theory of Hoskins and Bretherton (1972) and Hoskins (1975). In recent years as many as three types of major theory of frontal instability have been proposed.

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Thornicroft and Hoskins (1990) analysed the development of an isolated second-generation cyclone event along a fully three-dimensional, curved, active cold front which had resulted from the nonlinear growth of a mid-latitude baroclinic wave on the sphere. They modelled the dynamics by making use of the adiabatic primitive equations which they discretized on the sphere and truncated at T95. The occurrence of the wave takes place in the late stage of the life cycle of the cold front and involves a potential vorticity extrusion from the polar reservoir at the upper levels. The jet structure at that time is such that it cuts off the potential vorticity from its source. This upper-level anomaly then interacts with the surface front. The fundamental ideas behind this kind of interaction are discussed by Hoskins et al. (1985) The nonlinear nature of the mechanism sets it apart from the other two types of theory, which are more the classical linear instability theories that neglect the curvature of the fronts supposing them to be two-dimensional. At first glance, a superficial difference between these two theories can be assigned to the dynamical assumptions: one theory is based upon the primitive equations, the other makes the geostrophic momentum approximation at the outset. But it is our belief that the genuine difference consists in the choice of basic states.

In work based solely on linear calculation, Moore and Peltier (1987, 1990) obtained an instability with a 1000 km along-front wavelength which developed along the ‘uniform potential vorticity deformation front’, considered stationary. This wave was filtered by the geostrophic momentum approximation and existed only by virtue of the primitive equation dynamics.

In the second approach, used basic states were used that were in better agreement with observations, in that they took into account diabatic effects during the primary frontal development. Schär and Davies (1990) made use of a front resulting from purely analytic formulae, whose interior potential vorticity is uniform but which is characterized by a low-level warm anomaly inspired by the observed so-called ‘warm conveyor belt’ (Browning 1985). On the other hand, Joly and Thorpe (1990), hereafter referred to as JT90) used a simple parametrization to represent the condensation in their semi-geostrophic model. In this way they obtained two-dimensional, steady-shear fronts with a positive potential vorticity anomaly near the surface. In both these cases, the linear stability analysis was able to give one instability maximum having a wavelength within the limits of the subsynoptic scale.

Schär and Davies (1990) also extended their linear analyses by simulating the nonlinear growth of the most unstable normal mode superimposed on their basic state, performed within the framework of the semi-geostrophic theory, and found that significant development of their unstable mode occurred prior to nonlinear saturation. Conversely, the ‘ageostrophic mode’ of Moore and Peltier is unable to grow to a significant amplitude (Peltier and Polavarapu 1991). On the other hand, JT90 undertook only linear studies.

Our purpose here is to examine the behaviour, in the nonlinear regime, of the linear modes found by JT90. Special attention will be paid to the criterion which is most familiar to forecasters and synopticians, namely, the pressure fall within deepening lows, as exemplified by Roeber (1984). Recent papers on cyclogenesis (e.g. Thornicroft and Hoskins 1990) concentrate on the deformation of surface isentropes as the criterion for development, although the actual pressure deepening is not always stated, and appears modest on investigation. Section 5 below shows that the different ways of looking at the results are not equivalent.

The nonlinear experiments which are described in detail here must be considered as following directly from the study of frontal instabilities generated by the lower troposphere potential vorticity anomalies of JT90; however for the nonlinear simulations,
we have abandoned the semi-geostrophic theory and opted for a primitive equation model. The reasons for this choice are discussed in section 3(b).

In section 2 we briefly present basic states with a potential vorticity anomaly near the surface, and also recall the results of the linear stability analysis for two different two-dimensional shear fronts. In section 3, we describe the nonlinear model and present our reasons for choosing the primitive equations. Section 4 provides details on the comparison between SG normal modes and the PE normal modes used as suitable initial conditions for the main experiments with the nonlinear model. Section 5 is reserved for a detailed presentation of the nonlinear simulations, and, in section 6, attempts are made to understand the important parameters of the initial state that determine the nature of the solution at nonlinear saturation. The results of sections 5(a) and 5(b) are revisited in section 7 in the light of shallow-water experiments.

2. SHEAR FRONTS MODIFIED BY CONDENSATION: A SUMMARY OF THE LINEAR STUDY

In the study by JT90, the geostrophic momentum approximation is introduced during both the basic frontal evolution and the linear stability analysis, and a simple parametrization allows condensation only in regions of ascent (Emanuel, Fantini and Thorpe 1987). The evolution of a two-dimensional baroclinic Eady wave (after Eady 1949) leads to a steady shear front at some arbitrary time at which it is deprived of its main energy source. The effects of both non-stationarity and condensation in the linear analysis have been left out of this paper, as they were by JT90, except in section 6(b). However, a more complete linear theory for time-dependent frontal instability has been propounded by Joly and Thorpe (1991).

The presence of a local maximum in the zone of latent-heat release is associated with the formation of a positive potential vorticity (PV) anomaly beneath it in the vicinity of the surface. The amount of condensation occurring during front formation, as measured by the non-dimensional moist PV, highly influences the resulting fields in the frontal region and, ultimately, the stability of the front. In particular, in the case of intense condensation, the ascent is narrowed in the cross-frontal direction. A wider anomaly occurring because of slower condensation processes gives a signature in the static stability rather than in vorticity, according to the invertibility principle (Hoskins et al. 1985).

In the following, the two fronts already studied by JT90 serve as examples. Their characteristics are, briefly, these: Front 1 (Fig. 1) is characterized by a relatively large anomaly (width 330 km, height 2000 m, maximum amplitude of the potential vorticity anomaly, $\delta P V_{\text{max}}$, of $0.85 \times 10^{-6} \text{Km}^2\text{kg}^{-1}\text{s}^{-1}$, or 0.85 PVU, and a maximum absolute vorticity in geostrophic space of 7.6$f$, where $f$ is the Coriolis parameter). Front 2 (not shown) results from a more intense condensation, so that the anomaly is narrower (width of the anomaly 150 km, depth 1800 m; $\delta P V_{\text{max}}$ 0.91 PVU (potential vorticity units), maximum absolute vorticity 6.7$f$). Note that for the JT90 experiments a free surface (i.e. $\omega = \frac{dp}{dt} = 0$ at the upper boundary) is imposed at the top. The simulation in section 5(a) has a similar upper boundary; but in subsection 5(c), a pseudo-stratosphere is introduced at the top.

Note that in the series of operations leading to the nonlinear results, condensation is included only in the two-dimensional basic-state computation. These studies concentrate on how the presence of the potential vorticity anomaly affects the dynamics. It is the anomaly itself that is important here, rather than the way in which it has been generated. Condensation is one process, among others, that has been used to produce structures and amplitudes for this anomaly, which are not totally unreasonable.

The linear stability analysis shows two maxima in the growth-rate curve plotted
Figure 1. (a) Vertical cross-section of Front 1. Across-front wavelength 3250 km, depth of troposphere 8 km. Ticks are set every 500 km horizontally, 2 km vertically. (i) Potential vorticity, PV, contour interval 0.2 PVU with thicker contour 0.3 PVU, dash-dotted contour 0.2 PVU and dashed contour 0.4 PVU. (ii) Thick solid and dotted lines: along-front wind \( V_x \), contour interval 4 m s\(^{-1}\), negative contours dotted. Light dashed lines: potential temperature \( \theta \), contour interval 4 K. (b) Growth-rate \( \sigma \) (top panel) and phase speed (lower panel) as a function of the along-front wavenumber \( \ell \) of a normal mode as given by linear semi-geostrophic calculations, after JT90. Dashed line: as by JT90 including the Boussinesq approximation. Solid line: non-Boussinesq result. Dash-dotted line: non-Boussinesq, with a realistic tropopause. (c) Same as (b) but for Front 2. Boussinesq and non-Boussinesq curves are the same. All phase speeds are identical.

against the along-front wavenumber (Fig. 1(b, c)). The deep normal mode, corresponding to the first maximum, is the classical baroclinic instability model, while the second mode has a shorter wavelength in agreement with observed frontal cyclone wavelengths (about 1800 km for Front 1; only 800 km for Front 2). The mode with the larger frontal wavelength remains fairly similar to a baroclinic wave and extends throughout the troposphere, energy being drawn from the frontal surface jet, with a significant contribution from the sloping isentropes; it is a mixed barotropic/baroclinic mode. The other
mode is trapped at low levels, developing mainly by drawing the kinetic energy from the basic state through horizontal shear (the so-called 'barotropic instability' mechanism).

