On the dynamics of convection waves

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(Received 18 December 1991; revised 16 November 1992)

SUMMARY

The linearized system of atmospheric equations is solved analytically for a two-layer model to study the characteristics of convection waves. The solutions show that convection waves induced by a disturbance are composed of trapped waves with selected wave numbers that are determined by the interaction between the convective boundary layer and the overlying stable layer. The explicit relation of wave-number selection defines the pattern of the vortices in the convection waves, which may appear as cloud streets, cellular convections or convective thermals. The wave-number selection explains the formation of longitudinal and cellular convections and the transition between them. The analytical solutions of the convection waves reveal, theoretically, the structure of the large vortices in the convective boundary layer, and bring to light the effects of the upper stable layer on these vortices.

1. INTRODUCTION

Widespread gravity waves, known as convection waves or thermal waves, have frequently been found to exist over the convective boundary layer. Three kinds of atmospheric convective activity discussed in the paper are related to these waves: organized thermals, convective cloud streets, and mesoscale cellular convections. In the present study a common form of analytical solution for all three phenomena will be sought and the atmospheric conditions for the different types of convection will be discussed.

The thermal waves or convection waves, long recognized by glider pilots (Jaechisch 1968), have been investigated in observations (Kuettnner 1971), in laboratory tank experiments (e.g. Deardorff et al. 1969) and in numerical simulations (Clark et al. 1986; Hauf and Clark 1989; Mason and Sykes 1982). In the National Center for Atmospheric Research (NCAR) Convection Waves Project (Kuettnner et al. 1987) the aircraft measurements over a flat land surface in convective conditions revealed that the frequency of the wave activities ranged from 67% to 90% of the time. According to the observations, these convection waves had horizontal wavelengths of 5 to 15 km, vertical motion amplitudes of 1 to 3 m s⁻¹, and extended vertically throughout the whole troposphere and even part of the stratosphere. Gravity waves have also been studied theoretically. Townsend (1966) demonstrated that convection impinging on a capping inversion can excite horizontally propagating gravity ripples, while the study by Stull (1976) showed that penetrative convection stimulates both horizontally propagating interfacial waves at the inversion and vertically propagating waves in the stable layer above the inversion. In their two-dimensional numerical experiments Clark et al. investigated the interaction between the convective eddies in the boundary layer, and gravity waves in the overlying stable layer. Their simulation showed that the gravity waves, initially excited by the boundary-layer convection, acted as a feedback mechanism to organize the convective activities, and to tune up the spacing of the convective eddies. They came to the conclusion that the overlying stable layer plays an important role in the organization of the convective activities in the boundary layer.

Convection waves frequently occur over the oceans in addition to over flat land surfaces. The regular convective cloud patterns over vast ocean surfaces, which are often

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seen in satellite images, suggest that the formation of these clouds must be controlled by wave activities. These convective cloud patterns can be either two-dimensional cloud streets oriented along the mean wind direction, or three-dimensional cellular cells. Both patterns of cloud are usually associated with the flow of cold air over warm water, such as mid-latitude cold-air outbreaks.

Convective cloud streets have been extensively studied observationally (Kuettnner 1959; Brown 1980; Le Mone 1973) and numerically (Mason 1985). A variety of instability theories, such as thermal instability of shear flow (Asai 1972) and inflection point instability (Brown 1970), has been developed to explain the formation mechanism of the convection cloud streets. In most of the studies, however, the layer-interaction has been left out of the consideration.

Three-dimensional cellular convection may form open or closed cloud patterns with diameters of 15–50 km and depths of 2–3 km (Agee et al. 1973; Agee 1987; Rothermel and Agee 1980, 1986). Gradual interchange of the cloud patterns between the streets and the cells as revealed by satellite images (Scorer 1986) suggests that they have similar mechanisms and conditions for their formation. As pointed out by Kuettnner (1971) in the theoretical study, the three-dimensional cells may be considered to be a special case of the two-dimensional ones. He derived an expression for the growth of a disturbance in a convective boundary layer and gave the relation between the convective-layer depth and the spacings of two- or three-dimensional cells. In his model a rigid lid was set as the top boundary layer, thus the effects of the overlying layer were excluded.

In previous theoretical studies either band convection or cellular convection was treated as a phenomenon restrained in the boundary layer. The interaction between the boundary layer and the overlying stable layer, which might account for the horizontal spacings of the convections, was not taken into consideration. Using a two-layer model, in which the lower layer is thermally unstable and the upper one is stable, the author derived an analytical solution (Sang 1991) for convective vortices. In the solution a disturbance at the interface between the two layers can excite gravity waves, propagating horizontally in the upper stable layer, and tuning up the convections in the downstream lower layer. The effects of the thermal and flow structures in the two layers on the orientation, spacing and intensity of the roll vortices were also discussed. At proper thermal and flow conditions the model produced a secondary circulation similar to that found in the convective cloud streets.

In the present study an analytical solution for a more general form of the convection waves (not only for convective cloud streets) is obtained. In the model the disturbance source is set at the bottom of the lower layer, like a thermal caused by solar heating. The wind direction shear is included in the solution. The forms of the solution are discussed in a wide range of the large-scale mean meteorological conditions. The formation conditions for different types of convection waves, such as cloud streets, cloud cells and organized thermals, are obtained. The exchange between the roll patterns and the cellular patterns, as observed from satellite images, is also discussed in terms of the alteration of the parameters in the solutions.

2. The model

It can be seen from satellite pictures that a certain pattern of convective streets or convective cells can last at least a few hours without changing their appearance and intensity. During the period of their existence the whole system presents a steady state. The convection bands neither disappear nor develop into deep convection. According to the observations (e.g. Le Mone 1973) the phase speed of the convection bands is much
smaller than the mean speed of the background flow. Then, in the total derivative, the
local change rate, $\partial/\partial t \sim c/L$, is much smaller than the advective derivative, $u \partial/\partial x, v \partial/\partial y \sim U/L$, since $|c| \ll U$, where $c$ and $U$ are wave phase speed
and mean flow speed, respectively, and $L$ is horizontal wavelength.

