Oscillations in the equatorial components of the atmosphere’s angular momentum and torques on the earth’s bulge

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SUMMARY

The angular momentum of the atmosphere is the sum of the wind term, \( W \), due to the winds relative to the earth’s surface, and the matter term, \( M \). It is shown that if the earth were an oblate spheroid with an equipotential surface, the atmospheric torque on the earth would be \( -\Omega \wedge M \), \( \Omega \) being the rotation rate of the solid earth. As a result, the equatorial components of \( M \) for a wave propagating at azimuthal angular velocity \( \sigma \) without changing shape are \( (\Omega - \sigma)/\Omega \) times the equatorial components of \( W \). Laplace’s equation for tidal motions ‘on a sphere’ strictly applies to motions on such an oblate spheroid and its solutions apply a torque on the earth equal to \( -\Omega \wedge M \). It is shown that the resulting relationship between \( M \) and \( W \) also implies that in any separable wave solution of the tidal equations the surface wind is simply related to the vertical integral of the wind and the equivalent depth.

Analyses of the equatorial components of the matter term by the weather forecast systems of the European Centre for Medium-range Weather Forecasts and the Meteorological Office are dominated by chaotic oscillations with periods of between 8 and 10 days that are well forecast out to 5 days ahead. It is argued that these are essentially free solutions of the tidal equations which exert considerable torques on the earth. The main feature of the equatorial components of the wind term is a seasonally modulated diurnal oscillation. Analyses and 2-day forecasts of this phenomenon are in less good agreement. It is argued that it is thermally forced and depends on the compressibility of the atmosphere.

1. INTRODUCTION

On a planet which is rotating with (nearly) constant angular velocity, \( \Omega \), the angular momentum of the atmosphere

\[
A = \int_{\text{atmos}} \rho r \wedge (u + \Omega \wedge r) \, d\tau
\]

where \( r \) is the position vector from the planet’s centre of mass of the volume elements \( d\tau \) which have densities \( \rho \) and velocities \( u \) relative to the planet’s surface. The contribution to \( A \) from the winds relative to the underlying planet is usually called the wind term, \( W \), and the contribution from the rotation of the atmosphere with the angular velocity of the planet the matter term, \( M \):

\[
W = \int_{\text{atmos}} \rho r \wedge u \, d\tau \quad M = \int_{\text{atmos}} \rho r \wedge (\Omega \wedge r) \, d\tau.
\]

\( A \), \( W \) and \( M \) are often expressed as components (e.g. \( A = (A_1, A_2, A_3) \)) relative to a body-fixed frame with the 3rd component aligned with \( \Omega \) and hence referred to as the axial component. On the earth the two other components lie in the equatorial plane, the first ‘equatorial component’ along the Greenwich meridian and the second along 90°E.

The total torque on the atmosphere at the earth’s surface produces fluctuations in the atmosphere’s angular momentum (AAM) and equal and opposite changes in the angular momentum of the underlying planet. Fluctuations in \( A_3 \), together with tidally induced changes in the earth’s moment of inertia, have been shown (see Barnes et al. 1983; Rosen et al. 1987) to account for most of the variations in the rotation rate of the solid earth, and hence of the length of the day, over time-scales of between a few days

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and a few years. Most of the fluctuations in $A_3$ occur in $W_3$ despite the fact that $M_3$ (which is nearly constant) is about 60 times larger than $W_3$.

Evidence for the hypothesis that the equatorial components of AAM play a similarly dominant role in exciting the wobble of the earth’s pole of rotation about the principal axis of its moment of inertia for similar time-scales is, as yet, much less clear-cut.

Brzezinski (1987) has shown that $M_1$ and $M_2$ exhibit strong oscillations, with periods of between 8 and 12 days and chaotically varying amplitude. Figure 1 presents a time series of $M_2$ for 1990. The fluctuations in Fig. 1 are entirely typical of those in both $M_1$ and $M_2$, their amplitude varies little with season and they are forecast quite accurately out to 5 days by both the Meteorological Office (UKMO) and the European Centre for Medium-range Weather Forecasts (ECMWF) forecast systems (Bell et al. 1991). Similar fluctuations are not apparent in $W_1$ and $W_2$—see for example Fig. 2. Indeed the data

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**Figure 1.** Time series of $\chi^P_2$ evaluated from the UKMO analyses valid at 00 GMT (full line) and 12 GMT (dashed line) for 1990. The matter term $M_2$ is related to (the non-dimensional) $\chi^P_2$ by $M_2 = (C - A)\Omega \chi^P_2$.

**Figure 2.** Time series of $\chi^W_2$ evaluated from the UKMO analyses valid at 00 GMT for 1990. The wind term $W_2$ is related to (the non-dimensional) $\chi^W_2$ by $W_2 = (C - A)\Omega \chi^W_2 / 1.43$. 
appear to be very noisy and for some time there seemed to be no discernible agreement between evaluations of \( W_1 \) and \( W_2 \) based on analyses from several major forecast systems. Eventually T. M. Eubanks (private communication) inferred that \( W_2 \) has a diurnal variation with a seasonal modulation. This is clearly apparent in Fig. 3 which presents evaluations of \( W_2 \) at 00 GMT (full line) and 12 GMT (dashed line) from (a) ECMWF data for 1988 and (b) UKMO data for 1990. Similar evaluations of \( W_1 \) are presented in Fig. 4. A small diurnal variation is apparent in the UKMO data (Fig. 4(b)) and an even smaller one in the ECMWF data (Fig. 4(a)).

The fluctuations in \( M \) and \( W \) just described are so rapid that, unless they are aliased into longer periods, they do not greatly excite the wobble of the pole (whose resonant period is of the order of 430 days). The understanding of their nature is, nevertheless, the subject of this paper.

(a)

![Graph](image)

(b)

![Graph](image)

Figure 3. Time series of \( \chi_2^W \) evaluated at 00 GMT (full line) and 12 GMT (dashed line) from (a) uninitialized ECMWF analyses for 1988 and (b) UKMO analyses for 1990. The time series have been smoothed using an 11-day running-mean filter.
Figure 4. Time series of $\chi_{1}^{w}$ evaluated at 00 GMT (full line) and 12 GMT (dashed line) from (a) uninitialized ECMWF analyses for 1988 and (b) UKMO analyses for 1990. The time series have been smoothed using an 11-day running-mean filter.

