Passage of a frontal zone over a two-dimensional ridge

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SUMMARY

The passage of a frontal zone over a two-dimensional triangular ridge is investigated using a semigeostrophic model. The model atmosphere consists of three layers of uniform density and potential vorticity. The densest (coldest) layer represents the airmass behind the cold front, the next layer the frontal zone itself, and the least dense (warmest) layer represents the flow ahead of and above the frontal zone. At ground level, the frontal zone is contained between two surface fronts. A prescribed pressure field imposes translation at a speed $U$, and confuence with a deformation rate $\alpha$. A one-layer version of the model has also been considered. The main conclusions are that (a) the mountain-induced shift of the position of a front does not depend on $U$ or $\alpha$; it is mainly negative on the windward slope and the upper part of the leeward slope, but is positive in the downstream plain and on the lower part of the leeward slope (there are no effects far-ahead), and (b) frontal collapse must occur for $\alpha > 0$. The collapse is delayed over most of the mountain, compared to the no-mountain case, but occurs earlier downstream from the mountain. Prior to the collapse, there is up-slope frontogenesis when part of the frontal zone is in the plain. Frontolysis prevails if all of the frontal zone is on the up-slope and if $\alpha$ is not too large. Frontogenesis occurs on the down-slope and in the plain.

1. INTRODUCTION

The motion of cold fronts over mountains is a fully three-dimensional process (see Smith 1986, and Egger and Hoink the reviews); nevertheless, the two-dimensional aspects of the problem have attracted considerable attention. In this case the front is supposed to pass over an infinitely extended ridge—a problem which is attractive because of its relative simplicity. Besides, much has been learned about the interaction of fronts and orography from analytic and numerical work on the topic of this highly idealized situation. Davies (1984) considered the passage of a shallow-water front over a ridge and discovered an analytic solution to the problem. The front is retarded when moving up-slope and accelerated when moving down-slope. Schumann (1987) arrived at the same conclusions when re-analysing Davies's problem using a numerical model with high-horizontal and high-vertical resolution. Blumen and Gross (1987) proposed that a weak front, at least, will be advected like a passive scalar over a mountain. Using analytic techniques Blumen and Gross showed that acceleration occurs on the upwind slope in partial contradiction of Davies’s result (see section 3 for further comments on this discrepancy). In addition, frontolysis is predicted on the windward side of the mountain, a feature which is excluded a priori in Davies’s model. Williams et al. (1992) tested the suggestions put forward by Blumen and Gross (1987) by integrating in time a two-dimensional model of a front passing over a ridge. By and large, the fronts behaved as predicted by the passive-scalar approach, at least on the up-slope (see Fig. 15 of Williams et al.).

Bannon (1983) introduced an interesting complication through supposing that the front and the mountain were embedded in a deformation flow, so that the enforced frontogenesis is affected by the orography. Bannon was able to provide analytic solutions to the problem of quasi-geostrophic frontogenesis in a deformation flow over shallow topography. Since then many versions of this problem have been studied (see Williams et al. 1992 for a succinct review; see also section 5). Williams et al. (1992), for example, found an almost linear superposition of frontogenesis induced by the deformation field

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with frontolysis on the up-slope and frontogenesis on the down-slope due to the mountain, provided that the mountain was sufficiently wide. Gravity waves might distort the frontal structure significantly on the lee side (see also Keuler et al. 1992).

In this paper we shall study the two-dimensional problem of the frontal passage over a mountain in deformation flow, using a three-layer version of Davies's (1984) model, in which the angular momentum and potential vorticity are conserved. This model is well suited for tackling some unresolved issues. For example, the role of mountains in deformation flow needs to be clarified, because pure deformation flow does not satisfy the continuity condition in the presence of a two-dimensional ridge. The lateral (zonal) inflow towards the mountain is not automatically balanced by the meridional outflow along the ridge (see Egger and Hoinka 1992 for a more detailed discussion of this problem). In the past various devices have been tried to deal with this situation (see also section 5), but here we are able to present analytic solutions to the problem. It is a characteristic feature of our model that the width of the frontal zone is well defined, so allowing us to study in detail the frontogenetic impact of orography in deformation flow. In particular, the delay or acceleration of frontal collapse due to the presence of the mountain will be investigated. Such problems are intractable when tackled by aid of a gridpoint model (see, however, Garner 1989, who used a Lagrangian grid to study frontal collapse in the absence of topography).

2. THE MODEL

The model used here is identical to the one described by Egger (1992), except that advection by a deformation field has been added to the flow equations. The structure of this two-dimensional model atmosphere is depicted in Fig. 1 and is seen to be composed of three dynamically active layers of homogeneous inviscid fluid underneath a fourth passive layer. The interface between the two most dense (coldest) layers 1 and 2 intersects the surface at \( x = x_1 \) (surface front 1), and that between layers 2 and 3 at \( x = x_2 \) (surface front 2). Layer 2 is called 'the frontal zone' of width \( w = x_2 - x_1 \) at the surface. The third layer represents the flow ahead of and, of course, also above the frontal zone. Suppose now that there is deformation flow and a zonal mean wind in the fourth layer, i.e.

\[
\begin{align*}
  u_4 &= U - \alpha(x - x_0), \\
  v_4 &= \alpha y,
\end{align*}
\]

where \( U \) and \( x_0 \) are constants. The peak of a triangular mountain of width \( d_e + d_w \) and height \( H_1 \) is located at the origin (\( \eta \) is the height profile and \( \gamma = d\eta/dx \) the slope). In