For an inviscid, non-rotating incompressible atmosphere, the three-dimensional,
linearized, steady-state atmospheric equations can be written as

\begin{equation}
U \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x}
\end{equation}

\begin{equation}
U \frac{\partial v}{\partial x} + V \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\end{equation}

\begin{equation}
U \frac{\partial w}{\partial x} + V \frac{\partial w}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - \frac{\rho g}{\rho}
\end{equation}

\begin{equation}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\end{equation}

\begin{equation}
U \frac{\partial \theta}{\partial x} + V \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = -\frac{\partial H}{\partial z}
\end{equation}

where $U, V, \overline{\theta}$ and $\overline{\rho}$ are the mean values of the wind components
along the $x$ and $y$ axes, the potential temperature and the density, and $g$ is the gravitational
acceleration. $u, v, w$ are the perturbed wind velocity components, and $p, \theta,$ and $\rho$ are the perturbed
values of the pressure, potential temperature and density.

The term on the right-hand side of Eq. (5) represents a thermal forcing. As noted
by Kuettner (1959), convective streets may originate from a heat source upstream, such
as an industrial area, a fire or a heat island. In this study the thermal forcing is assumed
to be convergence of temperature flux, $-\partial H/\partial z$, in which the flux $H = \overline{w} \overline{\theta}'$
(primes denote eddy components) decreases linearly with the height and quadratically with
the distance from its centre. Thus the flux can be written as the form

\begin{equation}
H(x, y, z) = H_0 (1 - z/h) \frac{a^2}{(a^2 + x^2)} \frac{b^2}{(b^2 + y^2)}
\end{equation}

if

$$-\infty < x < \infty, \quad -\infty < y < \infty, \quad z < h$$

and

$$H(x, y, z) = 0 \quad \text{if} \quad z > h$$

where $H_0$ is the flux value at the surface in the centre of the disturbed heating area, $h$ is
the depth of the boundary layer, and $a$ and $b$ are the horizontal measures of the area.

Each disturbed variable is assumed to consist of Fourier components in the wave-
number space

\begin{equation}
\varphi = \overline{\varphi} \exp(i k x + i l y)
\end{equation}

$$-\infty < l < \infty, \quad k \geq 0$$

where $i = \sqrt(-1), k$ and $l$ are the wave numbers along the $x$ and $y$ axes, and $\varphi$ can be
$u, v, w, p, \theta$ or $H$. The Fourier transform of $H$ has the form

$$\tilde{H} = \tilde{H}_0 (1 - z/h) a b e^{-ak} e^{-bl}.$$
From the Fourier transforms of Eqs. (1)–(5) and the potential-temperature equation, the equation for the Fourier components of the vertical velocity can be obtained as

\[
\frac{\partial^2 \tilde{w}}{\partial z^2} + \left\{ \frac{k^2 + l^2}{(U_k + V_l)^2} \frac{k^2 + l^2}{\partial z^2} \right\} \tilde{w} = \frac{k^2 + l^2}{(U_k + V_l)^2} \frac{gH_0}{h} \frac{\partial \theta}{\partial z} e^{-ak} e^{-b|l|}
\]

(8)

where the static stability \( \beta = -1/\rho_0 \partial \rho/\partial z \) in the fluid corresponds to \( 1/\rho_0 \partial \bar{T}/\partial z \) in the atmosphere. The term including \( \partial^2 U/\partial z^2 \) and \( \partial^2 V/\partial z^2 \) can be neglected since it is normally much less than the static stability term (Smith 1979).

The model consists of two layers with an unstable lower layer, a stable upper layer and an interface at \( z = 0 \). The \( x \) axis of the system is set along the mean wind direction of the lower layer. Thus in the lower layer we have \( V_1 = 0 \) and \( \beta_1 < 0 \). If

\[ -\beta_1 g/U_1^2 = n^2 > 0 \]

(9)

Eq. (8) in the lower layer, \( -h \leq z \leq 0 \), becomes

\[
\frac{\partial^2 \tilde{w}_1}{\partial z^2} - \lambda^2 \tilde{w}_1 = \frac{k^2 + l^2}{k^2} E \frac{\lambda^2}{\lambda^2} \frac{gH_0}{U_1^2} e^{-ak} e^{-b|l|}
\]

(10)

where \( E = gH_0/(U_1^2 \bar{U}_h) \) and \( \lambda^2 = (k^2 + l^2) (n^2 + k^2)/k^2 \).

In the upper layer, \( z > 0 \), we have

\[ \beta_2 g/U_2^2 = m^2 > 0 \]

(11)

and

\[
\frac{\partial^2 \tilde{w}_2}{\partial z^2} + (k^2 + l^2) \left\{ \frac{m^2}{(k + c)^2} - 1 \right\} \tilde{w}_2 = 0
\]

(12)

where \( \alpha = V_2/U_2 \). The subscripts 1 and 2 in Eqs. (9)–(12) represent the variables in the lower and upper layer, respectively.

The solutions of Eqs. (10) and (12) have the forms

\[
\tilde{w}_1 = A \cosh(\lambda z) + B \sinh(\lambda z) - \frac{1}{\lambda^2} \frac{k^2 + l^2}{k^2} E \frac{\lambda^2}{\lambda^2} \frac{gH_0}{U_1^2} \frac{\partial \bar{T}}{\partial z} e^{-ak} e^{-b|l|}
\]

(13)

\[
\tilde{w}_2 = C e^{-\mu z}, \quad \text{if} \ (k + c)^2 > m^2
\]

(14)

and

\[ \tilde{w}_2 = C e^{i\nu z}, \quad \text{if} \ (k + c)^2 < m^2 \]

(15)

where

\[ \mu^2 = (k^2 + l^2) \left\{ 1 - \frac{m^2}{(k + c)^2} \right\} \]

and

\[ \nu^2 = (k^2 + l^2) \left\{ \frac{m^2}{(k + c)^2} - 1 \right\} \]