Eubanks et al. (1988) suggested that the fluctuations in $M_1$ and $M_2$ could be interpreted as free-travelling-wave solutions of Laplace's tidal equations, since the period of the wave with largest projection on these components of AAM is approximately 8 to 10 days. Whilst this seemed plausible (and I will argue that their interpretation is correct) I was unclear how free (unforced) solutions of Laplace's equations on a sphere could either provide the torque to the underlying earth which must accompany the observed fluctuation, or maintain themselves whilst doing so.

The seasonal modulation of the diurnal variation in $W_2$ strongly suggests that it is thermally forced. Since the torques accompanying an oscillation of the observed amplitude in $W_2$ alone would be large (section 6) it seemed likely that the diurnal fluctuation consists of a travelling wave with a diurnal period that is stationary in inertial space. Originally my examination of solutions of the tidal equations was motivated by desires to support this interpretation and to explain the sharp difference between the periods of
fluctuations dominating the wind and matter terms. Figure 5, however, practically settles the first of these two issues, presenting evaluations of (a) $W_1$ and (b) $W_2$ at 06 GMT and 18 GMT from the ECMWF data for 1988 which show that $W_1$ has a seasonally modulated diurnal oscillation. Figures 3(a), 4(a) and 5 together show that the diurnal oscillation moves westward and is stationary in inertial space.

The evaluations of $W_1$ at 00 GMT and 12 GMT from UKMO data for 1988 presented in Figs. 14 and 15 of Bell et al. (1991) (note that the scaling on the ordinate of these figures should be marked as $10^{-7}$ not $10^{-6}$) show a diurnal oscillation in $W_1$ between 00 GMT and 12 GMT comparable with that in $W_2$. Of the $W_1$ data calculated from the UKMO, ECMWF, US National Meteorological Center and Japanese Meteorological Agency analyses between 1979 and 1988, only the UKMO data had such a large seasonal cycle in the difference between 00 GMT and 12 GMT. The UKMO data for 1989 (not shown) and 1990 (Fig. 4(b)) display a markedly smaller oscillation. The only major

![Figure 5](image-url)  
Figure 5. Time series of (a) $\chi_1^W$ and (b) $\chi_2^W$ evaluated from uninitialized ECMWF analyses at 06 GMT (full time) and 18 GMT (dashed line) for 1988. The time series have been smoothed using an 11-day running-mean filter.
change to the UKMO system during this period was the introduction of a new analysis system on 30 November 1988 (Lorenc et al. 1991). Whilst this improvement in agreement between centres is encouraging, it is by no means certain that the analyses are now accurate. The amplitude of the seasonal cycle in the change in $W_2$ in the ECMWF 60-hour forecasts started at 12 GMT in 1988 was almost twice that of the change between the 00 GMT and 12 GMT analyses (see Bell et al. (1991), Fig. 12). Understanding of the nature of the diurnal oscillation should facilitate the assessment of the accuracy of the analyses and forecasts.

The earth's bulge, being about 20 km high, is the most significant topography on the planet. The mountain torque on the atmosphere due to the bulge is calculated in section 2 and shown to equal $\Omega \wedge \mathbf{M}$. This result leads to a simple expression (20) for the ratio of the equatorial matter and wind terms for a travelling wave disturbance of a given period, which explains in large part the sharp difference between the significant periods in the equatorial wind and matter terms. Section 3 shows that, as first established by Lamb (1932), Laplace's tidal equations on a sphere strictly apply to motions on a spheroid of constant geopotential (e.g. the earth with its bulge). Their solutions thus contain implicit mountain torques equal to $\Omega \wedge \mathbf{M}$. Section 4 summarizes results on free and forced linear solutions of the tidal equations in preparation for sections 5 and 6. Section 5 discusses the interpretation of the matter-term oscillations, the magnitude of the associated torques and the accuracy of the approximations implicit in the linear solutions. Lindzen's (1965) solution for diurnal oscillations is presented in section 6, and the wind-term oscillation associated with it is calculated and compared with Figs. 3 to 5. Conclusions are drawn in section 7.

2. The Atmosphere's Angular Momentum and the Earth's Bulge

Expressions for the equatorial components of $\mathbf{M}$ and $\mathbf{W}$ in the body-fixed Cartesian coordinate system introduced in section 1 are easily found from (2). In spherical polar coordinates with $r$ denoting radial distance, $\phi$ latitude, $\lambda$ longitude (increasing eastward from zero at the Greenwich meridian), $\hat{\phi}$ and $\hat{\lambda}$, the corresponding unit vectors, and $u$ the eastward and $v$ the northward components of velocity, the wind term

$$\mathbf{W} = \int_{\text{atmos}} (\rho u \hat{\phi} - \rho v \hat{\lambda}) \, d\tau$$

(3)

and the matter term

$$\mathbf{M} = \int_{\text{atmos}} \rho \Omega r^2 \cos \phi \, \hat{\phi} \, d\tau.$$  

(4)

Using the following expression for the horizontal coordinate of any vector $\mathbf{V}$ with $V_r = 0$,

$$
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix} =
\begin{pmatrix}
-\cos \lambda \sin \phi & -\sin \lambda \\
-\sin \lambda \sin \phi & \cos \lambda
\end{pmatrix}
\begin{pmatrix}
V_\phi \\
V_\lambda
\end{pmatrix}
$$

(5)

and the hydrostatic relation,

$$\frac{\partial p}{\partial r} = -\rho g$$

(6)

where $p$ is pressure and $g$ gravity, in (3) one finds that $W_1$ and $W_2$ are given by
\[(W_1, W_2) = \frac{a^3}{g} \int_0^\rho \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \{u \sin \phi \cos \lambda, \sin \lambda\} + v(-\sin \lambda, \cos \lambda)\} \cos \phi \, d\phi \, d\lambda \, dp \]

where \(a\) is the earth's radius (at 45°N) and \(\rho\) is the surface pressure.