![Figure 1. Structure of the model.](image-url)
most cases we suppose that the mountain profile is symmetrical, i.e. \( d = d_c = d_w \); \( \gamma = \gamma_c = \gamma_w \). We also suppose that the potential vorticity is given by the expression

\[
P_i = f(H_i - H_{i-1})^{-1} \quad (i = 1, 3)
\]

for each layer, where \( H_i - H_{i-1} \) is the upstream depth of layer \( i \) \( (H_0 = 0) \). This model possesses sufficient flexibility to enable the problems outlined in the introduction to be tackled. Moreover, the number of dynamically active levels can be reduced, so that we can also investigate the behaviour of fronts in, say, a one-layer model.

Following Davies (1984) we stipulate that there is semigeostrophy, so that the alongfront flow velocity is given by the equation

\[
\nu_i = \sum_{j=i}^{3} f^{-1} g \frac{\partial h_j}{\partial x}
\]

where \( h_i(x) \) is the interface height, \( \rho_i \) the layer density, \( g_y = g \Delta \rho_y \), \( \Delta \rho_y = (\rho_i - \rho_{i+1})/\rho_i \), \( g \) the acceleration due to gravity, and \( f \) the Coriolis parameter. Since

\[
P_i = \left( \frac{\partial \nu_i}{\partial x} + f \right) (h_i - h_{i-1})^{-1} = f(H_i - H_{i-1})^{-1}
\]

where \( h_0 = \eta \) for \( x < x_1 \), \( h_1 = \eta \) for \( x \geq x_1 \), \( h_2 = \eta \) for \( x \geq x_2 \),

it follows that

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} (g_{11} h_1 + g_{12} h_2 + g_{13} h_3) - P_1 f(h_1 - \eta) &= -f^2 \\
\frac{\partial^2}{\partial x^2} (g_{22} h_2 + g_{23} h_3) - P_2 f(h_2 - h_1) &= -f^2 \\
\frac{\partial^2}{\partial x^2} g_{33} h_3 - P_3 f(h_3 - h_2) &= -f^2.
\end{align*}
\]

(4)

At the boundary the following conditions apply:

\[
\begin{align*}
h_i &\to H_i \quad \text{as} \quad x \to -\infty, \quad h_1 = \eta \quad \text{at} \quad x = x_1, \\
h_2 &= \eta \quad \text{at} \quad x = x_2, \quad h_3 \to H_3 - H_2 \quad \text{as} \quad x \to \infty.
\end{align*}
\]

Equations (4) can be solved if the positions \( x_1, x_2 \) of the surface fronts are known. The resulting heights, \( h_i \), are composed of a steady part, \( h_{is} \), and an unsteady part, \( h_i^* \), representing the advancing front, i.e.

\[
h_i = h_{is} + h_i^*.
\]

In particular, near the mountain, \( h_i = h_{is} \) as \( x \to x_1 \) and \( x_2 \to \infty \). In other words, \( h_{is} \) is the solution of (4) for steady-state flow over the mountain. Note that the parameters \( U \) and \( \alpha \) do not enter into (4), nor are they in the boundary conditions. Note also that (4) is valid only in the domain \( x \leq x_1 \). Corresponding solutions in the other cases will be denoted by \( h_{3i}, h_{3i} \) \((i = 1, 3)\), where the first index gives the number of layers. Corresponding two-layer equations are valid for \( x_1 \leq x \leq x_2 \) \((h_{2i}, h_{2i}, i = 2, 3; h_{21} = h_{21} = 0)\), and the one-layer equation, viz.

\[
\frac{\partial^2}{\partial x^2} g_{33} h_{13} - P_3 f(h_{13} - \eta) = -f^2
\]

(5)
must be solved for \( x > x_2 \) (\( h_{13}, h_{s13}, h_{12} = h_{11} \equiv h_{s12} = h_{s11} = 0 \)). Homogeneous solutions are of the type \( \exp(\pm \lambda x) \), where \( \lambda \) is a root of the characteristic equation of set (4) or of the related two-layer (one-layer) set of equations. Evaluation of the steady-state solutions \( h_{s10}, h_{s21}, h_{s31} \) is simple since the orography varies linearly with \( x \). However, we have to match analytic solutions at the kinks of the orographic profile because there are no derivatives of \( \eta \) at \( x = -d_w, 0, d_e \). We require continuity of \( h_{sij} \) and \( \partial h_{sij} / \partial x \) at these matching points and so for \( x \to \pm \infty \) we impose the following boundary conditions

\[
\begin{align*}
h_{s13} &\to H_3 - H_2, & h_{s22} &\to H_2 - H_1, \\
h_{s23} &\to H_3 - H_1, & h_{s34} &\to H_i.
\end{align*}
\]

Moreover, we also require continuity of \( h_1, h_2, \partial h_3 / \partial x, \partial h_2 / \partial x \) at \( x = x_1 \), and of \( h_{31}, \partial h_3 / \partial x \) at \( x = x_2 \) and so further impose the conditions

\[
\begin{align*}
h_1^* &\to 0 \quad \text{as} \quad x \to -\infty, & h_1^* &\to 0 \quad \text{as} \quad x \to \infty.
\end{align*}
\]