The coefficients \( A, B \) and \( C \) are determined according to the boundary condition and the interface conditions. At the ground surface, \( z = -h \), the boundary condition is \( \tilde{w}_1 = 0 \), that is

\[ \tilde{w}_1 = 0. \]

(16)
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If the mean speeds across the interface are continuous, the first interface condition at
\( z = 0 \), according to Yih (1980), is

\[ w_1 = w_2, \quad \text{or} \quad \vec{w}_1 = \vec{w}_2. \]  

(17)

The second one is

\[ \frac{\partial \vec{w}_1}{\partial z} = \frac{\partial \vec{w}_2}{\partial z} + \frac{k^2 + l^2}{k^2} \gamma \vec{w}_1 \]  

where

\[ \gamma = \frac{g}{U^2} \frac{\Delta \bar{\rho}}{\rho} \quad \text{or} \quad \frac{g}{U^2} \frac{\Delta \bar{\theta}}{\bar{\theta}} \]

\( \Delta \bar{\theta} = \bar{\theta}_2 - \bar{\theta}_1 \) is the temperature jump across the interface.

Substituting Eqs. (13), (14) and (15) into Eqs. (16)–(18), we obtain the coefficients
\( A, B \) and \( C \), and then have the solutions of the Fourier components of the vertical
velocities. In the lower layer, the higher wave-number component of \( w_1 \) with
\( (k + a \ell)^2 > m^2 \) is

\[ \vec{w}_{1h} = \vec{w}_{1l} + \vec{w}_{\text{local}} \]  

(19)

where

\[ \vec{w}_{1l} = \frac{(E/(n^2 + k^2))a b e^{-ah} e^{-b\ell}}{\lambda \cosh(\lambda h) - \left(\frac{k^2 + l^2}{k^2} \gamma - \mu\right) \sinh(\lambda h)} \left\{ \lambda \cosh(\lambda z) + \left(\frac{k^2 + l^2}{k^2} \gamma - \mu\right) \sinh(\lambda z) \right\} \]

is the trapped wave component and

\[ \vec{w}_{\text{local}} = -(E/(n^2 + k^2))a b e^{-ah} e^{-b\ell} \]

the local disturbed component. The lower wave-number component of \( w_1 \) with
\( (k + a \ell)^2 < m^2 \) is

\[ \vec{w}_{1l} = \vec{w}_{1u} + \vec{w}_{\text{local}} \]  

(20)

where

\[ \vec{w}_{1u} = \frac{(E/(n^2 + k^2))a b e^{-ah} e^{-b\ell}}{\lambda \cosh(\lambda h) - \left(\frac{k^2 + l^2}{k^2} \gamma + i\nu\right) \sinh(\lambda h)} \left\{ \lambda \cosh(\lambda z) + \left(\frac{k^2 + l^2}{k^2} \gamma + i\nu\right) \sinh(\lambda z) \right\} \]

is the untrapped wave component.

The disturbed vertical velocity \( w_1 \) is then obtained from the integral of the components \( w_{1l} \) and \( w_{1u} \) with respect to the wave number \( k \) and \( l \),

\[ w_1 = \int_{-\infty}^{\infty} \left\{ \int_{0}^{k_c} \vec{w}_{1l} e^{ikx} dk + \int_{k_c}^{\infty} \vec{w}_{1h} e^{ikx} dk \right\} e^{ily} dl \]

\[ = \int_{-\infty}^{\infty} \left\{ \int_{0}^{k_c} \vec{w}_{1u} e^{ikx} dk + \int_{k_c}^{\infty} \vec{w}_{1l} e^{ikx} dk \right\} e^{ily} dl + \int_{-\infty}^{\infty} \left\{ \int_{0}^{k_c} \vec{w}_{\text{local}} e^{ikx} dk \right\} e^{ily} dl \]

\[ = \int_{-\infty}^{\infty} \int_{0}^{k_c} \vec{w}_{1u} e^{ikx} e^{ily} dk dl + \int_{-\infty}^{k_c} \int_{k_c}^{\infty} \vec{w}_{1l} e^{ikx} e^{ily} dk dl + \int_{-\infty}^{k_c} \int_{0}^{k_c} \vec{w}_{\text{local}} e^{ikx} e^{ily} dk dl \]  

(21)

where \( k_c \) is the cut-off wave number satisfying \( (k + a \ell)^2 = m^2 \).
The solution of \( w_1 \) can be divided into two parts: near-field solution and far-field solution. The untrapped wave
\[
\int_{-\infty}^{\infty} \int_{0}^{k_c} \tilde{w}_{1a} e^{ik_x x} e^{il_y y} dk dl
\]
and the local disturbance
\[
\int_{-\infty}^{\infty} \int_{0}^{\infty} \tilde{w}_{local} e^{ik_x x} e^{il_y y} dk dl
\]
are the near-field solutions. Since in their expressions the integrand includes the factor \( e^{-ak_x - bl_y} \) the main contribution of the integral comes from the small wave-number components near \( k = l = 0 \). So the near-field solutions are proportional to \((a^2 + x^2)^{-1}\) and \((b^2 + y^2)^{-1}\) (Scorer 1978). It means that the range of the flow fields influenced by this part of the solutions is limited within the horizontal extent of the forcing.

The trapped wave, which we are more interested in, is the far-field solution. It is
\[
w_{1t} = \frac{E}{(n^2 + k^2)^2} \left[ \frac{\lambda \cosh(\lambda h) + \left(\frac{k^2 + l^2}{k^2} \gamma - \mu\right) \sinh(\lambda h)}{\lambda \cosh(\lambda h) - \left(\frac{k^2 + l^2}{k^2} \gamma - \mu\right) \sinh(\lambda h)} \right] a e^{-ak_x y} e^{il_y y} dk dl
\]
where \( \text{Re} \) denotes the real part.