Using (4), (5) and (6) \(M_1\) and \(M_2\) can similarly be shown to be given by

\[(M_1, M_2) = -\Omega a^4/g \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} p_s \cos \phi \sin \phi (\cos \lambda, \sin \lambda) \cos \phi \, d\phi \, d\lambda.\]

The total torque on the atmosphere, \(\Gamma\), due to pressure gradient forces is:

\[\Gamma = -\int_{\text{surf}} r \wedge p \, dS = -\int_{\text{atmos}} r \wedge \nabla p \, d\tau\]

where \(n\) is the unit normal pointing into the earth. Using (5) once more, the equatorial components of the surface pressure torque may be shown to be expressible as

\[-\{(r \wedge \nabla p)_1, (r \wedge \nabla p)_2\} = \partial p/\partial \phi (-\sin \lambda, \cos \lambda) + \partial p/\partial \tan \phi (\cos \lambda, \sin \lambda).\]

Assuming that the earth is of uniform density and that its oblate surface coincides with a surface of constant geopotential, its radius \(R(\phi)\) is given by

\[-\frac{GM}{R} - \frac{1}{2}\Omega^2 R^2 \cos^2 \phi \approx -\frac{GM}{a} - \frac{\Omega^2 a^2}{4}\]

where \(G = 6.672 \times 10^{-11}\) is the universal gravitational constant and \(M\) is the mass of the earth. Hence

\[R(\phi) - a = \frac{1}{2} \Omega^2 a^4 \left(\cos^2 \phi - \frac{1}{2}\right).\]

The pressure torque on the earth's bulge may be calculated by transforming the expression for the pressure torque from spherical coordinates \((r, \lambda, \phi)\) to spheroidal coordinates \((r', \lambda', \phi')\) following the earth's surface;

\[r = r' \left\{1 + \frac{\Omega^2 a^2}{2g} \left(\cos^2 \phi' - \frac{1}{2}\right)\right\}; \quad \phi = \phi'; \quad \lambda = \lambda'.\]

In these coordinates the latitudinal pressure gradient

\[\frac{\partial p}{\partial \phi} = \frac{\partial r'}{\partial \phi} \frac{\partial p}{\partial r'} + \frac{\partial p}{\partial \phi'}\]

\[= \frac{\Omega^2 r^2}{g} \cos \phi \sin \phi \frac{\partial p}{\partial r'} + \frac{\partial p}{\partial \phi'}\]

\[= -\rho \Omega^2 r^2 \cos \phi \sin \phi + \frac{\partial p}{\partial \phi'}\]

(the last expression being obtained by using the hydrostatic relation (6)). The volume integrals of the horizontal-pressure-gradient terms cancel after integration by parts (the earth's surface being parallel to surfaces of constant \(r'\)) leaving

\[(\Gamma_1, \Gamma_2) = \Omega^2 a^4/g \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} p_s \cos \phi \sin \phi (\sin \lambda, -\cos \lambda) \cos \phi \, d\phi \, d\lambda.\]
This expression may be derived alternatively (without introducing spheroidal coordinates) by calculating the integral (9) of (10) directly. Integration by parts of the first term on the right-hand side of (10) with respect to \( \phi \) yields boundary contributions which may be written as

\[
(\Gamma_1, \Gamma_2) = -\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} p_i R^2 \cos \theta \frac{dR}{d\phi} \, d\phi \, (\sin \lambda, -\cos \lambda) \, d\lambda.
\]  

(15)

Equation (14) can be re-derived from (15) by using (11) and the shallow-atmosphere approximation. This result, (14), also holds to a similarly good approximation (i.e. of the order of the ratio of the maximum to the minimum radii of the earth) when the density of the earth is variable and its gravity field, \( \Pi \), non-radial, provided its surface follows the geopotential. This may be shown by again using coordinates which follow the earth’s surface

\[
gr' = \Pi(r, \lambda, \phi) - \frac{\Omega^2 a^2}{2} (\cos^2 \phi - \frac{1}{2}).
\]

In these coordinates, taking \( \partial p/\partial \phi \) as an example,

\[
\frac{\partial p}{\partial \phi} = -\rho \left( \frac{\partial \Pi}{\partial \phi} + \Omega^2 a^2 \cos \phi \sin \phi \right) + \frac{\partial p}{\partial \phi'}
\]

and the total force \(-a^{-1}(\partial p/\partial \phi + \rho \partial \Pi/\partial \phi)\) involves only the pressure gradient \( \partial p/\partial \phi' \) and the centripetal acceleration term (cf. (13)).

Comparison of (14) with (8) reveals that

\[
\Gamma_1 = -\Omega M_2, \quad \Gamma_2 = \Omega M_1.
\]  

(16)

So the pressure torque on the atmosphere due to the earth’s bulge is directly proportional to the matter term’s contribution to \( \mathbf{A} \), and indeed equals \( \Omega \wedge \mathbf{M} \).

Angular momentum is transferred between the atmosphere and the solid earth by surface wind stresses and orographic drag on various scales (Swinbank 1985; Palmer et al. 1986). In this paper these torques are neglected and the pressure torque due to the bulge (6) is taken to equal the rate of change of \( \mathbf{A} \) in the inertial frame:

\[
\Gamma = \left( \frac{d\mathbf{A}}{dt} \right)_I = \left( \frac{d\mathbf{A}}{dt} \right)_R + \Omega \wedge \mathbf{A}
\]  

(17)

where subscripts \( R \) and \( I \) denote derivatives in the planet’s rotating frame and any inertial frame respectively. The result of combining (16) with (17) is the following important equation of motion for the equatorial components of the angular momentum;

\[
\left( \frac{dA_1}{dt} \right)_R - \Omega W_2 = 0, \quad \left( \frac{dA_2}{dt} \right)_R + \Omega W_1 = 0.
\]  

(18)

Much of the theory of the motion of atmospheric tides is concerned with propagating-wave solutions of the form

\[
\alpha = \text{Re}\{\hat{\alpha}(\phi, z) \exp i(s\lambda + \sigma t)\}.
\]

(19)

in which \( \alpha \) is any property of the wave (e.g. its temperature), \( \hat{\alpha}(\phi, z) \) is a complex function of latitude \( \phi \) and height \( z \), \( \text{Re} \{ \} \) denotes the real part of the quantity in brackets, \( \sigma \) is the angular velocity of the wave and \( s \) its longitudinal wave number. The dependence
of the relative amplitudes of the wind and matter terms of such wave solutions on their frequency $\sigma$ follows immediately from (18).

\[(M_1, M_2) = \frac{(\Omega - \sigma)}{\sigma} (W_1, W_2).\]  

(20)

Hence diurnal oscillations, which are stationary in an inertial space and for which $\sigma = \Omega$, have null matter terms. Waves with periods of $N$ days have $(M_1, M_2) = (N - 1) (W_1, W_2)$. So waves with periods of many days have much larger matter terms than wind terms. The period and structure of any wave with a non-zero wind or matter term is evidently intimately linked with the pressure torque on the equatorial bulge and (18). Since the standard equations for tidal motion are formulated for a spherical earth, it might be thought that their representation of waves containing angular momentum would be incorrect. In the next section it is shown that the full tidal equations do satisfy (18), and hence can simulate angular-momentum variations faithfully.