The boundary and matching conditions are displayed in Fig. 2. More details on these standard procedures were given by Egger (1992). The advantage of prescribing linear slopes, compared to more complicated topographic profiles, is obvious. According to (4), there is the simple inhomogeneous solution

\[
h_1 = \eta + f/P_1, \quad h_2 = h_1 + f/P_2, \quad h_3 = h_2 + f/P_3.
\]

It is helpful to exploit the fact that ‘angular momentum’ given by the equation

\[
f_\xi = f(x - x_0) + \nu,
\]

experiences an exponential decay in pure deformation flow (e.g. Hoskins 1971). The equation of meridional motion is

\[
\frac{\partial v_i}{\partial t} + [u_i - \alpha(x - x_0)] \frac{\partial v_i}{\partial x} + \alpha v_i + fu_i = f U
\]

Figure 2. Matching conditions for interface heights and volume fluxes at \( x = x_i \), and boundary conditions at \( x = \pm \infty \). To avoid confusion, steady-state boundary conditions at the kinks of the orographic profile are omitted.
where \( u'_i = u_i + \alpha(x - x_0) \) and where we assume that the pressure-gradient force linked to (1) is the same in all layers in the sense of the Boussinesq approximation. Combining (6) and (7) we find

\[
\frac{d\xi}{dt} = -\alpha \xi + U.
\] (8)

We now apply (8) to the surface fronts. The speed of front \( i \) is denoted by

\[
c_i = \frac{dx_i}{dt}.
\] (9)

With

\[
\xi_i = (x_i - x_0) + f^{-1} v_{i|x=x_i} \] (10)

and with (8) we have

\[
-\alpha \xi_i + U = c_i + f^{-1} \frac{dv_i}{dt} \bigg|_{x=x_i}.
\] (11)

A solution of (4) provides us with analytic expressions for \( h_i(x, x_1, x_2) \) and for \( v_i(x, x_1, x_2) \) (see (2)). In particular, we have for the one-layer model

\[
\frac{dv_i}{dt} \bigg|_{x=x_1} = c_1 \frac{\partial v_i}{\partial x} \bigg|_{x=x_1}.
\] (12)

The speed, \( c_1 \), is obtained from (11), and (12), and the position of the front can then be predicted via (9). However, for the three-layer model an alternative approach is necessary, and for this we have to evaluate the volume flux \( M_i = u'_i(h_i - h_{i-1}) \) in each layer. As usual (see Davies 1984 and Egger 1992), the relevant equations are derived from (7) and the equations of continuity,

\[
\frac{\partial}{\partial t} (h_i - h_{i-1}) + \frac{\partial}{\partial x} M_i - \alpha(x - x_0) \frac{\partial}{\partial x} (h_i - h_{i-1}) = 0,
\] (13)

by imposing the geostrophic relation (2) on the tendencies of \( v_i \) and \( h_i \). Further details about the evaluation of volume fluxes are given in the appendix. Once the volume fluxes are available we have

\[
c_i = -\alpha(x_i - x_0) + \frac{\partial M_i}{\partial x} \left( \frac{\partial}{\partial x} (h_i - \eta) \right)^{-1} \bigg|_{x=x_i}
\] (14)

where l'Hôpital's rule is invoked to evaluate \( u'_i \) at the surface fronts.

In a standard run we adopt the following values for the parameters: \( H_1 = 3000 \text{ m}, \ H_2 = 4000 \text{ m}, \ H_3 = 8000 \text{ m}, \ U = 10 \text{ m s}^{-1}, \ \alpha = 10^{-5} \text{ s}^{-1}, \ f = 10^{-4} \text{ s}^{-1}. \) The density change, \( \rho_{i+1} - \rho_i \), is considered to be the same for all layers: \( \Delta \rho = (\rho_{i+1} - \rho_i)/\rho_2 = 0.03. \) A model with layers of constant density is isomorphic to an isentropic model provided there is also incompressibility. Thus \( \Delta \rho \) can be interpreted also as a typical normalized increment of potential temperature, \( \theta. \) In particular, making the choice \( \Delta \rho = 0.03 \) implies that we have a jump in \( \theta \) of about 10 K at an interface.

3. **One-layer model**

It is illuminating (see also Davies 1984, and Blumen 1992) to consider, first of all, fronts in a one-layer model which contains only layer 1 of upstream depth \( H_1 \) with a
surface front at position $x = x_1$. Layers 2 and 3 are passive and need not be taken into account. In this case the model equations are (5), (8) and (11). Given the position $x_1$ of the surface front, the height profile is given by the equation (see also Blumen 1992)

$$h_1 = H_1 [1 - \exp(\lambda(x - x_1))] + \tilde{h}_{1l} + \eta - \tilde{h}_{s1}(x_1) \exp(\lambda(x - x_1)), \quad \text{(15)}$$

where

$$\tilde{h}_{s1} = \eta + \tilde{h}_{s1} + H_1 \quad \text{(16)}$$

is the steady-state solution of (5) with the indices altered, viz.

$$g_{33} \rightarrow g_{11}, \quad h_{13} \rightarrow h_{11}, \quad P_3 \rightarrow P_1; \quad \lambda^2 = f^2/(g_{11} H_1).$$