The nonlinear simulations are done with a non-Boussinesq primitive equation model. The influence of the Boussinesq approximation on the features of the normal mode developing along Front 1 was tested (Fig. 1); this approximation is optional in the linear semi-geostrophic model. It appears that the growth rates are slightly larger in the non-Boussinesq case. The maximum growth occurs at the same wavenumbers in both cases.

The influence of the geostrophic momentum approximation on the subsynoptic normal modes is a more important question which is dealt with in sections 3(b) and 4.

3. THE PRIMITIVE EQUATION MODEL AND ITS RELEVANCE

(a) Description

ARPEGE is the global numerical weather prediction model currently being developed by Météo-France in collaboration with ECMWF (IFS project). ARPEGE is a hydrostatic primitive equation (PE) code, in which the horizontal discretization relies on the spectral technique and the vertical discretization on the finite-difference scheme of Simmons and Burridge (1981); it conserves globally energy and angular momentum (within the limits of the horizontal and time discretization) and is characterized by a hybrid vertical coordinate (proportional to \( p/p_s \) at the surface and to \( p \) higher up, where \( p \) is the pressure and \( p_s \) the surface pressure). The model was originally designed to be a global model on the sphere, allowing for a stretched horizontal coordinate resulting from a conformal transform (Courtier and Geleyn 1988), but for the nonlinear simulations described below, a doubly-periodic version was developed. In toroidal geometry, a constant Coriolis parameter, \( f \), equal to \( 10^{-4} \) s\(^{-1}\), applies over the whole domain. The essential difference between the global model and the limited-area version is localized in the spectral transform algorithm. The fast Fourier transforms used for the east–west projections are simply rescaled, and the Legendre transforms are replaced by matrix Fourier transforms; thus keeping the original code. All the gridpoint code is kept unchanged.

The nonlinear integrations appearing here use a leap-frog time-integration scheme combined with an Asselin filter with coefficient 0.05. A standard semi-implicit treatment of gravity-wave adjustment accompanies this scheme, while the horizontal diffusion scheme, when used, involves the Laplacian squared. Except in part of section 6(b), no other physical parametrizations have been added.

The spherical version has been validated carefully (Thépaut and Courtier 1991). The limited area modifications were validated by running experiments similar to the simulations of Hoskins and West (1979) with their parameter \( \mu \) set to 1. Although the original results were obtained in the semi-geostrophic context, the primitive equation integrations gave a very similar frontal organization. However, one important difference was noted in comparing the two results—the amplitudes of the low-pressure and high-pressure systems were not symmetric in the PE simulation, the depression being actually more pronounced than the anticyclone throughout the life cycle, unlike Hoskins and West's results. This feature is now well understood from the work of Davies et al. (1991) and Snyder et al. (1991), the first of which gives the key to this behaviour, which originates in a combined symmetry of the semi-geostrophic equations and of the mode and basic state. The second paper shows, by an argument based on the ageostrophic vorticity dynamics, that, even when the symmetry is broken, the semi-geostrophic model will underestimate the deepening.
(b) Reasons for choosing the primitive equations

As was mentioned in section 2, the unstable normal modes were first exhibited in a filtered, semi-geostrophic, linear model, but here, only the most unstable wavelengths are retrieved from the simplified dynamics. Normal modes are re-computed in the PE model prior to carrying out the nonlinear simulations we are interested in. This procedure is now justified. The fronts that have been considered already possess a large vorticity maximum, and as regards the linear theory, the validity of the semi-geostrophic theory is in some cases stretched somewhat to the limit. This is evident from one of the momentum equations linearized in the special case of a steady frontal flow:

\[
\left( \frac{\partial}{\partial t} + \bar{V}_g \frac{\partial}{\partial Y} \right) u'_k - f v'^*_a = 0
\]  

(1)

where \( \bar{V}_g \) and \( Y \) refer to the geostrophic mean wind and the horizontal coordinate along the front and \( u'_k \) to the linear perturbation of zonal wind. The notation \( v'^*_a \) was used by Hoskins and Draghici (1977):

\[
v'^*_a = v_a - \left( \frac{w}{f} \right) \frac{\partial u'_k}{\partial z}.
\]

Linearization of this expression gives \( v'^*_a = v'_a \). For a classic normal-mode perturbation, \( A' = A' \exp(\sigma t + iY) \), (1) reduces to

\[
(\sigma + i\bar{V}_g) u'_k - f \bar{\sigma}'_a = 0.
\]  

(2)

The growth rate, \( \sigma \), is a complex number, \( \sigma = \sigma_c + i\sigma_i \); \( l \) is the along-front wavenumber. Now, for consistency, the geostrophic momentum approximation requires that

\[
|v'^*_a| \ll |u'_k|
\]  

(3)

and the inequality (3) requires, everywhere, that

\[
\left( \frac{\sigma_c^2 + (i\bar{V}_g + \sigma_i^2)^2}{f^2} \right)^{1/2} \ll 1.
\]  

(4)

In the case of Front 1, evaluation of the square root gives a maximum value of 0.5. For Front 2, the larger wavenumber \( l = 8 \times 10^{-7} \text{ m}^{-1} \) is balanced by the smaller basic frontal wind, and the square root also reaches 0.5 in the frontal region.

It is worth noticing that these numbers were computed using the maximum value of the wind, \( \bar{V}_g \). They should be understood more as the extreme local values than as globally representative of the wave. The maximum amplitude of the frontal waves is, in fact, obtained in the zone of maximum shear, slightly on the cold-air side of the low-level jet.

In other words, the linear semi-geostrophic theory predicts ageostrophic motions as large as half the magnitude of their geostrophic counterpart. It is therefore possible that ageostrophic effects may rapidly become even more important as the wave grows. Going from linear to nonlinear evolution under such conditions provides one reason for abandoning the balanced jet. It turns out that ageostrophic effects in the grown frontal wave are related to curvature effects, which is a difficulty of the semi-geostrophic equations that has already been analysed elsewhere (Snyder et al. 1991). One further reason for changing follows from the fact that vorticity within the basic frontal band is initially quite large. As frontogenesis is activated again, it can soon reach a value beyond which the geostrophic momentum approximation becomes unquestionably unrealistic.
4. Initial conditions for the nonlinear model

The two-dimensional fronts computed in geostrophic space are transformed to physical space, all the information being carried to the PE model by the geopotential field alone. Details of the transformations are given in Appendix 1. If no further changes are added to these structures, they represent stationary solutions of the PE model, as has been checked by integrating forward for up to ten days.

The wavelength along the front is fixed at a value given by the SG linear stability analysis, then the PE normal mode is constructed by integrating the model forward in time starting from an arbitrary very-small-amplitude sinusoidal wave. The required amplitudes and phases are provided by a random-number generator. This noise is added to the basic geopotential field prior to the computation of temperature and wind. A similar technique was used by Thornicroft and Hoskins (1990). The amplitude is monitored and kept small, and at a chosen time within the period of linear growth the perturbation is extracted from the output of the nonlinear model by subtraction of the along-front mean fields; it can be re-scaled for use in nonlinear experiments. In this section, the raw structure is compared to that of the mode produced by the semi-geostrophic analysis.

The PE mode is characterized by the maximum value of $u'_g$ at the surface, so the SG mode is normalized to this amplitude. The properties that give the best summary of the mode structures are the mean-wave fluxes, which are defined as follows:

$$\overline{u'v'}(x, z) = \frac{1}{L_y} \int_0^{L_y} u'v' \, dy,$$

the so-called barotropic correlation, $\overline{w'\theta'}$ the conversion internal and $\overline{u'\theta'}$, the baroclinic correlation. They provide an excellent picture of the modal length scales and position with respect to the front.