3. ANGULAR MOMENTUM AND THE TIDAL EQUATIONS

The standard equations for tidal motions on a sphere are often derived on the assumptions that the ellipticity of the earth and the centripetal acceleration of fluid parcels are negligible. Here the momentum equations for tidal motions are derived by transforming the full equations of motion into spheroidal coordinates. The resulting equations may be viewed as being defined in spherical coordinates for a spherical earth with negligible error. It is also demonstrated that the angular momentum of motions governed by these equations satisfies (18).

The latitudinal and longitudinal components of the momentum equation for an inviscid fluid are:

\[\rho \left( \frac{dv}{dt} + \frac{v w}{r} + \frac{u^2 \tan \phi}{r} + 2\Omega u \sin \phi + \Omega^2 r \sin \phi \cos \phi \right) = -\frac{1}{r} \frac{\partial p}{\partial \phi} \]  

(21)

\[\rho \left( \frac{du}{dt} + \frac{u w}{r} - \frac{u w \tan \phi}{r} + 2\Omega w \cos \phi - 2\Omega \nu \sin \phi \right) = -\frac{1}{r \cos \phi} \frac{\partial p}{\partial \lambda} \]  

(22)

where $w$ is vertical velocity and $d/dt$ the Lagrangian time derivative. On transforming the latitudinal component of the momentum equation into spheroidal coordinates the centripetal-acceleration term is directly balanced by the transformation term of (13). Hence

\[\rho \left( \frac{dv}{dt} + \frac{v w}{r} + \frac{u^2 \tan \phi'}{r} + 2\Omega u \sin \phi' \right) = -\frac{1}{r} \frac{\partial p}{\partial \phi'} \]  

(23)

The longitudinal component is unaltered. (22) and (23) are exact in spheroidal coordinates. The geometric differences between the spherical and spheroidal surfaces are of order $\Omega^2 a/g$ and can safely be neglected for motions on the earth. (That the earth's bulge can nevertheless be dynamically significant is due to the fact that vertical pressure gradients are much larger than horizontal ones.) (22) & (23) may then be viewed as defined in spherical coordinates and appropriate for a spherical earth. The familiar momentum equations for small-amplitude tidal motions follow from them by linearizing about a state of rest, neglecting vertical velocities and making the shallow-atmosphere approximation (setting $r = a$);
\[
\frac{\partial u}{\partial t} - 2\Omega v \sin\phi = - \frac{1}{\rho_0 a \cos\phi} \frac{\partial p}{\partial \lambda} \\
\frac{\partial v}{\partial t} + 2\Omega u \sin\phi = - \frac{1}{\rho_0 a} \frac{\partial p}{\partial \phi}.
\] (24)

That (22) and (23) for spherical coordinates on a spherical earth together satisfy (18) can be shown directly as follows. The angular momentum of a fluid parcel (see (3) and (4)) is

\[
a = \rho d\tau ((\Omega r^2 \cos\phi + ru) \hat{\phi} - rv \hat{\lambda}).
\] (25)

Using (17) and the identities \(\partial \hat{\phi}/\partial\phi = -\hat{r}, \partial \hat{\lambda}/\partial\phi = 0, \partial \hat{\phi}/\partial\lambda = -\sin\phi \hat{r} + \sin\phi \hat{\phi}, u = r \cos\phi \, d\lambda/dt, v = r \, d\phi/dt,\) in (25), \((da/dr)_1\) may be calculated to be

\[
\left( \frac{da}{dr} \right)_1 = \rho d\tau \left( \frac{du}{dt} + \frac{uw \tan\phi}{r} + 2\Omega w \cos\phi - 2\Omega v \sin\phi \right) \hat{\phi} - \\
- \rho d\tau \left( \frac{dv}{dt} + \frac{vw \tan\phi}{r} + \frac{u^2 \tan\phi}{r} + 2\Omega u \sin\phi + \Omega^2 r \sin\phi \cos\phi \right) \hat{\lambda}.
\] (26)

(22) and (23) may evidently be interpreted as statements concerning the angular momentum of the fluid parcels. Substituting them into (26) shows that

\[
- \left( \frac{da}{dr} \right)_1 = \rho d\tau \left\{ \frac{\partial p}{\partial \lambda} \hat{\phi} + \left[ \frac{-1}{\rho r} \frac{\partial p}{\partial \phi} + \Omega \frac{a}{r} \sin\phi \cos\phi \right] \hat{\lambda} \right\}.
\] (27)

That the implied torque (17) is related to the matter term as in (16) and hence that (18) holds can be established by integrating (27) over the whole (spherical) ‘atmospheric’ shell, using (6) and then integrating the pressure-torque terms by parts.

4. A SUMMARY OF SOLUTIONS OF THE TIDAL EQUATIONS

The following account of thermally forced atmospheric tides and solutions of Laplace’s tidal equation on a sphere summarizes results given in Chapman and Lindzen (1970) and Longuet-Higgins (1968). These works can be consulted for further algebraic details.

The tides are assumed to be describable as small wave-like motions in a thin layer of an inviscid perfect gas covering the rotating earth. Investigations are limited to propagating linear-wave solutions of the form of (19). The basic state of the atmosphere on which the tidal motions are superposed is taken to be a vertically stratified state of rest with no horizontal thermal variation, and the tidal motions and rest state of the atmosphere are taken to be in hydrostatic balance. Thus, denoting the density and pressure of the rest state by \(\rho_0(z)\) and \(p_0(z)\) respectively and those of the motions by \(\delta\rho\) and \(\delta p,\)

\[
\frac{\partial p_0}{\partial z} = -\rho_0 g \quad \text{and} \quad \frac{\partial \delta p}{\partial z} = -\delta \rho \, g.
\] (28)

The horizontal motions are governed by (24). For motions of the form (19), the velocities \(u\) and \(v\) can be expressed in terms of the pressure field \(\delta p.\) The horizontal divergence of the velocities,
\[ \nabla_h \cdot u_h = \frac{i\sigma}{4a^2\Omega^2} F(\delta p/\rho_0) \]  

(29)

where \( F \) is the differential operator given by

\[ F = \frac{1}{\cos \phi} \frac{d}{d\phi} \left( \frac{\cos \phi}{\eta^2 - \sin^2 \phi} \frac{d}{d\phi} \right) - \frac{1}{\eta^2 - \cos^2 \phi} \left( \frac{s}{\eta} \frac{\eta^2 + \sin^2 \phi}{\eta^2 - \sin^2 \phi} \cos^2 \phi \right) \]  