The integrand of (22) has singularities at which \( k \) and \( l \) satisfy the relation
\[
\lambda \cosh(\lambda h) - \left(\frac{k^2 + l^2}{k^2} \gamma - \mu\right) \sinh(\lambda h) = 0
\]
(23)
or approximately
\[
\lambda - \left(\frac{k^2 + l^2}{k^2} \gamma - \mu\right) = 0
\]
(24)
since in most atmospheric conditions \( \lambda h \) is large enough to satisfy \( \cosh(\lambda h) = \sinh(\lambda h) \). Then (22) can be integrated by the residue theorem. At first the integration is carried out with respect to \( l \) on a deformed contour which stretches from \(-\infty\) to \(+\infty\) along the real axis and along an upper half-circle with infinite radius. This involves half the residues at the singularities of the integrand.

\[
\nu_{1t} = \text{Re} \left[ \pi i \int_{k_c}^{\infty} \frac{E/(n^2 + k^2)}{G(k, l)} \frac{a e^{-ak_x y} e^{il_y y} \sin(\lambda(h + z))}{\sinh(\lambda h)} e^{il_y y} e^{ik_x x} dk \right]
\]
where
\[
G(k, l) = \frac{\partial}{\partial l} \left\{ \lambda \cosh(\lambda h) - \left(\frac{k^2 + l^2}{k^2} \gamma - \mu\right) \sinh(\lambda h) \right\}
\]
and all \( k \) and \( l \) satisfy the relation (23). After rearrangement we have
\[
w_t = -\pi \int_{k_c}^{\infty} \frac{E/(n^2 + k^2)}{G(k, l)} \frac{a e^{-ak_x y} e^{il_y y} \lambda \sin(kx + ly) \sin(\lambda(h + z))}{\sinh(\lambda h)} dk.
\]
(25)
Similarly we have the trapped wave part of the vertical velocity in the upper layer

\[ w_{2l} = -\pi \int_{k_c}^{\infty} \frac{\{E/(n^2 + k^2)\} a e^{-ak} e^{-b|l|} \lambda \sin(kx + ly)}{G(k, l)} e^{-\mu z} \, dk \]  

(26)

where \( k \) and \( l \) in (26) also satisfy relation (23).

This part of the solution includes the factor \( e^{-\mu z} \) in the upper stable layer. It means that the wave energy is trapped within the lower part of the atmosphere and the waves can only propagate downstream. In the real atmosphere the convection waves, i.e. the convective streets or cells, may extend for hundreds to thousands of kilometres and present clear trapped-wave characteristics. During their downstream propagation the trapped waves tune up the convective activities in the boundary layer and give the wavelength selection. Then the orientation and spacing of the convective patterns in the boundary layer are conditioned by the matching relation (23) of the wave numbers \( k \) and \( l \) in the gravity waves. Thus the discussion in the following paragraphs will be focused on the trapped gravity waves.

From Eqs. (1)-(5) we have the Fourier components of \( u, v \) and \( \theta \)

\[ \bar{u} = -\frac{ik}{k^2 + l^2} \frac{\partial \bar{W}}{\partial z} \quad \bar{v} = -\frac{il}{k^2 + l^2} \frac{\partial \bar{W}}{\partial z} \quad \text{and} \quad \bar{\theta} = -\frac{i\bar{W}}{Uk + Vl} \frac{\partial \bar{\theta}}{\partial z} \]

as well as the vertical displacement of the interface

\[ \bar{z}_e \bigg|_{z=0} = -\frac{i\bar{W}}{Uk + Vl} \bigg|_{z=0} \]

Integrating the above expressions with respect to \( k \) and \( l \), we obtain

\[ u_{1l} = -\int_{k_c}^{\infty} \frac{\pi \{E/k\} a e^{-ak} e^{-b|l|} \cos(kx + ly) \cosh(\lambda(h + z))}{G(k, l)} \sinh(\lambda h) \, dk \]  

(27)

\[ v_{1l} = -\int_{k_c}^{\infty} \frac{\pi \{E/k^2\} a e^{-ak} e^{-b|l|} \cos(kx + ly) \cosh(\lambda(h + z))}{G(k, l)} \sinh(\lambda h) \, dk \]  

(28)

\[ \theta_{1l} = -\int_{k_c}^{\infty} \frac{\pi E}{n^2 + k^2} \frac{\lambda \delta \bar{\theta}}{kU} a e^{-ak} e^{-b|l|} \cos(kx + ly) \sin(\lambda(h + z))}{G(k, l)} \sinh(\lambda h) \, dk \]  

(29)

\[ \bar{z}_e \bigg|_{z=0} = \int_{k_c}^{\infty} \frac{\pi E}{n^2 + k^2} \frac{\lambda}{kU} a e^{-ak} e^{-b|l|} \cos(kx + ly)}{G(k, l)} \, dk \]  

(30)

The other expressions, such as \( u_{2l}, v_{2l} \) and \( \theta_{2l} \) are omitted here.

It can be seen from the above expressions that the solutions are composed of wave components whose wave numbers are determined by the relation (23). The contribution of different components to the composite waves is not the same. Since the amplitudes of the components include the factors such as \( e^{-ak}, e^{-b|l|}, k^{-2}, k^{-1}, \) etc., they decrease rapidly with the wave numbers. So the main contribution to the composite waves comes from the wave-number domain near the cut-off wave numbers \( k_c \) and \( l_c \). That is, the pattern of the convection waves is to a large extent determined by \( k_c \) and \( l_c \), and then, through the relation (23) and \((k_c + \alpha l_c)^2 = m^2\), by the atmospheric conditions such as \( n, \gamma, h \) and \( \alpha \). Thus the pattern of the trapped convection waves, like that of mountain
lee waves, is more a function of the atmospheric conditions rather than that of the size and shape of the forcing. The parameters of the forcing, such as \(a, b\) and \(H\), mainly influence the amplitudes of the wave components.