The meridional velocity at the front is given by the equation

$$v_1(x_1) = g_{11} f^{-1} \left\{ -\lambda H_1 + \left( \frac{\partial \tilde{h}_{s1}}{\partial x} + \frac{\partial \eta}{\partial x} - \lambda \tilde{h}_{s1} \right) \right\}_{x = x_1}. \quad \text{(17)}$$

We demonstrate the mountain effect by comparing the positions of two fronts both of which are located at $x = x_2$ at time $t = 0$ and of which one which has reached position $x = x_1$ at time $t$ is moving in the presence of a mountain while the other, with no mountain present, has reached the position $x_{1n}$ (index $n$ for no-mountain) at the same time, so that $\delta x_1$, the shift of the frontal position induced by the mountain, is

$$\delta x_1 = x_1 - x_{1n} \quad \text{(18)}$$

We suppose, moreover, that $x_0$ is sufficiently far upstream so that $\tilde{x}_1 = \tilde{x}_{1n}$ at time $t = 0$. It follows then, that

$$\begin{align*}
\tilde{x}_{1n} &= x_{1n} - x_0 - g_{11} f^{-2} \lambda H_1 \\
\tilde{x}_1 &= x_1 - x_0 - f^{-1} v_1(x_1).
\end{align*} \quad \text{(19)}$$

Since $\tilde{x}_1 = \tilde{x}_{1n}$ at any time (see (8)), we arrive with (17) at the equation

$$\delta x_1 = g_{11} f^{-2} \left( \frac{\partial \tilde{h}_{s1}}{\partial x} - \lambda \tilde{h}_{s1} \right)_{x = x_1}. \quad \text{(20)}$$

The shift $\delta x_1$ does not depend on $U$ or on $\alpha$. These parameters regulate the speed of the no-mountain front via (8), but the displacement $\delta x_1$ of the orographic front is independent of this speed. Note, in particular, that $\delta x_1 \to 0$ as $x \to -\infty$ wherever $h_{1l} = H_1$. In Fig. 3 we present $\delta x_1$ as a function of $x_{1n}$ for the case of a symmetrical mountain. In this case

$$\tilde{h}_{s1} \begin{cases} = A \exp(-\lambda |x|) & \text{for } |x| \geq d \\ = B \exp(-\lambda |x|) + C \exp(\lambda |x|) & \text{for } |x| < d \end{cases} \quad \text{(21)}$$

where

$$\begin{align*}
A &= \gamma \lambda^{-1} (\cosh(\lambda d) - 1) \\
B &= \gamma (2\lambda)^{-1} \{ \exp(-\lambda d) - 2 \} \\
C &= \gamma (2\lambda)^{-1} \exp(-\lambda d).
\end{align*} \quad \text{(22)}$$

Substituting (21) and (22) in (20) yields

$$\delta x_1 \begin{cases} = 0 & \text{for } x_1 < -d \\ = g_{11} f^{-2} \gamma \{ \exp(\lambda (d + x_1)) - 1 \} & \text{for } -d \leq x_1 \leq 0 \end{cases} \quad \text{(23)}$$
The front lags \( x_{in} \) as soon as it begins to ascend the slope (see also Fig. 3). This implies that there is a sudden drop in the speed of the front at the upwind base of the mountain. Note, that the derivative \( d(\delta x_1)/dx_1 \) does not exist at \( x = -d \). The lag increases as the distance to the foot of the mountain decreases, reaching a maximum at a point corresponding with the top. We have in fact

\[
\delta x_1 = g_{11} f^{-2} \gamma [1 + \{\exp(-\lambda d) - 2\} \exp(-\lambda x_1)] \quad \text{for} \ 0 \leq x_1 \leq d
\]

\[
\delta x_1 = 2g_{11} f^2 \lambda (\cosh \lambda d - 1) \exp(-\lambda x_1) \quad \text{for} \ x_1 > d.
\]

The front with mountain is catching up when on the down-slope, compared with the no-mountain front, and \( \delta x_1 = 0 \) at position

\[
x_{1c} = \lambda^{-1} \ln[2 - \exp(-\lambda d)].
\]

For \( x_1 > x_{1c} \), the front with orography is leading the no-mountain front and, in particular, \( \delta x_1 > 0 \) for \( x_1 > d \), but there is no shift far afield because \( \delta x_1 \to 0 \) as \( x_1 \to \infty \). So there is a downstream effect due to the mountain. There is a rather simple interpretation of (23), (24) and (25). The front with orography is not as steep at the up-slope as in the no-mountain case. Conservation of angular momentum requires that \( \delta x_1 < 0 \) (see (6)). On the other hand, the front is steeper in the presence of the mountain for \( x_1 > x_{1c} \) and must, therefore, be in advance of the no-mountain front. In Fig. 3, also, we give the shifts, \( \delta x_1 \), for asymmetrical mountains. The up-slope retardation is slightly larger the steeper the slope.