Figure 2 illustrates examples of fluxes in Front 1. Clearly the overall structure is very similar in both models, which implies, in particular, that no new mechanism is able to modify the energetics of the mode in an obvious way in the less stringent PE approximation. This is confirmed by computing the energy conversions, as discussed below.

Nevertheless, differences do exist, but they are restricted to details. The frontal slope, for instance, is better represented in the SG model (the resolution is better in this zone), and is apparent from the slope towards the cold air of the barotropic flux. The internal conversion reveals the deep nature of the mode in both models. One of the key features of this kind of subsynoptic instability, noted by JT90, is that it is negative at the front, where the barotropic process of deriving energy from the low-level jet is most effective. The heat flux is positive at the upper level, near the upper-front jet. However, in the PE model, unlike its SG counterpart, the zone of positive heat flux does not extend downwards along the original cold front. On the contrary, the region of negative heat flux is broader at low levels and includes a portion of the positive $\partial \theta'/\partial x$ zone. The horizontal heat flux (not shown), negative ahead of the front, extends deeper in the PE model, but, mostly, the latter does not have the zone of positive flux at the upper lid. The structural differences are even less visible in the second frontal mode (not shown). In both models this mode turns out to be quite shallow, with a definitely negative internal conversion.

Correlations are also useful in relation to the wave energetics. Multiplied by appropriate functions related to the frontal structure, these correlations become energy conversions, energy in this case being averaged over the whole domain. All necessary definitions can be found in Appendix 2. The result of the PE normal-mode extraction is
Figure 2. Vertical sections across the front displaying the mean-wave correlations of the normal modes growing along Front 1 with wavenumber $3.5 \times 10^{-6} \text{s}^{-1}$ obtained respectively with the primitive equations (top panel of each figure) and under the semi-geostrophic approximation (lower panel). Both modes are normalized so that the maximum geostrophic wind at the surface is 0.0971 m s$^{-1}$. (a) 'Barotropic' correlation $u'v'$. Contour interval $0.7 \times 10^{-3}$ (PE) and $10^{-2}$ (SG) m$^2$ s$^{-2}$. (b) Internal conversion $w'\theta'$. Contour interval $0.7 \times 10^{-7}$ (PE) and $10^{-6}$ (SG) km s$^{-1}$. Ticks are set every 100 km horizontally, 2 km vertically. The bold solid line is the zero contour. Negative lines are dashed.

summarized in Fig. 3. These box diagrams should be compared to Figs. 15 and 16 of JT90. Both modes appear essentially barotropic in nature, that is most of their energy is derived from the frontal jet kinetic energy through the horizontal shear. This result also holds for the SG modes. Both modes, however, derive some energy from the frontal baroclinicity, something the SG mode along Front 2 is unable to achieve; also, whereas the first frontal SG mode has, on the average, a small positive internal conversion, the PE mode has a negative one, the reversal of sign occurring in the SG analysis at a smaller wavelength. However, this difference may be due to numerical differences, as the mean values shown by the box diagrams result from two distinct regions where this conversion has opposite signs, as shown by the cross-sections of Fig. 2.

Therefore the intrinsic structures of SG and PE modes are quite similar. Using the procedure described in Appendix 1, it turns out that both the SG modes and the PE modes can be used as initial conditions for the nonlinear simulation. Further discussion of the linear properties of PE and SG normal modes is beyond the scope of the present paper; however, a detailed study of this kind, based on a fully linearized PE model, is the subject of current work.
Front 1 PE mode $l = 3.5 \times 10^{-6} \text{ m}^{-1}$

\[
\begin{align*}
\overline{A} & \quad \overline{K} \\
0.6 \downarrow & \quad 0.6 \uparrow^{(v)} \quad 28.6 \downarrow^{(l)} \\
A' & \quad K'
\end{align*}
\]

Front 2 PE mode $l = 8 \times 10^{-6} \text{ m}^{-1}$

\[
\begin{align*}
\overline{A} & \quad \overline{K} \\
0.1 \downarrow & \quad 0.1 \uparrow^{(v)} \quad 7.0 \downarrow^{(l)} \\
A' & \quad K'
\end{align*}
\]

Figure 3. Energy box diagram for the PE modes computed, respectively, along Front 1 at wavenumber $l = 3.5 \times 10^{-6} \text{ m}^{-1}$ and along Front 2 at wavenumber $l = 8 \times 10^{-6} \text{ m}^{-1}$. Conversions are normalized by the internal conversion. (See Appendix 2 for definitions.)

5. Nonlinear simulations

(a) Evolution and structure

The initial conditions for the reference nonlinear experiments combine one of the fronts shown in section 2 with a normalized normal mode computed as indicated in the previous section. The results of the nonlinear integrations made over a period of 10 days are stored every 12 hours and form the data-base for all the necessary diagnostics. For convenience, it is supposed that the initial two-dimensional front is orientated along a north–south axis, however, it will be remembered that this is an arbitrary choice without physical meaning in these $f$-plane doubly-periodic simulations.

For the Front 1 simulations, the horizontal wavelengths are 3250 km in the acrossfront direction and 1800 km along the front; the lid is at 8 km above the surface. A spectral discretization of $21 \times 11$ (i.e. the smallest represented wavelength is 160 km) and 20 levels in the vertical are used. Nonlinear terms are evaluated on a $64 \times 36$ point grid to avoid aliasing of the quadratic terms. The normal mode is characterized with a maximal $u'$ perturbation of 0.5 m s$^{-1}$.

The perturbation grows during the first 4.5 days. Then, the eddy kinetic energy starts decreasing (see subsection 5(b) below). As nonlinear saturation is achieved the growing phase is replaced by equilibration rather than decay. Figures 4 and 5 show the fields of potential temperature, $\theta$, vorticity, $\zeta$, and surface pressure, $p_s$, during the growth stage. After 24 hours of integration, an undulation appears along the primary front, the surface low starts to deepen, and in the north-west part of the low a region of strong frontogenesis coincides with the maximum of vertical velocity (Fig. 6). This zone of frontogenesis remains active during the 10 days of integration. Another less intense
region is observed in the south-east of the depression. From the point of view of vertical velocity and frontogenesis, this organization is the very image, on a smaller scale, of a classical baroclinic wave in which the $\theta$ and $\zeta$ surface fields roll up around the low with the action of frontogenesis. At day 4, the potential temperature takes on the shape of a comma, very similar to the one shown by Schär and Davies (1990). After two days of integration a band of large vorticity is observed to form on the west side of the domain and to persist throughout the 10 days, while another band, present at the time of saturation in the south-east of the low, later dissociates itself from the system. The
maximal surface wind increases to 33 m s\(^{-1}\), which is 10 m s\(^{-1}\) stronger than the initial surface wind. The geostrophic component of the wind strongly overestimates the wind near the ground, but the strong curvature component of the ageostrophic wind compensates for this geostrophic circulation. In altitude, in the vicinity of the lid, a very intense jet develops (40 m s\(^{-1}\)). However, the somewhat artificial upper boundary condition is suspected of significantly amplifying the wind at this level, therefore the introduction of a more realistic tropopause at the top of the domain is a valuable improvement. The results are given in section 5(d).

The initial positive surface PV band organizes itself into a positive circular anomaly centred on the low (Fig. 7), in much the same way as was expected by Kleinschmidt (1950), who proposed the theory that the core of any finite amplitude frontal wave was an airmass with high potential vorticity. He also showed in observed case-studies how
the generation of rain in a region of frontal ascent induced an increase in potential vorticity. Nevertheless, it is not clear in Kleinshmidt's work how he imagined a vortex to form out of the generation of high PV air by moist processes. Our results explicitly detail one possible way: the band of high PV air generated during frontogenesis wraps itself up through the realization of an instability mechanism. This kind of secondary wave event, besides being in good agreement with theory, could in a more practical framework also be used by forecasters to diagnose the appearance of frontal waves over the Atlantic ocean.