3. Discussion

Equation (23) is an important relation between the wave numbers \(k\) and \(l\), which should be met for the trapped waves. It is determined by the parameters \(m, n\) and \(\gamma\), that is, the thermal stabilities \(\beta_1\) and \(\beta_2\), the mean wind speeds and wind turning \(U_1, U_2\) and \(V_2\) (or \(a\)) in the boundary layer, and the overlying stable layer as well as the temperature jump, \(\Delta \bar{\theta}\), across the interface. Various atmospheric conditions may make up different patterns of propagating trapped waves. They may be the convective cloud streets, the convective cells as the cold air breaks out over a warm surface, or the organized thermals over a heated flat land surface. In the following paragraphs the formation conditions and the characteristics of the different convection patterns in the convective boundary layer will be discussed.

(a) Wave-number selection and wind-turning effect

The integrals (25)-(30) can be calculated numerically. At first Eq. (23) is solved by iteration to obtain the corresponding wave numbers \(k\) and \(l\) for the trapped wave. Then substituting \(k\) and \(l\) into these integrals, integrating them numerically, we have the values of \(w_1, w_2, u_1,\) etc. For instance, if the atmospheric conditions are set as \(\partial \bar{\theta}_1/\partial z = -0.3 \times 10^{-2} \text{ K m}^{-1}, \partial \bar{\theta}_2/\partial z = 0.8 \times 10^{-2} \text{ K m}^{-1}, U_1 = 8 \text{ m s}^{-1}, U_2 = 16 \text{ m s}^{-1}, h = 1000 \text{ m}, \Delta \bar{\theta} = 4 \text{ K} \) and \(V_2 = 2 \text{ m s}^{-1},\) then \(m = 0.001 \text{ m}^{-1}, n = 0.0012 \text{ m}^{-1}, \gamma = 0.0016 \text{ m}^{-1}\) and \(\alpha = 0.125.\) Equation (23) gives the relation of the trapped wave numbers \(k\) and \(l,\) shown as line 1 in Fig. 1. The cut-off wave numbers \(k_c\) and \(l_c\) are determined by the relation

\[
(k_c + \alpha l_c)^2 = m^2
\]

where \(k_c\) and \(l_c\) are the minimum wave numbers of the trapped waves below which the excited gravity waves can no longer propagate horizontally. The corresponding relations for wind turning in the upper layer, for example with \(\alpha = 0.5\) or \(\alpha = 1,\) are also shown in Fig. 1 as lines 2 and 3 respectively. It can be seen that the wind turning reduces the cut-off wave numbers. This means that more components with longer wavelength contribute to the trapped waves. The relative magnitudes of the integrand of the integral (25) for different wave numbers are indicated on the curves. These values show that as wave numbers \(k\) and \(l\) increase, the intensity of the components decreases rapidly. This means that the trapped wave is made up of a rather narrow band of wave-number components.

Line 4 in Fig. 1 shows the wave-number relation from (23) and (31) with negative \(l\) in the situation of wind turning. It can be seen from the curves that the absolute values of the negative \(l\) are larger than those of the positive ones, while the amplitudes of the wave components with negative \(l\) are smaller. Then we may assume that the wave components with weak intensity and short wavelength have less probability of surviving in a convective boundary layer owing to the strong turbulent mixing.

Thus the convection waves have two parts propagating in different directions. The part composed of the components with positive \(l\) propagates into the first quadrant from the source, i.e. the region of \(x > 0\) and \(y > 0,\) while the one with negative \(l\) propagates into the fourth quadrant, i.e. \(x > 0, y < 0.\) If \(\alpha > 0,\) i.e. wind backing with height, the
Figure 1. The wave-number selection relation, $k$ and $l$ are wave numbers in the $x$ and $y$ directions respectively. The atmospheric conditions for line 1 are: $n = 0.0012 \text{ m}^{-1}$, $m = 0.001 \text{ m}^{-1}$, $\gamma = 0.0016 \text{ m}^{-1}$, and $\alpha = 0.125$. The atmospheric conditions for lines 2 and 3 are the same as those for line 1 except that $\alpha = 0.5$ for line 2 and $\alpha = 1$ for line 3. The dashed line indicates the positions of the cut-off wave numbers $k_c$ and $l_c$. For example, for line 1 $k_c = 0.00093 \text{ m}^{-1}$ and $l_c = 0.00057 \text{ m}^{-1}$. Line 4 is the same as line 1 but for negative wave number $l$. The numbers on the lines indicate the relative amplitudes of the wave-number components.

former part dominates. If $\alpha < 0$, i.e. wind veering, the wave-number relation satisfying (23) and (31) can also be illustrated by the lines in Fig. 1, but the sign of $l$ for lines 1 to 3 should be negative and that for line 4 positive. In this case the part propagating into the fourth quadrant will dominate. In the study we will only discuss the part with $\alpha > 0$ and $l > 0$.

(b) Organized thermals over heated land surface

An original thermal disturbance, which may be an active thermal, is supposed to be at $x = 0$ and $y = 0$ with a strong intensity of surface temperature flux in its centre. The dimension of the disturbance shown as Eq. (6) is set as $a = b = 500 \text{ m}$ though $a$ and $b$ can be different. The atmospheric conditions are assumed as a vigorous convective boundary layer, capped with an overlying stable layer. The wind in the lower layer is weak to moderate, while in the upper layer the wind is stronger with moderate directional turning. Then the atmospheric parameters $m$, $n$, $\gamma$ and $\alpha$ can be set as the same as those of the example in the previous section. The relation of the trapped wave numbers $k$ and $l$ is shown as curve 1 in Fig. 1. The cut-off wave numbers are $k_c = 0.00093 \text{ m}^{-1}$, $l_c = 0.00057 \text{ m}^{-1}$ and $\lambda_c = 0.00178 \text{ m}^{-1}$. If $\cosh(\lambda h) = \sinh(\lambda h)$ then the denominator of the integrand in (25) can be written as

$$G(k, l) = \frac{\partial}{\partial l} \left[ \lambda \cosh(\lambda h) - \left( \frac{k^2 + l^2}{k^2 - \gamma - \mu} \right) \sinh(\lambda h) \right]$$

$$\approx \left[ \frac{l}{k^2 + n^2} - 2l\gamma - \frac{l}{k^2} \right] + \left( 1 - \frac{m^2}{(k + \alpha)^2} \right) - \frac{k^2 + l^2}{(k + \alpha)^3} \frac{m^2 \alpha}{\mu} \cosh(\lambda h).$$
Putting these parameters into (25), (26), (27), (28) and (30) etc., we obtain the disturbed $w_{1t}, w_{2t}, u_{1t}, u_{2t}, v_{1t}$ and $v_{2t}$ as well as the vertical displacement of the interface $\zeta|_{z=0}$, induced by the trapped waves.