(30)

with \( \eta = \sigma/2\Omega \). The three-dimensional (3D) divergence of the 3D velocity field is related to the material time derivative of the density by the continuity equation appropriate for a compressible fluid:

\[ \frac{Dp}{Dt} = -\rho_0 \left( \nabla_h \cdot u_h + \frac{\partial w}{\partial z} \right) \]  

(31)

and the system is closed by the thermodynamic equation appropriate for a compressible perfect gas:

\[ \frac{Dp}{Dt} = \gamma gH \frac{Dp}{Dt} + (\gamma - 1)\rho_0 J. \]  

(32)

In these equations \( \gamma = c_p/c_v \) (the ratio of specific heats at constant pressure and volume respectively) = 1.4, \( H(z) = p_0(z)/(\rho_0(z)g) \), \( J \) is the distribution of diabatic heating and,

\[ \frac{D}{Dt}(p, \rho) = \frac{\partial}{\partial t} \left( \delta p, \delta \rho \right) + w \frac{d}{dz} (p_0, \rho_0). \]  

(33)

Equations (28)–(33), for modes of the form (19), may be reduced to a single equation for

\[ G = -\frac{1}{\gamma \rho_0} \frac{Dp}{Dt} \]  

(34)

which is essentially the pressure tendency following the motion: from (29), (31), (33) and (34) one can show that

\[ \frac{\gamma}{g\rho_0} \frac{\partial}{\partial z} (p_0 G) = -\nabla_h \cdot u_h = \frac{-1}{4a^2\Omega^2} F \left( \frac{i\sigma}{\rho_0} \right) \]  

(35)

and a second expression for \( i\sigma \delta(\delta p/\rho_0)/\partial z \) in terms of \( G \) and \( J \) can be found from (33) using (31), (32) and (35); combining the vertical derivative of the first expression with the second gives

\[ \frac{H\delta^2 G}{\partial z^2} + \left( \frac{dH}{dz} - 1 \right) \frac{\delta G}{\partial z} = \frac{g}{4a^2\Omega^2} F \left\{ \left( \frac{dH}{dz} + \kappa \right) G - \frac{\kappa J}{\gamma gH} \right\} \]  

(36)

where \( \kappa = (\gamma - 1)/\gamma \).

This equation is clearly suited to solution by the method of separation of variables. The heating function, \( J \), and \( G \) are expanded as series,

\[ (J, G) = \sum_m (\tilde{f}_m(z), \tilde{G}_m(z)) \Theta_m(\phi) \exp (i\delta \lambda + \epsilon t) \]  

(37)

where \( \Theta_m(\phi) \) are eigenfunctions of the operator \( F \);

\[ F[\Theta_m] = -\varepsilon \Theta_m; \quad \varepsilon = \frac{4a^2\Omega^2}{g\lambda_m}. \]  

(38)
The horizontal structure of the $w$ and $\delta p$ fields is identical to that of $J$ and $G$. The vertical structure equation is

\[ \frac{Hd^2G_m}{dz^2} + \left( \frac{dH}{dz} - 1 \right) \frac{dG_m}{dz} + \left( \frac{dH/dz + k}{h_m} \right) \dot{G}_m = \frac{\kappa \dot{F}_m}{\gamma g H h_m}. \]  

(39)

For unforced (resonant) solutions of (38) and (39) with $\dot{F}_m = 0$, the separation constant $h_m$ is an eigenvalue of (39) and the frequency $\sigma$ of the oscillation is determined by (38) (through the dependence of $F$ on $\eta$ and hence $\sigma$; see (30)). For solutions forced at a given frequency (by $\dot{F}_m$), $h_m$ is determined by (38).

Love's method for the solution of (38) (Lamb 1932 p. 350) is particularly relevant to the discussion of angular-momentum fluctuations. Love re-expressed the velocities $u$ and $v$ in terms of a stream function $\Psi$ and velocity potential $\Phi$:

\[ au = \frac{1}{\cos \phi} \frac{\partial \Phi}{\partial \lambda} + \frac{\partial \Psi}{\partial \phi}, \quad av = \frac{\partial \Phi}{\partial \phi} - \frac{1}{\cos \phi} \frac{\partial \Psi}{\partial \lambda}. \]  

(40)

He used the curl of the momentum equations (24) to obtain one relation between $\Phi$ and $\Psi$ and its divergence to obtain a second relation between $\Phi$, $\Psi$ and $\delta p/\rho_0$. For pressure fields, $\delta p$, which are eigenfunctions of $\mathcal{F}$,

\[ \nabla^2 \Phi = \nabla_h \cdot u_h = \frac{i \sigma}{4a^2 \Omega^2} \mathcal{F}(\delta p/\rho_0) = - \frac{i \sigma}{gh_m} \delta p/\rho_0 \]  

(41)

(see (25) and (38)), so $\delta p/\rho_0$ can be eliminated from the second relation. On expanding $\Phi$ and $\Psi$ as series of spherical harmonics $P_n^m$ (Longuet-Higgins 1968)

\[ (\Phi, \Psi) = \sum_{n,s} (A_n^s, iB_n^s) \hat{Y}(z) P_n^m(\sin \phi) \exp i(s \lambda + \sigma t) \]  

(42)

the relations reduce to recurrence relations between the coefficients $A_n^s$ and $B_n^s$. The solutions fall into two sets distinguished by the symmetry of their motions about the equator. The set which contains the angular momentum satisfies

\[ \begin{pmatrix} L_s & r_{s+1} & 0 & 0 & \cdots & B_n^s \\ q_s & K_{s+1} & r_{s+2} & 0 & \cdots & \cdots \\ 0 & q_{s+1} & L_{s+2} & r_{s+3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \end{pmatrix} \begin{pmatrix} A_{s+1}^1 \\ A_{s+2}^2 \\ \vdots \\ A_{s+1}^s \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \end{pmatrix}. \]  

(43)

Here $K_s$, $L_s$, $r_s$ and $q_s$ are numerical constants which depend only on $n$, $s$, and $\epsilon$. For future reference (in section 5) we note that the first of the relations in (43) is

\[ A_2^1 = \frac{5}{9} \frac{(\Omega - \sigma)}{\Omega} B_1^1. \]  

(44)