It is of interest to translate these significant structural changes into the evolution of a parameter much favoured by forecasters, i.e. the minimum pressure. The pressure fall reaches 9.8 mb over the first five days, the most rapid change over the 24 hours occurring between days 3 and 4, but remaining modest, with a fall of 3.3 mb in 24 hours. According to Roebber's (1984) study, a deepening of this amount is extremely common. However, it is clear that such an ordinary, spontaneous cyclone cannot, by itself, turn into a 'bomb', despite the large initial baroclinicity.

The development of Front 2 is less dramatic. For this experiment, the wavelength is still 3247 km in the across-front direction, but is reduced to 785 km in the along-front direction. For this smaller-scale instability, a better spectral resolution of $42 \times 10$ modes is employed (the smallest wavelength represented is 75 km). The maximal wind amplitude for the normal mode is also 0.5 m s$^{-1}$. Near the surface, fields evolve in a manner remarkably similar to those for other, larger-scale cyclogenesis events (and are therefore not shown). At day 2, an efficient frontogenetic organization of deformation leads to enhanced surface vorticity, as can be observed to occur temporarily at the ground (Fig. 8). After day 3, however, this pseudo-front disappears as the wave saturates, and the solution depicts a quasi-circular vortex being passively advected by the flow, and compos-
from the decay of any organized signal in the low-level vertical-velocity field while the upper-level basic-state circulation, on the other hand, remains practically unaffected. Despite its well-developed organization—the surface front is broken up into a vortex—the amplitude of the depression remains small, reaching only 1.6 mb while the wind field reveals only a 1 m s$^{-1}$ increase. The contrast between the impact on the flow and the signature on the pressure field should be noted.

(b) Energetics of the nonlinear evolution

Some insight into the mechanisms behind the horizontal structural changes that have been shown can be gained by considering the evolution in time of the energy reservoirs and conversions. Definitions are given in Appendix 2. The perturbation fields $u'$, $v'$, $w'$ and $\theta'$ are, strictly speaking, only well defined at the beginning of the integration, when
Figure 8. Results from the nonlinear simulation with Front 2 at 48 hours (along-front wavelength 800 km, across-front wavelength 3250 km with only the central 1625 km shown). (a) Potential temperature at 850 mb, contour interval 1 K. Q vector. (b) Vertical velocity at 850 mb, contour interval 1 mm s$^{-1}$. Ticks every 500 km.

the linear development prevails. In this subsection, the perturbation is defined as the departure from the current mean fields $U$, $V$ and $\theta$ along the initial front direction.

The life cycle of the wave along Front 1 is summarized in Fig. 9(a), showing the evolution of the wave's kinetic and potential energies for the first five days of integration. The wave growth appears unchecked up to day 4, then first the potential energy then the kinetic energy begin to decrease. This decrease is itself temporary, the wave energies oscillating about half their peak values after day 6. Figure 9(b) reveals, however, that the growing phase of the mode along Front 1 can be split between two well-defined and distinct phases. Following on from the linear analysis, the main wave energy source remains the kinetic energy of the frontal jet up to day 2.5. This phase, which is the distinctive feature of a sub-synoptic frontal wave such as this, is absent (or very small) in the primary baroclinic waves (see e.g. Simmons and Hoskins 1978). Afterwards, a
second stage occurs consisting of a classical baroclinic source, and gains efficiency until day 4. This corresponds to the period of maximum deepening rate. The barotropic source still contributes to the growth between days 2.5 and 3.5. Then, and this time in a way very similar to the life cycle of larger-scale baroclinic waves, the barotropic conversion becomes the main sink of energy and the cause of wave equilibration. During the whole period, little energy is derived from the vertical wind shear. Note, also, that these experiments were run without horizontal diffusion, so that dissipation is also negligible.

The life cycle of the mode along Front 2 is, by contrast, shorter and simpler. As a sign of its much less significant development, note that in Fig. 10(a) the values reached by the energies are about 40 times smaller than those obtained by the normal mode of Front 1. The most interesting point, however, appears in Fig. 10(b): unlike the mode superimposed on Front 1, the second, baroclinic stage does not occur. Instead the
barotropic conversion remains the only source up to day 2.5, which is the time of nonlinear saturation, and until this time the kinetic energy derived from the frontal jet supplies both reservoirs of the wave. Then, the barotropic conversion becomes once more a mechanism through which a loss of wave energy occurs without any significant compensating contribution from the frontal baroclinicity. Thus, this type of subsynoptic frontal wave decays in the same way as large-scale baroclinic waves, by increasing the basic-state kinetic energy. What is specific in their case is that the same energy conversion term initiates and sustains their growth. But then, depending on the mode or frontal structures—an important point that will be clarified in section 6 (a)—some of these waves can also derive energy from the sloping frontal isentropes, but others cannot, and simply break up the frontal flow near the surface. Only the frontal waves that go through the two-stage-growth process can deepen significantly. Exactly the same behaviour, with very little quantitative difference, is observed when the model is initialized with the SG normal modes already added to the front.
It is now interesting to translate the meanings of these few numbers into structural changes, especially in the vertical. This can be done by looking at the evolution of both the basic front and the wave correlations. But before this, a new upper boundary condition must be introduced to avoid the unrealistic effects of the upper boundary that the domain average, used for the energy analysis, easily smoothes out.

(c) A more realistic upper boundary condition

The upper boundary condition employed in the experiments of sections 5(a) and 5(b) is a pseudo-rigid lid defined by \( w = 0 \) at the top. This artificial tropopause leads to unrealistic structures at the upper levels in the case of nonlinear simulations with Front 1. In particular, a region of marked static instability appears in the top levels as early as the first day of integration (not shown), giving an unrealistic circulation at this height. Values of PV increase considerably in the upper 2 km from about 0.5 PVU, initially, to values greater than 5 PVU at day 4.

The interest in a top layer that may serve to absorb the upward propagating waves is then obvious. The basic state used in this section now possesses a supplementary layer above the troposphere, having a depth of 8 km and characterized by strong static stability. At the tropopause, the potential temperature profile of the original Front 1 is imposed \( (\theta_{\text{imposed}}) \); otherwise, the internal troposphere (Front 1) is untouched. The height of the tropopause is calculated using an iterative version of the semi-geostrophic invertibility method.

The linear analysis of the resulting basic state, done with the SG model, gives an instability maximum at a larger wavenumber than in the case with a rigid lid \( (4 \times 10^{-6} \text{ m}^{-1} \text{ as compared with } 6 \times 10^{-6} \text{ m}^{-1}, \text{ see Fig. 1}) \). The barotropic mechanism in the energy cycle of the most unstable mode is stronger than in the case without a tropopause. However, the mode selected to be superimposed on the basic state for the following nonlinear simulation has the same wavenumber as in section 5(a), for the sake of continuity. Its growth rate is also very close to \( 10^{-2} \text{ s}^{-1} \).

The observed nonlinear evolution near the ground, in fact, depends little on the presence of the modified upper boundary condition. With the more realistic tropopause and 40 levels, saturation is also reached between 4.5 and 5 days. The deepening of the low is only 7 mb (against 10 mb for the experiment of section 5(a)), but the maximum surface wind increases from 23 m s\(^{-1}\) to 30 m s\(^{-1}\) at day 5 in both cases. Shapes and maximal values of the low-level fields of \( \theta, \zeta \) and PV vary very little when the upper boundary condition is modified. Frontogenesis remains active below the tropopause but is without the unreasonable amplification instability for the results of section 5(a) (Fig. 11).

(d) Fluxes—basic state evolution

The main findings of this study, revealed in section 5(b), and of course still valid for the simulation with a tropopause, are now examined from the point of view of their effect on the vertical structure of the wave and of the initial front. Fluxes \( u'v', u'\theta' \) and \( w'\theta' \) are considered only through their means along the front, i.e. as the correlations \( u'\bar{v}', u'\bar{\theta}' \) and \( \bar{w}'\bar{\theta}' \)—all functions of \( x, z \) and \( t \).