Figure 2 shows the distribution of the horizontal disturbed velocities in the convective boundary layer in the same atmospheric conditions as that shown as line 1 in Fig. 1. The disturbed flow fields appear as parallel convergence lines, which present an oblique angle with the mean wind direction. Figure 3 shows the disturbed flow on the vertical cross-section perpendicular to the mean wind direction. The vertical displacements of the interface are also shown on the section. The behaviour of the gravity waves can be made

![Figure 2](image1.png)  
Figure 2. Distribution of the disturbed horizontal velocities $(u, v)$ in the convective boundary layer. The grid interval is 500 m. The atmospheric conditions are the same as those for line 1 in Fig. 1.

![Figure 3](image2.png)  
Figure 3. Distribution of the flow field $(\nu, w)$ on the vertical cross-section $(y, z)$. The horizontal and vertical grid intervals are 500 m and 50 m respectively. The dashed line is the vertical displacement of the interface. The atmospheric conditions are the same as those for line 1 in Fig. 1.
clear from these figures. The trapped components of the waves excited by a disturbance propagate horizontally downstream with characteristic spacing and orientation determined by atmospheric conditions. The maximum amplitudes of the waves occur near the interface between the two layers, diminish with the distance downward and upward, and vanish at the ground surface and at the upper limit of the atmosphere, respectively. These waves induce and organize the local convection in the lower layer, and tune in the convection spacing to that of the upper waves. Since the flow fields are made up of different wave-number components the flow pattern varies spatially, depending on the combination of the waves. The spacing of the convergence bands shown in Fig. 2 ranges from 5 to 7 km along the mean wind direction of the lower layer, while the width of the strongest convergence belt, which can be taken as the area occupied by the thermal, is only 1 km or so. The longest wavelength along the mean wind direction is about 7 km, corresponding to the cut-off wave number \( k_c = 0.00093 \) \( \text{m}^{-1} \) as shown in line 1 of Fig. 1. As seen in Fig. 3 the intensities of the vortices and their intervals in the cross-wind direction are also varied. There are two updraught areas, three downdraught areas and four vortices among them in this cross-section. The first two vortices on the left have stronger intensities and narrower widths than those of the third one. The strongest vortex can be thought to be made up of multiple wave-number components, while the weakest and wider one is mainly contributed to by the components with the wave number near the cut-off \( l_c = 0.00055 \) \( \text{m}^{-1} \). The multi-wave-number structure of the vortex may be one of the causes why the convections in the convective boundary layer are not evenly distributed.

Once the local convection occurs it also launches both untrapped and trapped waves. The former causes a near-field disturbed flow, while the latter superimposes on the previously existing trapped waves and makes up resonant oscillation. The amplitude of the superimposing waves may be much larger than that of the waves excited by a single disturbance. Then the mechanism of the organized convection waves are supposed to be as follows. The orientation and the spacing of the convection waves are controlled by the wave numbers of the trapped waves depending on the whole atmospheric conditions, while the wave energy is supplied by the local convections, which are induced and tuned up by the waves.

Figure 4 shows the vertical displacement of the interface. From this figure the wave structure can be made clear. The contour of the interface in this figure is similar to that of the isentropic surfaces simulated by Hauf et al. (1989) in their three-dimensional numerical experiments of convection waves. In the simulation the vertical displacements of the potential-temperature surface in the overlying stable layer present a distinct wave structure, while in the boundary layer the outlines of the waves are obscured by the strong turbulent mixing. Thus the convergence lines of the flow fields shown in Fig. 2, and the wave-like structure of the vertical motions in Fig. 4, will not be so clear in the real atmosphere. The convergence lines are subjected to be cut off into separate closed spots by horizontal wind shear and turbulent mixing, since these lines present a large angle with the mean flow direction.

(c) Effects of the overlying stable layer

Relation (23) reveals that the structure of the large eddies in the convective boundary layer is determined not only by the atmospheric conditions in the boundary layer but also by that of the upper layer.

Setting the same lower-layer atmospheric conditions, say \( n = 0.0012 \) \( \text{m}^{-1} \), one finds that the relation of wave numbers \( k \) and \( l \), as well as \( \lambda \) in three-dimensional space, varies with atmospheric conditions, such as wind speed and temperature lapse rate, in the upper
stable layer as well as the temperature jump at the interface. Figure 5 shows the wave-numer selections with different upper-layer conditions. Line 1 represents the conditions: $\Delta \theta_z/\alpha z = 0.8 \times 10^{-2}$ K m$^{-1}$, $U_2 = 16$ m s$^{-1}$, $V_2 = 4$ m s$^{-1}$; curve 2: $\Delta \theta_z/\alpha z = 0.45 \times 10^{-2}$ K m$^{-1}$, $U_2 = 20$ m s$^{-1}$, $V_2 = 5$ m s$^{-1}$ (i.e. $m = 0.0006$ m$^{-1}$, $\alpha = 0.25$); curve 3: $\Delta \theta_z/\alpha z = 1 \times 10^{-2}$ K m$^{-1}$, $U_2 = 12$ m s$^{-1}$, $V_2 = 3$ m s$^{-1}$ (i.e. $m = 0.0015$ m$^{-1}$, $\alpha = 0.25$).