It is clear from (8) that only the $P_{1/2}^1$ ($\sin \phi$) $\exp i \lambda$ spherical harmonic of the surface pressure field contributes to the matter terms. (41) shows that the Laplacian of the velocity potential is directly proportional to the pressure field, $\delta p$, if $\delta p$ is an eigenfunction of (38). Hence $A_2^1$ is directly proportional to the matter term. Furthermore only the $P_{1/2}^1$ ($\sin \phi$) $\exp i \lambda$ component of the stream function, with coefficient $i B_1^1$, contributes to the wind terms as may be shown by integrating (7) by parts:
\[ (W_1, W_2) = \frac{2a^2}{g} \int \int \int \Psi \cos \phi (\cos \lambda, \sin \lambda) \cos \phi \, d\phi \, d\lambda \, dp. \]  \tag{45}

5. INTERPRETATION OF MATTER-TERM OSCILLATIONS

Since the oscillations in the equatorial matter terms lack a consistent amplitude and long-term stability of phase it is natural to suppose that they may be unforced, resonant solutions of the tidal equations. The range of periods of angular momentum oscillations to be expected from the unforced eigensolutions of the tidal equations can be inferred from the work of previous authors such as Madden (1979). For small values of \( \varepsilon \), the solutions of the tidal equations divide cleanly into gravity waves and planetary waves. The planetary waves have

\[
\begin{align*}
\sigma & \to \frac{2s}{n(n+1)} \\
\Omega & \to 0 \\
\Phi & \to 0 \\
\Psi & \to iB^s_n \tilde{V}(z) P^s_n (\sin \phi) \exp i(s\lambda + \sigma t).
\end{align*}
\]  \tag{46}

Figure 3 of Madden (1979) (which is based on Fig. 2(b) of Longuet-Higgins (1968)) shows how the eigenfrequencies \( \sigma \) of solutions with \( s = 1 \) decrease as \( \varepsilon^{-1/2} \) decreases (i.e. \( h^{1/2} \) decreases). The set of solutions which contain angular momentum (see (43)) have even values of \( n - s \) so solutions with \( n = 1 \) and \( n = 3 \) are likely to contain the most angular momentum.

The equivalent depth, \( h_m \), is an eigenvalue of Eq. (39) with \( \tilde{J}_n = 0 \), solved subject to suitable boundary conditions. For an isothermal atmosphere the deepest equivalent depth has

\[ h = \frac{H}{(1 - \kappa)}, \quad H = \frac{p_0}{\rho_0 g}. \]  \tag{47}

According to Madden, calculations of \( h_m \) with realistic altitude-dependent temperatures differ from 10 km by less than 20%. Within this range (8 km < \( h_m < 12 \) km) the period of the \( n = 3 \) oscillation lies between 8 and 9 days, which is in remarkably good agreement with the observational data. The \( n = 1 \) oscillation has a period of about 1.2 days and thus, as discussed at the end of section 2, will appear more dominantly in the wind term than the matter term.

The neglect of the atmosphere’s mean zonal winds and horizontal thermal variations from these calculations may seem crude, but the rapidity of the phase speed of the oscillations (50 m s\(^{-1}\) at the equator) makes these approximations less significant than one might at first suppose (see Madden (1979) for a summary of numerical calculations concerning this point). The least accurate of the assumptions used in deriving the tidal equations are probably the neglect of thermal variations in heating between continents and oceans and the torques associated with the principal orography other than the equatorial bulge (see Hsu and Hoskins (1989)).

The peak minus trough fluctuation in \( A_1 \) over a few days is typically about \( 8 \times 10^{24} \) kg m\(^2\) s\(^{-1}\). Such oscillations are due to fluctuations in the \( P^1_1 \) spherical harmonic of the surface pressure with half amplitude in the fluctuation at 45°N of only 2 mb. The torque on the earth associated with these oscillations (see (16)) is nevertheless large compared with the net torques which produce the largest 40–60 day oscillations in the axial component of the AAM. The largest changes in \( W_3 \) over the course of 20 days are
of the order of $2.5 \times 10^{25} \text{kgm}^2\text{s}^{-1}$. From (16) the equatorial matter-term torques are $16 \times 10^{14} \text{kgm}^2\text{s}^{-1}$ per day and hence are consistently 10 times as large as those of the 30–60 day oscillation.

It is easy to overlook the fact that much larger torques are required to maintain $M_2$ at its non-zero mean value (see Fig. 1). The climatological maps of dynamic height at 1000 mb presented in Hoskins et al. (1989) show that the main reason why $M_2$ is positive is that the Siberian high (centred near 90°W) is not balanced by a correspondingly dominant high-pressure system near 90°E. It would be inaccurate to say that the positive value of $M_2$ is due to the Siberian high alone, because the former is a global quantity and the latter a local one. Indeed the seasonal evolution of $M_2$ cannot be described simply in terms of the Siberian high. But providing these remarks are borne in mind one can consider the Siberian high to be able to rotate with the earth because of its torque on the earth’s bulge.

The relationship between (20) and (44) is of some interest. Using (42) in (45) to express $(W_1, W_2)$ as a function of $B^1$ and $\hat{V}(z)$, and (41) with (42) in (8) to express $(A_1, A_2)$ as a function of $A^1$ and $\hat{V}(z)$, one finds that

$$
(M_1, M_2) = \frac{24\pi i\Omega a^2}{5\sigma} h_m(p_0 \hat{V})_{z=0} A^1(1, i) \exp(i\sigma t)
$$

$$(W_1, W_2) = \frac{8\pi a^2 i}{3g} \int_0^{r_p} \hat{V} \, dp \, B^1(1, i) \exp(i\sigma t). \tag{48}$$

Combining these with (44) gives

$$(M_1, M_2) = \frac{(\Omega - \sigma)}{\sigma} g h_m(p_0 \hat{V})_{z=0} \left( \int_0^{r_p} \hat{V} \, dp \right)^{-1} (W_1, W_2). \tag{49}$$

Finally, comparing this last equation with (20) one infers that

$$\int_0^{r_p} \hat{V} \, dp = g h_m(p_0 \hat{V})_{z=0}. \tag{50}$$

In a barotropic fluid (50) merely states that $\rho_0 g h_m = p_0$ (i.e. the equivalent depth $h_m = H$). But (50) is true for all thermal profiles $T(z)$ and heating profiles $\hat{f}(z)$ for all separable solutions (of the form (37)) whether they contain angular momentum or not. This point may be proved as follows. From (35) and (41), for a single mode,

$$\gamma \frac{\partial}{\partial z} (p_0 G_m) = \frac{i\sigma}{g h_m} \delta p_m \tag{51}$$

$$\Rightarrow \int_0^\infty \delta p_m \, dz = \frac{h_m}{i\sigma} [\gamma p_0 G_m]_0. \tag{52}$$