Until day 2, the amplitude of \( \bar{u}'\bar{v}' \) increases and the zone of maximum correlation remains large in the frontal region, near the surface, gaining slowly in width and depth. The situation during the phase of barotropic growth remains similar to the normal-mode organization. Between days 2 and 3, the absolute value of \( u'v' \) starts to decrease rapidly at low levels and \( \partial\bar{V}/\partial x \), the frontal shear, flattens suddenly, indicating that the 'barotropic saturation' stage has been reached.
The correlation $\bar{u}'\bar{\theta}'$ is initially localized at low levels in the frontal region; but during the phase of baroclinic growth, it extends rapidly throughout the depth of the troposphere (Fig. 12) and widens, but only in the 'cold front' region. In consequence, the gradient of $\bar{\theta}$ decreases at low level at day 2, an effect which, with time, propagates significantly towards the tropopause. This is different from the synoptic-scale baroclinic waves, where the effect of the wave on the mean baroclinicity is limited to the lowest levels (see Simmons and Hoskins 1978). After day 3, when the barotropic source disappears, another signature of a classical baroclinic mechanism becomes visibly active: the direct circulation implied by the baroclinic mechanism with $w'\theta' > 0$. A zone such as this was not visible during the first two days, but appeared between days 2 and 3 west of the warm band and amplified significantly between days 3 and 4. At this final stage, the initial zone of negative $w'\theta'$ had vanished.
The final touch to this change in the structure is seen by returning to $u'v'$. The energy analysis revealed the importance of the loss of energy to the basic flow, but did not tell which characteristic of the initial jet was implicated. In fact, the perturbation reinforces the basic meridional wind westward and vertically downward of the initial upper-level jet, where $u'v'$ is strongly positive from day 3 onwards (Fig. 13). The jet is indeed deeper, with lesser gradients and magnitude at day 4. This reversal of the barotropic correlation soon occurs over the whole depth of the troposphere.

The evolution of the basic state during the life cycle can be summarized as follows. The initial 'front' is weakened and moves westwards: it is a continuous change but the most significant step is made between days 2 and 3. The frontal $\bar{\theta}$-gradient is then reduced to 0.8 K/100 km and the $\bar{\theta}$-field is modified over a depth of 7 km. All these changes are
summarized in the evolution of the along-front mean potential vorticity. The initial anomaly has its cross-front width widened from 130 to 400 km at day 5, and its amplitude reduced. The reduction in the mean amplitude is the mark of a significant roll-up of the initial band of high PV.

The mode along Front 2 exists, only thanks to the barotropic correlation and frontal horizontal shear. Up to day 2, this process gains efficiency and depth, with an overall structure similar to that of the normal mode. However, for a similar gradient of $\frac{\partial}{\partial t}$, the correlation $u' v'$ has a minimum of $-0.4$ m$^2$s$^{-2}$ at day 3, against $-16$ m$^2$s$^{-2}$ in the above experiment. Between days 2 and 3, the surface frontal zone that initiated the growth becomes a region of loss of energy with shallow, positive correlations of $u' v'$ in a region of positive shear, while a zone of negative $u' v'$ persists at mid level. The internal conversion remains always negative, without penetrating to the upper levels and, just in the same way as the barotropic correlation, it becomes cut off from the surface after a while. However, a region of conversion from $A$ into $A'$ appears in the cold frontal region in the lower half of the troposphere (Fig. 14), but seems unable to extend throughout the depth of the domain. At day 4, the perturbation starts to give energy back to the mean state. Because the perturbation remains trapped in the lower troposphere throughout its life cycle, the new upper boundary was not required here.

![Figure 14](image)

Figure 14. As in Fig. 12 but the nonlinear experiment with Front 2, after 72 hours. One third of the total domain is shown, with tick marks every 200 km horizontally and 2 km vertically. $u' \theta'$ every 0.03 m s$^{-1}$ K (solid lines, negatives contours dashed). $\theta'$ (dotted lines, contours every 2 K).

If observed over the whole life cycle, the mean frontal wind is seen to be slightly modified only at low levels. The PV anomaly extends from 110 to 185 km—the largest step in a continuous widening from a characteristic width of 80 to 210 km in 3 days. Most remarkably, the mean $\theta$-field is only slightly modified right at the surface front; elsewhere it is not changed at all.

6. Sensitivity Experiments on the Evolution and Nonlinear Saturation of the Waves

This section describes a number of experiments that were designed to understand the key features which in the initial conditions or in the basic state determine the behaviour of the waves, with emphasis on the mechanisms for growth and the control of the final equilibrated state.
(a) The respective influence of front and mode on the growth mechanisms

The main finding of section 5 lies in the two different development histories of the waves along Front 1 and Front 2. The first undergoes a two-stage growth period and achieves a reasonable, if 'ordinary', deepening. The second, on the other hand, remains purely barotropic (in the energy sense) and does not show any significant pressure signature. In this subsection, two experiments are presented with the idea of distinguishing what in this result is due to the basic states, and what originates from the waves.

The distinctive features of each experiment are as follows. Front 1 possesses a large horizontal gradient of potential temperature, reaching 1.8 K/100 km, together with the large vorticity accompanying the potential vorticity anomaly. In this case, the kinetic energy reservoir represented by the frontal jets is supplemented by an important potential energy reservoir. Front 2, on the other hand, is partly deprived of the latter, with a maximum horizontal gradient of potential temperature approximately half of that in the former case.

The mode growing along Front 2 (also called hereafter Wave 2) can be regarded as the prototype of a perturbation capable of retrieving energy from a region of large horizontal shear—a nice example of a barotropically unstable mode in a moderately baroclinic environment. The deep structure of the mode growing along Front 1 (rather than the main energy cycle), or Wave 1, reveals a potential for this kind of wave to interact with the baroclinic zone, mostly at upper-levels and also with the shear near the surface.

Because only normal modes have been considered thus far, it seems that wave and frontal structures are closely related: one may wonder how a finite amplitude shallow wave sensitive to horizontal shear will behave in a baroclinically stronger front, or what a deep perturbation, also with a finite amplitude, can do on a front having strong shear and weak horizontal temperature gradients. To answer these questions, the PE modes and fronts have been interchanged. To begin with, Wave 2 with an along-front wavelength of 800 km is superimposed on Front 1. The time evolution and deepening are very similar to what occurs in the experiment with Wave 2 in its original environment. This perturbation thus appears unable to take advantage of the increased baroclinicity (Fig.

![Figure 15](Image)  

Figure 15. Same as Fig. 9(b), but the modal wave along Front 2 (Wave 2) with an amplitude of \( u \) of 0.5 m s\(^{-1}\) is used as a non-modal initial condition perturbing Front 1. Note that the energy scales are different. This figure is directly comparable to Fig. 10.
15). When Wave 1 ($L_y \approx 1800$ km) is used as a non-modal initial condition on Front 2, both the barotropic and baroclinic development phases are observed. However, the resulting wave grows more slowly and the maximum deepening is reduced to 5.7 mb at day 7, the maximum deepening rate being 1.9 mb/24 hours between days 2.5 and 3.5.

The interpretation of this exchange of initial conditions is clear. The basic energy-retrieving mechanisms are attached to the waves themselves depending on their structure and wavelength, which explains the qualitative similarity of their behaviour. The frontal structure, on the other hand, depends on the quantitative aspects such as the rate of development, the equilibrated state, and the maximum amplitude reached. The mean shear is weaker along Front 1 than along Front 2, which explains the less intense development. Similarly, elementary arguments based on baroclinic instability theory indicate that the growth rate scales as $(g/\theta_0)(\partial \bar{\Theta}/\partial x)/N$. The reduction of baroclinicity when going from Front 1 to Front 2 explains quite well the slowness of the baroclinic phase followed by the transplanted mode.

These experiments suggest the following generalization of the result of section 5. While a deep initial perturbation can hope to interact with the surrounding baroclinicity, a shallow surface-trapped, short wavelength barotropically unstable wave can not. This way of looking at barotropic instability, through its effect (or rather the absence thereof) on the pressure field in terms of the resulting deepening, will be returned to later.