Figure 3 allows us to demonstrate qualitatively the impact of the mountain on fronts for any choice of \( U \) or \( \alpha \). We have

\[
\begin{align*}
x_{1n} &= x_e + Ut & \text{for } \alpha = 0 \\
x_{1n} &= x_e \exp(-\alpha t) + (x_0 + U \alpha^{-1} + \lambda g_{11} H_1 f^{-2} \{1 - \exp(-\alpha t)\}) & \text{for } \alpha \neq 0.
\end{align*}
\]

In Fig. 3 we depict the time-trace (27) of the position of the no-mountain front for a situation in which \( \alpha = 0 \), \( x_e = -3 \times 10^5 \text{m} \), \( U = 10 \text{m s}^{-1} \). The time-trace is a straight line,

\[
\begin{align*}
\alpha = 10^{-5} \text{ s}^{-1} & \quad \alpha = 0 \\
\text{t [h]} & \quad 24 \\
\delta x_1 [10^3 \text{m}] & \quad -4 \quad -2 \quad 0 \quad 2 \quad 6 \\
0 & \quad 10 \quad 20
\end{align*}
\]

\[x_{1n} = -1.16 \times 10^5 \text{m} \quad (10^5 \text{m} \text{ m})\]
of course, and we obtain the position $x_1$ corresponding with a time $t$ by adding $\delta x_1$ to the position $x_{1n}(t)$. It is obvious from Fig. 3 that the up-slope retardation is greater the faster the front is moving (see also (30)); but the lag $\delta x_1(x_{1n})$ is, of course, not affected by the choice of $\alpha$ and $U$. Also shown is the case if $\alpha = 10^{-5} \text{s}^{-1}$, $U = 10 \text{ m s}^{-1}$ and if $x_0$ is chosen such that the no-mountain front reaches a stagnation position at $x_{1sn} = 100 \text{ km}$ (Fig. 3). The front with mountain lags behind by an amount $x_{1a}$ at any time and the stagnation position, $x_{1s}$, is to the west of $x_{1sn}$. In this case the mountain affects the long-time behaviour of the front.

From (11), (12), (15) and (17) we obtain the phase speed, given by

$$c_1 = \{-\alpha(x_1 - x_0) - \alpha \xi^{-1} v_1(x_1) + U\left[1 + H_f^{-1} \left(\hat{h}_{1s} - \lambda^{-1} \frac{d\hat{h}_{1s}}{dx_1}\right)\right]_{x=x_1}\}^{-1}.$$  \hfill (29)

If there is no mountain (29) yields

$$c_1 = U + \alpha \lambda^{-1} - \alpha(x_1 - x_0).$$  \hfill (30)

The front is moving faster than the imposed flow. The negative velocities at the front provide a positive contribution to the term $-\alpha \xi$ in (8) which must be balanced by enhanced zonal propagation of the front. The front becomes stationary at position $x_{1sn} = x_0 + U \alpha^{-1} + \lambda^{-1}$.

With $\alpha = 0$, (29) reduces to

$$c_1 = U \left(1 + f^{-1} \frac{dv_1}{dx_1}\right)^{-1}.$$  \hfill (31)

In Fig. 4 (dots) we show $c_1$ calculated from (31) for a mountain with $H_f = 1000 \text{ m}$, $d = 200 \text{ km}$. There are retardation jumps of $c_1$ at $|x| = d$, and acceleration jumps at $x = $.

![Figure 4](image-url)  

Figure 4. Speed $c_1$ (m s$^{-1}$) of the front in the one-layer model as a function of $x$, for passage over a symmetrical mountain. No-deformation flow: dots. Deformation flow: continuous line; shown is $c_1 = c_1 + \alpha(x - x_0)$. Passive scalar approach for $\alpha = 0$: crosses. $H_f = 1000 \text{ m}$; $d = 200 \text{ km}$; $H_f = 4000 \text{ m}$; $\Delta \rho = 0.03$; $\lambda = 0.33 \times 10^{-5} \text{ m}^{-1}$. $U = 10 \text{ m s}^{-1}$; $x_0 = 0$; $\alpha = 10^{-5} \text{s}^{-1}$; mountain top at $x = 0$. 
0. There is slight acceleration on the way up the slope and strong deceleration on the way down the slope. The solution in Fig. 4 is close to breakdown. It is easy to show that the relevant criterion for breakdown is $B \leq -\frac{1}{2} H_1$ (see (22)). One may object that jumps or extremely rapid changes of the phase-speed of a front are not acceptable in a semigeostrophic model based on the supposition that zonal accelerations are small. However, numerical integrations with a shallow-water-front model by Haderlein (1989) showed that such rapid changes of the front’s speed did indeed occur at the foot of a triangular mountain, given the combinations of parameters as are in use in semigeostrophic theory. Davies (1984) and Blumen (1992) found up-slope retardation as well as down-slope acceleration in the case of a bell-shaped mountain. However, their solutions do not exhibit discontinuities in $c_1$ because the term $d\eta/dx$ exists everywhere for their mountain profiles.

One may contrast (31) with the frontal speed predicted by the passive scalar approach, viz.

$$c_1 = UH_1/(h_{k1} - \eta).$$

(32)

According to (32) there is slight deceleration when $x \leq -d$ (crosses in Fig. 4), with acceleration on the windward slope and deceleration on the leeward. So there is some qualitative agreement between (31) and (32) for $|x| \leq d$. Moreover, (32) correctly predicts the absence of effects far afield on the front’s position. However, the passive scalar approach, of course, misses the jumps in the front’s speed at $x_1 = -d$, 0, $d$ and predicts an upstream influence of the mountain which does not exist. Moreover, (32) is symmetrical with respect to the peak, while (31) is not. On the other hand, the jump in the frontal speed at $x = -d$ can be shown to be equal to

$$\Delta c = -\lambda U/(\lambda H_1 + \gamma).$$

Therefore, $\Delta c \rightarrow 0$ if $\lambda H_1 \rightarrow \theta$, i.e. there is no jump for a very deep front or for a rather weak front ($g_{k1} \approx 0$). The passive-scalar approach and Davies’s model both predict rather similar frontal motions in these limiting cases (see also Blumen 1992).