The cut-off wave numbers $k_c$ and $l_c$ represent the maximum and also the dominant wavelengths in the $x$ and $y$ directions, respectively. The ratio $k_c/l_c$ can be taken as a measure of the angle between the convergence lines and the mean wind direction. Then the wavelength in the $x$ direction may vary from 5 km in curve 3 to 11 km in curve 2. The waves may line up longitudinally as in curve 3, or transversally as in curve 2. Thus the different atmospheric conditions of the overlying stable layer may result in a variety of convection-wave patterns even though the atmospheric conditions in the boundary layer are the same. In order to study the large eddy structure of the convective boundary layer, either observationally or numerically, the effects of the overlying stable layer, or the layer interaction as stated by Clark et al. (1986), must be taken into consideration.

(d) Longitudinal convections over the ocean surface

Longitudinal convections, sometimes visualized as cloud streets, are often observed by satellite over the vast ocean surface. Typical conditions for the formation of longitudinal convection are cold-air outbreaks over a warm water surface. In these conditions the lower layer of the atmosphere is unstable because of the temperature difference between the cold air and the warm water, and the wind speeds in the whole troposphere are strong. For instance, suppose in the boundary layer $h = 1000$ m, $U_1 = 20$ m s$^{-1}$, $\Delta \theta_z/\alpha z = -0.2 \times 10^{-2}$ K m$^{-1}$; in the overlying stable layer $U_2 = 40$ m s$^{-1}$, $V_2 = 10$ m s$^{-1}$, $\Delta \theta_z/\alpha z = 0.25 \times 10^{-2}$ K m$^{-1}$ and $\Delta \theta = 2.5$ K. Then we have the parameters $m = 0.0002$ m$^{-1}$, $n = 0.0004$ m$^{-1}$, $\gamma = 0.0002$ m$^{-1}$ and $\alpha = 0.25$ approximately. Substituting these parameters into Eq. (23) we obtain the relation of the trapped wave numbers shown as line 1 in Fig. 6. The cut-off wave numbers $k_c$ and $l_c$ are about 0.000123 m$^{-1}$ and 0.000365 m$^{-1}$ respectively. Assuming a thermal disturbance with its dimension $a = b = 1000$ m we have a distribution of the vertical motion induced by the trapped waves shown in Fig. 7. In this figure the convective bands line up approximately along the mean wind
Figure 5. The effects of the overlying stable layer on the wave-number selection relation. The lower-layer conditions and the wind turning for lines 1–3 are the same: \( n = 0.0012 \text{ m}^{-1}, \gamma = 0.0016 \text{ m}^{-1} \) and \( \alpha = 0.25 \), but the upper-layer conditions are different: \( m = 0.001 \text{ m}^{-1} \) for line 1, \( m = 0.0006 \text{ m}^{-1} \) for line 2 and \( m = 0.0015 \text{ m}^{-1} \) for line 3.

Figure 6. The wave-number selection for longitudinal and cellular convections. The atmospheric conditions for line 1 are: \( m = 0.0002 \text{ m}^{-1}, n = 0.0004 \text{ m}^{-1}, \gamma = 0.0002 \text{ m}^{-1} \) and \( \alpha = 0.25 \); for line 2: \( m = 0.0002 \text{ m}^{-1}, n = 0.0003 \text{ m}^{-1}, \gamma = 0.0003 \text{ m}^{-1} \) and \( \alpha = 1 \). The numbers on the lines indicate the relative amplitudes of the wave-number components.
direction with an angle of about 15°. If there is enough water vapour in the boundary layer the cloud streets may form near the top of the boundary layer. The orientation and spacing of the cloud streets are determined by the atmospheric conditions, such as wind speed and the thermal structure in the upper and the lower layers. They are independent of the characteristics of the disturbance. The disturbance might be a mechanical forcing, such as overshooting of a thermal into the stable layer, an obstacle effect of a cumulus cloud or a terrain rising, or a thermal forcing, such as latent-heat release of cloud condensation, cloud heating and surface heating by radiation. The horizontal homogeneity of the atmospheric conditions over the ocean ensures the identical wave-number composition of the trapped wave. Thus the trapped waves can organize all the forcing to supply the energy to the wave motions. However, an uneven land surface is likely to destroy such homogeneity; this may explain why cloud streets are more frequently found over the ocean.

(e) Cellular convections over the ocean surface

Images from satellites show that cloud streets often change gradually to cellular patterns. This suggests that these two patterns of convection may have common conditions for their formation, while the difference of the wavelength in the transversal direction may be caused by the transformation of the cold air mass. An interesting example of convective-cloud-pattern transition was given by Scorer (1986) in his book of cloud satellite pictures. Picture 6.4 of the book shows the cloud cover over the Greenland Sea and Norwegian Sea at 1236 GMT 2 March 1982. The cold air from the Arctic Ocean had burst southward to be over the North Atlantic Ocean. Typical cloud streets formed over the Norwegian Sea. With the southward movement, the cold air spread laterally and, as expected, became warmer. Then the wind speed decreased and the temperature dif-
ference between the air and the sea surface lessened. Correspondingly, in the front part of the cold air, for example near the coast of Norway, the size of the convection cells became larger. However, as the air passed over a large floe near Jan Mayen Island and cooled down again, the cells narrowed and the streets re-established. This example reveals that the transition of the convection patterns is related to the instability and wind speed of the boundary layer.

Suppose the atmospheric conditions in the upper layer are taken to be the same as those in the case of cloud streets, and in the lower layer the mean wind speed drops to 17 m s\(^{-1}\) from 20 m s\(^{-1}\), then the instability reduces from \(-0.2 \times 10^{-2}\) K m\(^{-1}\) to \(-0.08 \times 10^{-2}\) K m\(^{-1}\). The difference of the wind directions between the two layers is assumed to be larger because of the lateral spread of the air in the lower layer. Then the parameters will be \(m = 0.0002\) m\(^{-1}\), \(n = 0.0003\) m\(^{-1}\), \(\gamma = 0.0003\) m\(^{-1}\) and \(\alpha = 1\), and the relation of the trapped wave numbers is obtained as line 2 in Fig. 6. This curve indicates that the wave numbers \(k\) and \(l\) have the same order. This means that the convection has an approximate symmetric structure like that shown in the pictures of cellular convection. Another feature of the curve worth mentioning is that as wave numbers \(k\) and \(l\) increase, the amplitudes of the trapped wave components do not decrease so sharply as shown in line 2 in Fig. 6. This means that the wave components, which make up the cellular convection, have a wide spectrum. Figure 8 shows the distribution of the vertical velocity in this case. The pattern of vertical motion can be visualized as that of open cells if the large-scale subsidence suppresses the weak updraught at the periphery of the convections and leaves only narrow ascending bands.