(b) The controlling factors of the properties of the saturated waves

Both frontal waves' modes show themselves capable of inducing a major reorganization of fields like wind or potential temperature in the vicinity of the surface front. Seen from the point of view of a deepening within a low-pressure centre, however, our results look much less dramatic. Only the longest wavelength perturbation achieves a moderate deepening of 10 mb, with the major growing phase persisting for more than two days. Although these results fit the existing data on cyclogenesis at all scales (that is, not limited to the larger-scale cyclones), these pressure changes would not be regarded as significant by forecasters whose attention is mostly drawn towards depressions deepening at rates of 20 mb/24 hours up to 50 mb/24 hours and more. Such developments, according to them, have a timescale of about a day; it is therefore of interest to explore a number of more or less elaborate changes to the initial conditions and to study their effect on the solution at the time of nonlinear saturation to find if they are able to accelerate and/or amplify the deepening.

First, consider numerical aspects. It was observed that the results are not influenced by the use of a higher resolution. Furthermore, the effect of increasing the horizontal diffusion reduces the pressure fall by less than 5% at saturation time. The effects of having more physical changes in the model or initial conditions are now discussed.

(i) Initial normal-mode amplitude. The normal mode superimposed on the basic state may be characterized by its maximum across-front wind amplitude. Starting with a stronger mode amplitude should reduce the developing time of the cyclone; however, it should be kept in mind that if their amplitude is taken too large the initial conditions will contain some kind of finite amplitude perturbation which may draw on its own potential to grow, so violating linear theory. The reference experiment is made with a maximal wind, $u'$, of 0.5 m s$^{-1}$; amplitudes of 1 m s$^{-1}$ and 2 m s$^{-1}$ were also tried. For the strongest $u'$, saturation is reached a day sooner than for the reference; however, in all cases, the strongest deepening occurs after day 4. The structure of the perturbation at the time of saturation does not change, and the pressure perturbation changes by less than 1 mb with $u'(t = 0)$ varying from 0.5 m s$^{-1}$ to 2 m s$^{-1}$. 

(ii) Initial reservoir of energy. Perturbations reach saturation when they have drawn on all the available energy stored in the basic state. The question now is what happens if the initial available energy is increased without modifying the initial normal mode. Initial conditions suitable for this set of experiments are defined as follows, the domain being elongated in the meridional direction. The previous initial conditions (front + normal mode) defined for a normal-mode wavelength are extended using the front alone. Simulations with Wave 1 and three different domain sizes were done with the initial meridional wavelength multiplied by 1.2, 1.5 and 2. The maximum deepenings are then 14, 18 and 25 mb, However, they are reached, respectively, at days 6.6, 8 and 12.5. So, clearly, when the disturbance has more available energy to grow, the maximum deepening is actually increased, but the period of growth also increases; the waves do not grow faster.

(iii) Upper-level finite-amplitude vortices. The effect of introducing non-modal vortices artificially at the top of the domain above Front 1 (without the normal mode being superimposed) was tried, the idea being that it might be worth while taking account of the role of finite amplitude perturbations. Nonlinear simulations were run with a set of anomalies having a geopotential of Gaussian functional form, characterized by their amplitude, horizontal extension, depth and phase-shift in the along-front direction.

In every case, the solution of section 5(a) emerged. The total deepening of the surface pressure system and the wind increase are quite similar to what occurs in the normal-mode experiments. With anomalies of amplitude 500 J kg\(^{-1}\), the development remains slower than with the normal mode. So, the normal-mode solution seems very robust.

(iv) Experiments with latent-heat release. Latent-heat release during cyclogenesis has often been quoted as a mechanism able to enhance drastically the growth of a weaker system. Emanuel et al. (1987) have shown that the effects of latent heating in the case of linear theory are, in fact, relatively limited. It seemed logical enough to consider the effect of latent heating on our waves, given the fact that this process has been used to model a potential vorticity anomaly at the origin of the instability at reduced scales.

For these experiments an initial distribution for specific humidity must be added to the frontal basic state and the small amplitude normal mode. It is supposed that the air over the frontal zone is saturated within a band situated above the potential vorticity anomaly. Condensation is represented by the energy conserving scheme, including the evaporation of falling rain, used in the operational models in France. Here as in the former experiments, the impact is small. In the case of Front 1, the deepening is increased by 2 mb, and nonlinear saturation occurs at the same time as before; the surface wind, on the other hand, is increased by more than 5 m s\(^{-1}\). The effect on Front 2 and its normal mode is even weaker.

7. Discussion

Barotropic instability and pressure deepening

While the nonlinear simulations showed that both types of frontal wave evolved into neutral circular lows, a large difference appeared between the development of Front 1 and Front 2, the former deepening by 10 mb while the latter experienced only a reorganization of the wind field. It was then shown, in section 5(b), that the Front 1 and Front 2 evolutions clearly had two different life cycles in terms of energy. At the beginning of the nonlinear simulations, normal modes superimposed on Front 1 and also on Front 2 grow mainly through a 'barotropic mechanism'. After day 2 the two evolutions become
fundamentally different, one of them entering a second phase of growth and drawing available potential energy from the temperature gradient in the basic state. The baroclinic mechanism then seems to play an important role in the evolution of Front 1 from day 2 until nonlinear saturation.

So that this phase could be shown to be the really important one for getting any significant signature on the pressure field, purely barotropic experiments were set up. It has been shown by JT90 (see e.g. their Fig. 7) that in a stratified fluid, any significant instability (in terms of growth rate) is related to a large depth of penetration of the vorticity anomaly in the interior. However, we now show that this depth does not play any role in the nonlinear saturation process of the barotropic part of the development. Initial conditions for the barotropic experiments are extracted from the initial conditions of the PE simulations, at the level just above the surface, to be compared with the results of section 5. In the case of Front 2, the results are very close to those of the three-dimensional surface, the total deepening and the increase of wind having the same magnitude. The geopotential and vorticity fields have structures similar to those shown in section 5(a) (see Fig. 17(c)). A shallow-water experiment with Front 1 also leads to a similar reorganization of the fields (Fig. 16), but the variations of pressure and wind are three times smaller than in the baroclinic simulations.

![Figure 16. Shallow-water simulation for Front 1 at 72 hours. One half of the domain in the across-vorticity-strip direction is shown, with ticks every 500 km. Solid lines: absolute vorticity every 0.5f, the thickest line being the f-contour. Dashed lines: geopotential height, contour interval 200 J kg\(^{-1}\), the innermost contour being 760 J kg\(^{-1}\).](image)

The linear stability analysis revealed the importance of the barotropic conversion mechanism in the generation of sub-synoptic-scale frontal waves. Yet, the conclusion of the nonlinear simulations supported by this purely barotropic series of experiments is that this mechanism cannot lead to significant deepening of the perturbations. This is a property of finite amplitude barotropic instability that, to our knowledge, has not been pointed out so far.

Theoretical bounds on the growth of unstable waves have been proposed. In a recent stream of papers, in particular, there have appeared elegant discussions on the nonlinear saturation of balanced barotropic (Shepherd 1988a) and baroclinic instabilities (Shepherd 1988b, 1989). This approach is based on the nonlinear stability theorems of Arnol'd (1966). To summarize: when a flow is such that the necessary conditions for linear instability (Rayleigh 1880 for a balanced barotropic flow, Charney and Stern 1962 for a
baroclinic one) are not met, it is in fact nonlinearly stable in the sense that the norm of the amplitude of any perturbation is bounded by some finite number proportional to its initial value. By constructing a family of stable basic flows related to unstable ones, rigorous bounds on the amplification of unstable waves can be deduced. The main application of these ideas is in the turbulent breakup of vorticity strips and the parameterization of transient eddies. A result directly applicable to the present analysis states that, in the barotropic case, in the absence of any $\beta$-effect, the breakup of a vortex strip or sheet can be prevented only by boundary conditions (periodic ones, for example). Results relevant to the baroclinic case are mentioned at the end of this subsection.