We show, in Fig. 4 (continuous line), $c'_1 = c_1 + \alpha(x - x_0)$ for a situation for which the no-mountain front, for $a > 0$, is moving faster than it would for $\alpha = 0$ (dots) throughout the section of the $x$-axis as shown. The jumps in the speed of the front at $x = -d$, $x = 0$ are indeed slightly larger for $\alpha \neq 0$ than for $\alpha = 0$. However, the solid curve could have been obtained, to a good approximation, by adding to the dotted curve the speed $\alpha \lambda^{-1} \approx 3\, \text{m s}^{-1}$. In other words, the deformation acts on the front almost as if the mountain was not there.

4. THREE-LAYER MODEL

We expect that most of the features of frontal motion found for the one-layer model will also reappear in the three-layer model. However, it is the distinctly novel feature of frontogenesis which is being investigated. In particular, we expect that frontal collapse, with $w = x_2 - x_1 = 0$, will occur after a finite time if $a > 0$, just as in Hoskins’s (1971) model. From (8) we obtain immediately

$$\frac{d}{dr}(\xi_2 - \xi_1) = -\alpha(\xi_2 - \xi_1).$$

(33)

The height profiles, $h_n$, and the related velocities can be evaluated if $x_1$ and $x_2$ are known. Therefore, to determine the difference, $\delta \xi_c$, at the collapse point, from the known height
field is straightforward, viz.
\[
\delta \xi_c = (\xi_2 - \xi_1)|_{w=0} = f^{-1}(v_2(x_2) - v_1(x_2))
\]
(34)
where subscript 'c' denotes 'collapse'. Because \(v_1 < v_2 < 0\) if \(w = 0\), we have \(\delta \xi_c > 0\). Therefore, frontal collapse is inevitable and occurs at time \(t_c\) given by
\[
t_c = \alpha^{-1} \ln(\delta \xi_0/\delta \xi_c),
\]
(35)
where, for \(t = 0\), \(\delta \xi_0 = \xi_2 - \xi_1\). However, in (34) the mountain has an effect on the meridional flow. Thus, \(\delta \xi_c\) as evaluated at \(x = x_2\) in the presence of the mountain differs from \(\delta \xi_{cm}\) (subscript, 'm' denotes 'no-mountain'). We show in Fig. 5 \(\delta \xi_c\) and \(\delta \xi_{cm}\) (dashed) for a symmetrical mountain. It is seen that \(\delta \xi_c > \delta \xi_m\) for \(x < -d\) but that the differences are rather small. However, \(\delta \xi_{cm}\) decreases as we approach the top of the mountain and increases later on, so that \(\delta \xi_c > \delta \xi_{cm}\) on the lower part of the down-slope downstream from the mountain. To interpret Fig. 5 properly we have to compare frontal zones with and without the mountain. Suppose that both zones are identical at \(t = 0\) (i.e. the starting point is located far upstream). If collapse occurs, say, at the windward slope for the front with mountain, we have, from Fig. 5, \(\delta \xi_c < \delta \xi_{cm}\), therefore, the collapse of the no-mountain frontal zone must have occurred earlier. The collapse has been retarded by the mountain. On the other hand, if the frontal zone collapses in the downstream plain, this collapse happens earlier than in the case of the no-mountain front. The mountain affects the steepness of the frontal interfaces in much the same way as it does the steepness of the front in the one-layer model. Correspondingly, \(\delta \xi_m\) in Fig. 5 is rather similar to \(\xi x_1\) in Fig. 3.

In analogy with the one-layer case, we evaluate the shifts in the positions of both fronts, \(\delta x_1 = x_1 - x_{1m}\), \(\delta x_2 = x_2 - x_{2m}\), for the case of the no-mountain situation. Again we need not specify \(U\) or \(\alpha\). However, it is no longer possible to write down a formula like (20) for the shift. Instead a simple iterative method is used to find the positions \(x_{1m}\), \(x_{2m}\) of both fronts in the absence of the mountain, given the respective locations, \(x_i\), in the presence of the mountain. Again suppose that \(x_i = x_{1m}\), \(\xi_i = \xi_{2m}\) far upstream. The result is displayed in Fig. 6, where the continuous lines depict \(\delta x_1\) and the dashed lines \(\delta x_i\) are for a symmetrical mountain. However, Fig. 6 contains a slight inaccuracy, in that shifts \(\delta x_i\) do not exist if the front with mountain has already collapsed while the front without mountain has not yet collapsed. We see from Fig. 5 that this situation occurs for
$x_{2n} \approx x_{1n} \approx 10^5$ m. The corresponding boundaries of validity have not been inserted in Fig. 6 since they are too close to the line $x_{1n} = x_{2n}$.

If the confining interfaces 1 and 2 of the frontal zone do not affect each other, the isolines of the shifts $\delta x_1$, $\delta x_2$ will be parallel to the coordinate axes $x_{2n}$, $x_{1n}$ and similar to those of the one-layer model depicted in Fig. 3. In particular, one would have $\delta x_i = 0$ for $x_{in} < -d$. Indeed, this simple scheme of decoupled fronts explains the basic structure of Fig. 6. However, there are significant deviations; for example, we have $\delta x_1 > 0$ for $x_1 < -d$. Surface front 1 is accelerated in the upstream plain when front 2 is moving over the mountain. The more front 1 is retarded at the up-slope the broader the frontal zone. Finally, frontal collapse on the leeside takes place earlier in the presence of the mountain (see also Fig. 5).