(f) Transition of the convection patterns from cell to street structure

Many cloud pictures in Scorer's book reveal the transition of the convections from longitudinal to cellular structure. However, Picture 6.20 taken at 1347 GMT 13 March 1982

![Figure 8](image_url)

Figure 8. Distribution of the updraught area (\(w \geq 1\) m s\(^{-1}\)) of the cellular convections at the interface. The atmospheric conditions are the same as those for line 2 in Fig. 6. The grid interval is 8 km.
shows an opposite case. It shows the difference of the convection appearance in the same air mass over land and ocean. A cold air mass moved south-eastward across the North Atlantic Ocean and penetrated over the British Isles and France. A cellular pattern of clouds covered the ocean surface, while inland a cloud-street pattern dominated.

If the boundary layer is high enough to make \( \cosh(\lambda h) = \sinh(\lambda h) \), from Eq. (23) we have the cut-off transverse wave number

\[
l_c^2 = k_c^2(k_c^2 + n^2)/\gamma^2 - k_c^2
\]

that is \( l_c \) increases with parameter \( n \). Then the stronger instability and weaker wind speed, as the air mass moves across the land surface, will decrease the spacing of the wave train in the direction perpendicular to the mean wind speed.

(g) Ratio between spacing of wave bands and depth of the convective layer

The aspect ratio of the lateral width of convection waves to the depth of the convective layer is an important characteristic in the structure of the convection waves. There have been many investigations about the aspect ratio. Its value ranges from 2 to 10 for longitudinal rolls and from 10 to 40 for convective cells.

From satellite pictures and data from rawinsonde observations, Miura (1986) gave a relationship between the spacing of cloud bands and the height of cloud tops for longitudinal rolls, shown as the fitting curve in Fig. 9. The curve indicates that \( h \propto L_y^{3/5} \), where \( L_y \) is the lateral wave width.

![Figure 9. The relationship between the lateral wavelength \( L_y \) and the height \( h \) of the convective lower layer. Dot-dashed line: fitting curve according to the observed data by Miura (1986); dashed line: analytical relationship in this study with \( m = 0.0005 \, m^{-1}, n = 0.0005 \, m^{-1}, \gamma = 0.0005 \, m^{-1} \) and \( \alpha = 0 \); solid line: the same as the dashed one, except for \( \gamma \) increasing linearly with \( h \).](image)

In the present study the lateral wave width is determined by the combination of the wave components with the wave numbers given by relation (23). Among these components the one with the cut-off wave number \( l_c \) has the dominant amplitude. Thus the lateral width is to a large extent related to \( l_c \), which is given by

\[
l_c^2 = \frac{k_c^2(k_c^2 + n^2)}{\gamma^2} \coth^2(\lambda h) - k_c^2.
\]  

(32)

If the other atmospheric conditions such as \( m, n \) and \( \gamma \) are constant, the simple relation between \( l_c \) and \( h \) is as follows. As \( h \) increases \( \coth(\lambda h) \) and then \( l_c \) decreases, so the
minimum band width $L_y = 2\pi/l_c$ increases. For example, if the atmospheric conditions are assumed as $m = 0.0005 \text{ m}^{-1}$, $n = 0.0005 \text{ m}^{-1}$ and $\gamma = 0.0005 \text{ m}^{-1}$, the relationship between $L_y$ and $h$ is given as the dashed line in Fig. 9. As $h$ is large enough, $\coth(\lambda h)$ approaches unity, and $l_c$ as well as $L_y$ tends to a limitation. Then the relationship obtained is different from the observed one (dot-dashed line in Fig. 9).

In the evolution of the convective boundary layer the entrainment zone deepens with the growth of boundary-layer depth (Wilde et al. 1985). Thus the temperature jump across the interface increases with the boundary-layer depth. We may simply assume a linear relation between the temperature jump and the depth. For example, $\Delta \theta$ increases from 1 K at a depth $h$ of 500 m to 5 K at a depth of 2000 m. Putting these values into (32), we obtain the relationship between $L_y$ and $h$ shown as the solid line in Fig. 9. This curve is similar to the one fitting the observations. It may be speculated that the role of boundary-layer growth is at least two-fold. The growth provides a broad space for the development of a large roll, and increases the density discontinuity between the two layers. Both effects favour the widening of the convection waves.

4. CONCLUDING REMARKS

The analytical solutions of the convective waves in the two-layer model show that the dynamical characteristics of the cloud streets, the cellular convections and the convective thermals have something in common. All of them can classify as a certain type of convective wave, which is induced by thermal or mechanical disturbances, and is adjusted by the interaction of the convective lower layer and the stable upper layer. As suggested by the name, convection waves result from the joint activities of convection and gravity waves. The convection waves appear as large vortices whose pattern, such as their shape, size, intensity and orientation, defines the main feature of the structure of the convective boundary layer. Thus the convection waves should be one of the major topics in studies of the convective boundary layer.

As illustrated by the analytical solutions, gravity waves in the upper stable layer exert significant influence on the large vortices in the boundary layer. Thus in order to study the convection waves as well as the whole convective boundary layer, the scope of the research should extend to the tropopause and not just to the top of the boundary layer. The theoretical results in this study need verifying by further observational and numerical studies. Whatever the consequences, the solutions prove the importance of the layer interaction, which should be kept in mind in boundary-layer research, as stated by previous researchers (e.g. Kuettner et al. 1987). Further studies of the convective boundary layer might benefit from the concept.

ACKNOWLEDGEMENT

The Project was supported by the National Natural Science Foundation of China.

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