Otherwise, deductions on the bounds of the deepening do not obviously follow from the work of Shepherd, for two main reasons. The first is that the derivation depends crucially on the wind being non-divergent; the second is that, in any case, some geometrical information on the structure of the final state is needed to translate the maximum increase of enstrophy (say) into a pressure decrease. In the remainder of this subsection, we reinforce the idea that a wave developing from barotropic instability cannot have a marked signature in pressure, by considering an extremely simple, ad-hoc model, and adding only minimal constraints to get the uppermost values of deepening. The idea is to relate given simple initial and final states by some of the basic properties of shallow-water dynamics.

The initial state is modelled by a band of anomalous relative vorticity in geostrophic equilibrium, independent of $y$. The vorticity decreases linearly from $\zeta_0$ at $x = 0$ to zero at $x = \pm l$. The length of the band is $L$ (the period in our experiments). The geopotential away from the band is $\phi$, where relative vorticity is zero.

On an unbounded $f$-plane, according to Shepherd (1988a), the vortex completely breaks up the high-vorticity band before nonlinear saturation, and so provides us with a final state in which all the high potential vorticity is rolled up into an isolated, stationary, vortex. This extreme case overestimates the deepening since, in our two-dimensional experiments, complete breakup is prevented by the doubly-periodic boundary conditions.

Again, a linear decrease of relative vorticity from the new maximum, $\zeta_r$, at $r = 0$ to $\zeta = 0$ at $r = R$ is postulated, viz.

$$\zeta = \zeta_r (1 - r/R).$$

The real difference from the former case is that the vortex has to accommodate the geometry, as we have now assumed that there is gradient-wind equilibrium. The effect of the stretching allowed by shallow-water dynamics is represented by the expression

$$E = \frac{\zeta_r + f}{\zeta_x + f}.$$

What are the constraints that determine (and limit) its value? The first property summarizing the change from initial to final state, from the shallow-water dynamics, is the conservation of a form of potential vorticity i.e.

$$Q = \frac{\zeta + f}{\phi}$$

which, in this model, is a Lagrangian conservation property. The maximum potential vorticity, in particular, should be identical in both states (in the absence of any diffusive process). Potential vorticity, $Q$, is maximum, in both distributions, at $x = 0$ and $r = 0$, respectively. Values of the geopotential in those particular locations are, respectively, $\phi_{x0}$ and $\phi_{r0}$. This can be written, using the equilibration assumptions, as:
\[ Q_{\text{max}} = \frac{\zeta_x + f}{\phi - f \zeta_x l^2 / 6} = \]
\[ = \frac{\zeta_r + f}{\phi - (\zeta_r / 12) (\zeta_r / 2 + 5f / 3) R^2}. \]  

Equation (5) expresses the conservation of the product \( \zeta S_0 \), where \( \zeta \) is the mean vorticity over the area \( S_0 \) of the domain. This property is very similar to the circulation theorem (see e.g. Gill 1982, 7), except that it pertains here to a fixed domain instead of to a material circuit. This is necessary because the circuit formed by outer parcels is initially open. Applied to our model’s initial and final states, this becomes

\[ \zeta_x \mathcal{P} l = \frac{\pi}{3} \zeta_r R^2. \]

Equations (5) and (6) determine \( \zeta_r \) or \( E \). It is then possible to determine the maximum deepening \( \phi_{r0} - \phi_{x0} \). Table 1 displays some of the parameters deduced from these basic assumptions when applied to waves similar to those considered here. As expected, the decrease in vorticity and pressure is overestimated. The overestimation of the deepening would have been even greater if the balanced vorticity equation had been used (for which \( E = 1 \)), as they are in the upper-bound theorems. However, it seems reasonable to accept that Eqs. (5) and (6) give a simple summary of the minimal constraints determining the growth of a finite-amplitude vortex in a barotropic context. The addition of further limiting constraints, like the consideration of the finite extent of the domain, would bear directly on the final radius \( R \), which, here, is barely limited; however, reducing \( R \) would also immediately reduce the deepening. The simple model thus provides the extreme values of deepening that can be expected from barotropic instability, which are significantly smaller than 10 mb for the sub-synoptic range of scale.

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<tr>
<th>( \xi_x - f ) (f) ( \phi_{x0} ) (J/kg) ( l ) (km) ( \Delta \phi/8g ) (mb) ( k ) (km) ( u_{\text{max}} ) (m/s) ( \xi_r - f ) (f) ( E ) (J/kg)</th>
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Consideration of the equations governing the simple model indicates that, apparently, the decrease of vorticity and increase of radius allow for the simultaneous deepening and conservation only for a small ratio $f/\zeta R$. The changes in vorticity and radius must balance each other so as to leave $\zeta_0 R^2$ constant and conserve the circulation; however, the geopotential decreases as $\zeta_0^2 R^2$ because of curvature. Below a certain 'equilibrium vorticity', the pressure would decrease too rapidly for a given change of vorticity for potential vorticity to be conserved. This comes ultimately from the fact that air parcels are trapped in the vortex. To allow for convergence, the vortex must increase the curvature of its free surface. In this context, the role of baroclinicity can be viewed as allowing for changes in the circulation, but our results show that this is possible only if the perturbation is not fully trapped at the lower boundary. Another way of relaxing these constraints is to change the potential vorticity, which, in fact, is the role of the diabatic processes.

Thus, the simple ad hoc vortex model provides a rough interpretation of the lack of pressure change when a sub-synoptic wave grows solely because of the barotropic conversion process. It shows that, even in extreme barotropic events, when all the anomalous potential vorticity is concentrated within the low—something that is prevented in the cases simulated here—the pressure fall is small or moderate. Therefore, it is only if the perturbation becomes able to interact with the frontal temperature gradient that, through the baroclinic conversion, it may amplify in the more traditional manner of pressure deepening.

The experiments of section 6(a) indicated that the possibility of such an interaction depends on the wave rather than on the front. The distinctive features of Wave 1 are its longer wavelength and the fact that it is a deep mode, while Wave 2 is a small-scale and shallow mode. Green (1960) provided a parcel interpretation of baroclinic instability, expanding an earlier idea by Eady (1949), which is useful here. Potential energy from the sloping isentropes of a baroclinic zone can be converted to kinetic energy provided the parcel moves within a 'wedge of instability' so that the gain in potential energy more than compensates for the loss due to the work done against gravity. This idea leads to conditions on the slopes of the parcel trajectories, which is best translated in terms of scale. Then, (see e.g. Pedlosky 1987, 7.6), if $L$ is the horizontal scale of the wave and $L_R = N_0 H/f$ is the first internal radius associated with the isentropes sloping throughout the depth, with $N_0$ the background potential vorticity scaled as a buoyancy frequency, the parcel argument suggests that baroclinic instability is efficient when

$$L_R^2 \ll L^2.$$  

In our case, $L_R \approx 850$ km. This simple argument agrees well with the fact that Wave 1, with $L \approx 900$ km, does interact with the isentropes, while Wave 2, which has $L \approx 400$ km, does not. This suggests that, for the baroclinic part of the energy exchanges, the quasi-geostrophic scaling still holds in our complex basic states.

In conclusion it is worth returning to the study of nonlinear saturation of quasi-geostrophic baroclinic instability by Shepherd (1989), who also indicated that deep and shallow perturbations behave differently from each other. The limit on the growth of deep waves is simply the general conservation of potential enstrophy. For shallow waves, the potential enstrophy 'is bounded well below the total amount available'. The scaling defining the 'depth' of the waves is proportional to $\beta^{-1}$, where $\beta$ is defined by the horizontal changes of $f$, but is used, in fact, for scaling the horizontal changes of potential vorticity. Strictly speaking, in our simulations $\beta$ being zero all our waves are 'deep' ones; yet, our results are in rough agreement with Shepherd's, if it is accepted that $\beta$ be
replaced by a scale representative of the spatial change of the potential vorticity anomaly. Note, furthermore, that Shepherd insisted on the fact of there being no limit on the neutralization of surface potential temperature gradients: 'boundary' baroclinic instability ceases only when these are removed, as we observed to be the case at the lower boundary.