Suppose now that there is an unperturbed frontal zone for $\alpha = 0$. The width, $w_n = x_{2n} - x_{1n}$, does not change in time, and the point $x_{1n}$, $x_{2n}$ moves with speed $U$ along the line $w_n = \text{constant}$ (Fig. 6; circles at one-hourly distances, $U = 10$ ms$^{-1}$). The mountain-induced shifts of the positions of the fronts can be found immediately from Fig. 6. We see that upstream frontogenesis becomes strong when front 2 is on the up-slope while
$x_{1a} < -d$. There is frontolysis when both fronts are on the up-slope. Frontogenesis is strong if both fronts are on the down-slope (see also Egger 1992). Let us now add deformation. The crosses in Fig. 6 mark the time-trace of an unperturbed frontal zone at one-hourly distances in the case of deformation flow. The starting point is the same for both traces and is located slightly outside the figure. Of course, the frontal zone moves faster in deformation flow. Frontal collapse occurs in the downstream plain (cross and circle). Mountain-induced shifts, $\delta x$, occur at a faster rate, but otherwise differences of mountain effects, with and without deformation flow, are almost the same. However, frontal collapse occurs earlier in the presence of the mountain (star). This is in keeping with Fig. 5 where advanced collapse is predicted for the downstream plain.

We demonstrate the mountain effect again in Fig. 7 by presenting time-traces of frontal zones for $\alpha = 0$ (dashed) and $\alpha = 10^{-5} \text{s}^{-1}$. Also given is the frontal width, $w$. The dots give the width for deformation flow without a mountain. All features derived from Fig. 6 come out well in Fig. 7. In particular, the jumps in the speeds $c_i$ at $x = -d$, 0, $d$, can be clearly seen. The frontal collapse for the no-mountain case (cross) occurs later and farther downstream. The $w$-curves for $\alpha = 10^5 \text{s}^{-1}$ with and without mountain resemble closely those given by Williams et al. (1922, their Fig. 4; there is a misprint in the caption of their Fig. 4: the dashed curve represents the front starting at the foot of the mountain).

Although the model contains 16 parameters one would not expect to find dramatic changes if these are varied within reasonable limits. For example, collapse occurs later if $g_{33}$ is enhanced for given initial positions $x_{i0}$. The explanation is simple: meridional
winds at the fronts are larger if $g_{23}$ is enhanced and, in particular, if $\delta_2$ is increased; therefore, collapse occurs later. Figure 8 summarizes results from a series of experiments where the mountain height is varied from $H_T = 0$ to $H_T = 5000$ m. We display the change of width, $w$, in time. For $H_T = 0$ (case 1), there is frontogenesis caused by the deformation field, with collapse after 13 hours. The frontal zone is capable of crossing the mountain when $H_T = 500$ m (case 2), and we find all stages as discussed above: upstream frontogenesis followed by strong frontogenesis, when only part of the frontal zone is on the slope ($1.5 \text{ h} < t < 3 \text{ h}$). Afterwards, there is up-slope frontolysis, which is quite strong, when $x_2$ passes the crest ($5.5 \text{ h}$). Front 1 passes the crest at time $t = 7.2 \text{ h}$. Frontogenesis is rapid after that, with collapse on the down-slope. When $H_T = 1000$ m, front 2 passes the crest; but the solution breaks down when front 1 reaches the crest (case 3). Finally,

Figure 8. Width $w$ (km) of the frontal zone as a function of time for various mountain heights. (1) $H_T = 0$; (2) $H_T = 500$ m; (3) $H_T = 1000$ m; (4) $H_T = 2000$ m; (5) $H_T = 4000$ m; $H_1 = 3000$ m; $H_2 = 6000$ m; $H_3 = 9000$ m; $U = 10 \text{ m s}^{-1}$; $\alpha = 10^{-7} \text{s}^{-1}$; $\Delta \rho = 0.03$. 
breakdown occurs for \( x_2 = 0 \) and \( H_T = 2000 \) m (case 4), \( H_T = 4000 \) m (case 5). It is seen that the up-slope retardation is stronger, the higher the mountain.

5. Discussion

The key results of this study are based on the simple fact that the height field of the three-layer model can be determined if the positions of the fronts are known. We were able to deduce the mountain-induced shifts, \( \Delta x_a \), of the fronts by invoking the angular-momentum equation (8). These shifts do not depend on \( U \) or on \( \alpha \). The slopes of the frontal interfaces are not so steep over most of the mountain as they would be in the absence of orography, but are steeper on the leeward side. This implies that there are negative shifts on the up-slope and positive shifts in the lee. In particular, frontal collapse occurs earlier on the leeward side because of this steepening of the fronts.

Unfortunately it has not been possible to verify our results (and those of Zehrner and Bannon 1988 (ZB), Williams et al. 1992 (WPZ) and Keuler et al. 1992 (KKKS)) against observations. Although up-slope retardation of fronts is common (Egger and Hoinka 1992), frontogenetic effects are rather difficult to observe in mountainous terrain where even the tracing of a front becomes problematic. Moreover, there appears not to be any case-studies of frontal passages over mountains in large-scale deformation flow.