At the lower boundary $z = 0$, $w = 0$ and from (33)

$$-\gamma p_0 G_m = i\sigma \delta p_m(0). \tag{53}$$

So

$$\int_0^\infty \delta p_m \, dz = h_m \delta p_m(0). \tag{54}$$

Finally, from (24), $\hat{V}(z)$ is proportional to $\delta \hat{p}/\rho_0$ and (50) is easily derived.
The description of \( h_m \) as the equivalent depth is quite natural in (50), even when \( h_m \) is negative, and (50) provides an additional sense to the concept of the equivalent depth. Despite its simple form (50) does not appear to have been noticed before. It can be understood as being based on the properties of angular momentum which lead to (50) for oscillations containing angular momentum and the fact that the vertical-structure equation (39) and its boundary conditions (formed from (51) and (53)) are independent of \( \sigma \). For any solution of the form (37) with a given heating function \( \tilde{F}_m(z) \), a solution containing angular momentum with the same \( \tilde{F}_m(z) \) and vertical structure can be found at some frequency \( \sigma \) (which will usually be different from that for the first solution). Since (50) must hold for this second solution it must hold for the original solution and for all solutions whether they contain angular momentum or not.

6. INTERPRETATION OF WIND-TERM OSCILLATIONS

The locking of the main oscillation in the wind terms to the solar day and its seasonal variation makes it clear that the oscillation is principally thermally forced. For this reason we explore here the diurnally forced oscillation rather than the resonant solution whose period is slightly longer than a day. From (20) it is clear that the matter term of the diurnal oscillation is identically zero. The pressure torque on the equatorial bulge is hence zero, consistent with the angular momentum of the oscillation being stationary in an inertial frame.

Following the method outlined in section 4, the excitation of the thermal tide may be sought by calculating the projection of the heating function \( J \) onto each of the eigenfunctions of the operator \( F \). Heating with horizontal thermal structure \( \Theta_m(\phi)e^{i \Omega \lambda} \) excites the same horizontal component in the perturbation pressure \( \delta p \) (see (35) and (36)) and the Laplacian of the velocity potential \( \nabla^2 \Phi \) (see (41)). Hence according to (43) the \( P_1^1(\sin \phi) \exp(i \lambda) \) component of the stream function (the \( B_1^1 \) coefficient) is excited by heating functions with amplitude in \( P_{1_m}^1(\sin \phi) \exp(i \lambda) \) \( (n \neq 1) \). These heating functions are all antisymmetric about the equator. It is anticipated that of these heating functions \( P_1^1(\sin \phi) \exp(i \lambda) \) will have by far the largest amplitude and that its amplitude will have a seasonal cycle.

As Lindzen (1965) pointed out, however, the Hough functions \( \Theta_m(\phi) \exp(i \lambda) \), as given by (43), for the 'diurnal' tide with angular velocity \( \sigma = \Omega \) are all orthogonal to \( P_1^1(\sin \phi) \exp(i \lambda) \). This point arises from (44) which shows that for modes with \( \sigma = \Omega \), \( A_2 = 0 \). The Hough functions given by (43) are consequently not complete for the diurnal tide. They can, however, be supplemented by a solution, which can be regarded as an additional Hough function, for which the projection of \( G \) on \( P_1^1(\sin \phi) \exp(i \lambda) \) is zero. (In using this solution to describe the diurnal tide we neglect the slight difference between the solar period and the sidereal period.)

The simplest way to find the additional Hough solution is to look for one with

\[
\delta p = \hat{p}(z) \cos \phi \sin \phi \exp i(\lambda + \Omega t) .
\] (55)

Solving (24) for the velocities \( u \) and \( v \) one obtains

\[
\begin{align*}
u &= \frac{i \hat{p}}{\rho_0 a \Omega} \exp i(\lambda + \Omega t) .
\end{align*}
\]

\[
\begin{align*}
u &= \frac{i \hat{p}}{\rho_0 a \Omega} \exp i(\lambda + \Omega t) .
\end{align*}
\] (56)
This velocity field has zero horizontal divergence
\[ \nabla_h \cdot u_h = 0 \]
and is generated by
\[ \Psi = -\frac{\hat{p}}{\rho_0 \Omega} \cos \phi \exp i(\lambda + \Omega t). \]  
(57)

Thus from (29) for \( \delta p \) given by (55), \( \mathbf{F}(\delta p) = 0 \) and, by (35), \( G = 0 \). (33) then implies that
\[ w = i \Omega \frac{\delta p}{(\rho_0 \delta \rho)}. \]  
(58)

Using (31) and (32) with the results that \( G \) and the horizontal divergence are zero, one infers that
\[ gH \frac{\partial w}{\partial z} = \kappa J. \]  
(59)

Thus pressure perturbations of the form (55) are induced by diabatic heating with the same horizontal and temporal structure
\[ J = \tilde{J}(z) \cos \phi \sin \phi \exp i(\lambda + \Omega t). \]  
(60)

Since \( w = 0 \) at the surface, (58) implies that no surface-pressure (and hence no matter term) tide is excited. The pressure perturbation is related to \( J \) through (58) and (59)
\[ i \Omega \tilde{p}(z) = \rho_0(z) \int_0^z \frac{\kappa \tilde{J}(\xi)}{H(\xi)} d\xi. \]  
(61)

In summary, in this mode of motion the diurnal heating causes vertical expansion, which induces pressure fluctuations, but no horizontal divergence. The wave propagates horizontally like the pure barotropic Rossby wave with \( \varepsilon = 0 \) (see (46)). The standard equation (36) fails to describe the motion because \( F(U) = 0 \).