8. CONCLUSION

All these facts together point towards the following conclusion that a frontal wave with length scale smaller than the Rossby radius is a shallow wave, and that it can draw energy only from the wind shear; as a result, a wave such as this will be barely visible on a pressure chart. A frontal wave with length scale larger than the Rossby radius will have a signature reaching the upper levels, and will be able to derive energy from the sloping isentropes, so deepening significantly.

These conclusions agree with other theoretical studies, even though heretofore they have not been given much attention. Peltier and Polavarapu (1991) have shown that the shallow mode of Moore and Peltier (1987) does not amplify significantly in the nonlinear regime. The modes of Schär and Davies (1990) are, by construction, shallow waves whose deepening can be deduced from their Fig. 10 (depending on the dimensional constants, it is of 3 to 4 mb for the 'wide strong front' setting and 0.7 to 1 mb for the 'narrow weak front' setting). Our main conclusion seems to encroach on the non-normal-mode studies, since the deep wave studied by Thornicroft and Hoskins (1990) leads to a moderate deepening of 10 mb—very much like our own deep mode. In observational studies, our views are well represented by those of Reed (1979). On the basis of case-studies of polar-low developments, he found that the more intense lows have deep structures.

There is also another important conclusion to be noted. There is remarkable similarity in the horizontal low-level structure of all these waves (Fig. 17). From the large-scale baroclinic waves, as simulated by many authors, down to our last barotropic results derived from Wave 2, all display the formation of zones of enhanced vorticity in the same areas and in the same chronological order. This suggests that, for this kind of 'slow' dynamics, largely influenced by the earth's rotation, there exists strong invariance of finite-amplitude structures in terms of shapes. This similarity is to be contrasted with a variety of energy-converting mechanisms. In other words, the scales and shape of a mid-latitude cyclone observed superficially in temperature, pressure or even vorticity do not tell how this cyclone is growing or which mechanism is behind it. Ideas about the dynamics start to emerge only by studying higher-order fields like potential vorticity or fluxes.

An idea of the relevance of the idealized frontal waves discussed in this paper to an observed example is given by Fig. 18, which displays a comparison between the UK Meteorological Office's fine-mesh model analysis and the idealized simulation of Front 1. The structures observed on both low-level PV charts are remarkably similar and typical of these common non-explosive frontal waves. There is certainly, however, a lack of recent direct measurements within waves forming along primary cold fronts on the eastern side of storm tracks: this could be the objective of a future experiment.

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Figure 17. Example of similarity of frontal structures at different scales: Surface absolute vorticity at the saturation time for (a) a PE simulation from the Hoskins and West (1979) basic state (along-front wavelength $L = 4000 \text{ km}$), contour interval $f$, (b) a PE simulation for Front 1 ($L = 1800 \text{ km}$), contour interval $0.9f$, (c) a shallow-water simulation for Front 2 ($L = 800 \text{ km}$). Contour interval $0.5f$. Note the similarity of structure, as opposed to a variety of energy retrieving mechanisms for growth.

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**APPENDIX 1**

*Preparation of data from the SG model for the PE model*

The following explains how the PE model has been initialized with the SG normal modes in a number of experiments. The same operations were used for moving the
frontal basic states to the PE model before the normal modes were extracted. Transformation from geostrophic space back to real space is the first step in this sequence. It is essential for the method employed for this task that it be accurate enough so that the small amplitude normal mode is not masked by numerical errors. In order to avoid errors quickly nearing 20%, it is necessary to take into account the geostrophic wind perturbation in the transformation formula, viz. $X = x + V_y f + \frac{v_y f}{f}$, $Y = y - \frac{u_x f}{f}$. This system is solved by Newton's method with two unknowns. It allows the recovery of $(X, Y)$ knowing the $(x, y)$ coordinates of the given real-space gridpoint.
After the space transformation, the perturbation is taken to be geostrophic. This approximation simply means that the ageostrophic circulation \((u'_e, v'_e, w')\) is set to zero initially. This kind of dynamical initialization goes back to the beginning of numerical weather prediction, but is effective.

Further important sources of errors are eliminated by transforming the geopotential alone. It is projected on to the new spectral space and finally vertically interpolated. Afterwards all the other fields are derived on each new level using the PE model discretization.

Following the steps indicated above, a negligible ageostrophic wind was present in the initial conditions of the nonlinear integration. A final remark about the ‘geostrophic initial perturbation’ approximation: it is, in principle, possible to re-compute the ageostrophic circulation from the other fields. To that end, the linearized equations in physical space could be solved diagnostically; but, it is easier to let the primitive equation model itself restore the ageostrophic wind. This is the case if this geostrophic forcing on temperature is not zero initially; which can be checked on \(Q\) vector maps, where \(Q = \nabla \theta / \partial t\). As a result, the early nonlinear development of an SG mode is slower than its PE companion.

**APPENDIX 2**

**Definitions used for energy diagnostics**

Although ARPÈGE is a hybrid vertical coordinate model, all energy diagnostics were computed on files containing data post-processed on pressure levels. The diagnostics were, therefore, formulated using the pressure coordinate. Using the overbar definition given in section 4 (defining the mean value along the wavelength at a given \(x\) and \(p\)), the mean over the whole domain is defined as

\[
\langle Q \rangle = \frac{1}{L_x L_z \int_0^{P_{\infty}} \int_{P_{\infty}}^{P_{\infty}} Q(x, p) \, dx \, dp.
\]

Single quantities with an overbar correspond to an instantaneous basic state, and primed quantities to the departure from this basic state. The terms that are related to the change in time of the basic state are not considered in this study. Then, the kinetic and potential energy of the waves are given by

\[
\langle K' \rangle = \frac{1}{2} \langle u'^2 + v'^2 \rangle, \quad \langle A' \rangle = \frac{1}{2} \left( \frac{T'^2}{\overline{S}} \right) = \frac{1}{2} \frac{g^2}{[\theta]^2} \langle \theta'^2 \rangle,
\]

where \([\theta]\) and the related \([T]\) are reference profiles of potential temperature and temperature, respectively, which depend on \(p\) only. The static stabilities are

\[
\overline{N} = \frac{g}{[\theta]} \frac{\partial \theta}{\partial z}, \quad \overline{S} = \frac{[T]^2}{\overline{N}^2} \overline{N}^2 = \frac{\overline{T}}{c_p} = \frac{R}{\partial p}.
\]

where \(R\) and \(c_p\) are the perfect gas constant and specific heat capacity of dry air respectively.

Rescaling the pressure vertical velocity of the model as

\[
w' = -\frac{R}{p g} \omega',
\]
the following equations are easily derived:

\[ \frac{\partial}{\partial t} \langle K' \rangle = -\left\langle u'v' \frac{\partial \bar{V}}{\partial x} \right\rangle - \left\langle \omega' \frac{\partial \bar{V}}{\partial p} \right\rangle - \frac{R}{p} \langle \omega' T' \rangle, \]

the last term, the internal conversion, being also \( + \left( \frac{g}{\Theta} \right) \langle w' \theta' \rangle \). On the right-hand side, the first two terms are the horizontal and vertical components of the Reynolds stress terms, the first also being called the barotropic conversion term. For the potential energy,

\[ \frac{\partial}{\partial t} \langle A' \rangle = -\left\langle u'T' \frac{\partial \bar{T}/\partial x}{\bar{S}} \right\rangle + \frac{R}{p} \langle \omega' T' \rangle, \]

which is also equal term by term to

\[ \frac{\partial}{\partial t} \langle A' \rangle = - \frac{g^2}{[\theta]^2} \left\langle u' \theta' \frac{\partial \bar{\theta}/\partial x}{\bar{N}^2} \right\rangle - \frac{g}{[\theta]} \langle w' \theta' \rangle. \]

The first term on the right-hand side is the baroclinic conversion term. The \( (T', \bar{S}) \) form is used for the calculations, while the alternative form is useful for comparisons with, for example, JT90.

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