Figures 2 to 5 present \( \chi^w \) and \( \chi^w_2 \) which are non-dimensional functions introduced by Barnes et al. (1983). They are related to \( W_1 \) and \( W_2 \) by
\[ (\chi^w_1, \chi^w_2) = \frac{1.43}{\Omega^2 (C - A)} (W_1, W_2) \]  
(62)

where \( C \) is the major and \( A \) the minor principal moment of inertia of the solid earth. Substituting (45) into (62) and then using (57) and (61) the diurnal wind terms induced by heating in \( P_1 \) are
\[ (\chi^w_1, \chi^w_2) = \frac{2.86 \alpha^2 \lambda}{\Omega^2 (C - A)} \int \int \int \int d\xi \cos^2 \phi \exp i(\lambda + \Omega t) (\cos \lambda, \sin \lambda) \cos \phi \exp \lambda \exp dp. \]  
(63)

Several forms of heating could make significant contributions to \( J \); direct absorption of solar radiation by water vapour or ozone in the stratosphere, sensible heating in the surface layer, or latent-heat release. None of these appears to have been determined with much accuracy. The most studied contribution is that of water vapour which Lindzen (1967), following Siebert (1961), estimates to be given by
\[ J_{\text{water vapour}} = 0.085 \, S \, \frac{\Omega R_c}{\kappa} \exp \left( -\frac{x'}{3} \right) \quad (64) \]

where \( S \) varies sinusoidally with the season being 1 in the northern hemisphere summer, the phase of \( J \) has been chosen to be zero at 12 GMT, \( R_c = gH/T_0 \) is the gas constant, and \( x' \) is the vertical coordinate which is zero at the surface and has gradient \( dx'/dz = 1/H \) (with this coordinate \( p_\theta(z) = p_\theta(0) \exp(-x') \)). The vertical integrals can be evaluated by neglecting thermal variations in the vertical, in which case

\[
\int_0^{p_\theta} \int_0^z \exp \left( -\frac{x'}{3} \right) \, d\zeta \, dp = \frac{2}{3} H p_\theta(0).
\]

\( \chi_1^W \) and \( \chi_2^W \) are then found to be given by

\[
(\chi_1^W, \chi_2^W) = \frac{2.86a^2i}{\Omega^3(C - A)} \frac{H p_\theta(0)}{T_0} \int_0^{2\pi} \exp(i + \Omega t) \cos(\lambda) \sin(\lambda) \, d\lambda. \quad (65)
\]

Taking \( \Omega = 7.3 \times 10^{-5} \, \text{s}^{-1}, \quad C - A = 7 \times 10^{27}/300 \, \text{kg m}^2, \quad a = 6.4 \times 10^6 \, \text{m}, \quad T_0 = 300 \, \text{K}, \quad H = 10^4 \, \text{m} \) and \( p_\theta(0) = 10^5 \, \text{N m}^{-2} \) one finds that

\[
(\chi_1^W, \chi_2^W) \approx i 5 \times 10^{-7} \exp(i\Omega t) \, (1, i). \quad (66)
\]

The seasonal variation in \( \chi_2^W \) (00 GMT) \(-\chi_2^W \) (12 GMT) is hence estimated to be \( 4 \times 10^{-7} \) with its maximum during the northern hemisphere summer (July). \( \chi_1^W \) lags \( \chi_2^W \) by 6 hours; so \( \chi_1^W \) has its maximum value at 06 GMT in July.

The \( \chi_1^W \) and \( \chi_2^W \) 00–12 GMT differences presented in Figs. 3 to 5 appear to have good phase agreement with (66), but amplitudes between 50 and 100% larger than calculated. The amplitudes of the diurnal variations in the ECMWF 48- and 60-hour forecasts are about three times as large as suggested by (66). Both the UKMO and ECMWF forecast models use equations for compressible fluids, and should be able to represent solutions of the form discussed in this section very well. It seems most likely that the differences are due to deficiencies in the representation of diabatic heating.

A reviewer suggested that the shortfall in (66) might be accounted for by absorption of ultra-violet radiation by ozone in the stratosphere. Below 25 mb (10 mb), the highest level in the UKMO (ECMWF) model in 1990, zonal mean heating rates per day (i.e. \( J/c_p \)) are up to 1 (2) K per day (Mike Fisher, personal communication and Kiehl and Solomon 1986). This heating may well project closely onto the mode (60) being considered. Evaluating (63) with \( \tilde{\gamma} = 2S_c \) Joules per day over a slab of atmosphere between 50 and 25 mb, taking the depth of the slab to be 10 km and its temperature to be 220 K, one finds that

\[
\chi_1^W = \frac{2.86a^2i}{\Omega^3(C - A)} \iint_0^z \frac{k \tilde{\gamma}}{gH} \, dp \iint_0^\infty \cos^3 \phi \, d\phi \int \cos^3 \lambda \, d\lambda \exp(i\Omega t)
\]

\[
= 1.29 \times 10^{-8} \, iS \cdot 1.4/3 \cdot \pi \exp(i\Omega t)
\]

\[
= 5 \times 10^{-9} \, iS \exp(i\Omega t). \quad (67)
\]

Comparison with (66) suggests that ozone absorption in the two forecast models did not contribute significantly to the diurnal variation in the wind terms.

Fluctuations in the UKMO analyses up to 1988 of \( W_1 \) between 00 and 12 GMT were of concern because they suggested that the equatorial wind-term oscillation was not stationary in an inertial frame. The implied torques could be large if this were the case: if \( \chi_1^W \) oscillated as observed whilst \( \chi_1^W \) were constant the torque would be almost 10 times
larger than that involved in the 30–60 day oscillation. The accompanying wind-velocity errors could nevertheless be small; a vertically uniform fluctuation in the $P_1$ spherical harmonic of the stream function would give wind velocities with a half amplitude of only $0.1 \text{ m s}^{-1}$.

7. Conclusions

As noted by Lamb (1932), fluid motions on a spheroidal surface that follow an equipotential are well described by equations for motions on a spherical surface provided they omit the centripetal acceleration term (i.e. $\Omega^2 r \sin \phi \cos \phi$ in (21)). Thus Laplace's tidal equations and the primitive equations used by most weather forecast systems describe motions which (implicitly) exert a torque on the earth's bulge equal to $-\Omega \wedge \mathbf{M}$. The phase velocity of solutions of Laplace's tidal equations which contain angular momentum thus depends on the ratio of their wind to their matter terms according to (20). Solutions with null-matter terms exert no torque on the earth and are stationary in an inertial frame, whilst solutions with no wind term move with the earth.

The fluctuations of $M_1$ and $M_2$ illustrated by Fig. 1 can be interpreted as free (unforced) solutions of the tidal equations which are able to rotate rapidly in inertial space by the torque they exert on the earth's bulge. Their wind terms are small because $\sigma \ll \Omega$ (see (20)). A large mean value can be maintained in the $M_2$ component, which rotates with the earth, because of its torque on the bulge. This component reflects the large asymmetries in the global circulation associated with the Siberian high. The seasonally modulated diurnal fluctuations in the wind terms are thermally forced, virtually stationary in inertial space and hence have null-matter terms. Differences between the analyses and forecasts of this diurnal wave probably arise from problems in the simulation of processes that produce diabatic heating.

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