On the other hand, the simplicity of the semigeostrophic three-layer model allowed us to obtain a satisfactory physical understanding of the motion of a frontal zone over a mountain. It remains to be seen if our results are of any help towards a better understanding of the results obtained by ZB, WPZ and KKKS. There is a problem, however, that the deformation flow is treated differently in all these three papers. For example, ZB suppose that there is not an upstream influence of the mountain at the inflow boundary, unlike in our model where the mountain influence can be shown to extend far upstream. The deformation field of WPZ is periodic in space, moves with constant speed and varies with \( H/(H - \eta) \) (\( H \) is the height of the rigid lid). KKKS use open upper boundary conditions so that the imposed deformation convergence has an outlet above the mountain. The model of ZB is based on the geostrophic momentum approximation, while WPZ and KKKS use the primitive equations. Moreover, the models of ZB, WPZ and KKKS are based on gridpoint methods. Frontal collapse is not well defined in such models, neither is the location of the frontal zone (see, in particular, KKKS, where this point is discussed in some detail). ZB conclude that "Flow over the mountain ridge results in retardation of the surface cold front on the upstream side, while rapid advection of the front across the mountain top yields an advancement of the frontal position downstream. The combination of acceleration and deceleration produces a net 100 km advancement of the front far downstream compared with the front-only (no-mountain) case. The front is significantly weakened on the up-slope side, but reappears stronger in the lee." These conclusions are so close to ours, that we can state that the simpler three-layer model captures the essence of ZB's model. There is one major difference: ZB found a weakening of the front far downstream due to the mountain, whereas our model predicts an intensification which decreases with increasing distance towards the mountain, provided there is no collapse. We speculate that this difference of results is due to the fixing by ZB of upstream boundary conditions which enforce an enhanced ageostrophic outflow downstream (see also Fig. 5(b) of ZB and relevant comments on their p. 639). This interpretation is supported by WPZ in so far as they do not report a weakening of the front far downstream. Otherwise WPZ reconfirm the results of ZB. Thus WPZ's key results can be reproduced by our simpler model. KKKS report having obtained results similar to WPZ, but also to have found "the formation of a new frontal structure on the
leeward side of the mountain”, linked to the presence of a strong leewave—a process which is excluded \textit{a priori} in our semigeostrophic model.

Altogether we feel that the three-layer model helps towards a better understanding of the results that have already appeared in the literature. Moreover, we have been able to throw light on the effect of mountains on frontal collapse.

\section*{Appendix}

We split the volume flux $M_i = u_i'(h_i - h_{i-1})$ into a steady-state part

$$M_{sl} = u_{sl}'(h_{sl} - h_{sl-1})$$  \hfill (A.1)

and a rest-part $M_i^*$, so that

$$M_i = M_{sl} + M_i^*.$$  \hfill (A2)

It is also convenient to introduce a relative volume flux $M_i^r = M_i^r - U(H_i - H_{i-1})$. From (7), with $\partial v_{sl}/\partial t = 0$, we obtain

$$M_{sl} P_i = \alpha(x - x_0) \frac{\partial v_{sl}}{\partial x} - \alpha v_{sl} + fU.$$  \hfill (A3)

There is no impact of orography on the volume flux for $\alpha = 0$, i.e. $M_{sl} = fU/P_i$. However, (A3) shows clearly that the orographically induced steady-state flow affects $M_{sl}$ and, therefore, $M_i$. It is straightforward to show that $M_{sl}$ satisfies also the steady-state form of (13). Note that $M_{sl}$ is not differentiable in the lowest layer at $x = -d_w$, 0, $d_c$, since $\partial h_{0}/\partial x$ does not exist at these points. Obviously, $M_{sl} \to U(H_i - H_{i-1})$ as $x \to \pm \infty$.

Equations for the volume fluxes $M_i^*$ are derived by imposing (2) upon the tendencies in (6) and (13), with $h_0 = 0$. One finds

$$
\begin{align*}
&g_{33} \frac{\partial^2}{\partial x^2} (M_1^* + M_2^* + M_3^*) - M_3^* P_3 f = 2\alpha f v_3^*
\end{align*}
\hfill (A4)
\begin{align*}
&g_{22} \frac{\partial^2}{\partial x^2} (M_1^* + M_2^*) + g_{33} \frac{\partial^2}{\partial x^2} (M_1^* + M_2^* + M_3^*) - M_3^* P_2 f = 2\alpha f v_2^*
\end{align*}
\begin{align*}
&g_{11} \frac{\partial^2}{\partial x^2} M_1^* + g_{12} \frac{\partial^2}{\partial x^2} (M_1^* + M_2^*) + g_{13} \frac{\partial^2}{\partial x^2} (M_1^* + M_2^* + M_3^*) - M_3^* P_1 f = 2\alpha f v_1^*
\end{align*}
$$

where $v_i = v_{sl} + v_i^*$. The boundary conditions are: $M_i^* \to 0$ as $x \to -\infty$, $M_i^* = -M_{sl}$ at $x = x_1$, $M_i^* = -M_{sl}$ at $x = x_2$ (see (A2)).

Of course, (A4) is valid for $x \leq x_1$ only, and we introduce the notation $M_{3i}$, $M_{3i}$ for this part of the solution. The corresponding two-layer equations are valid for $x_1 \leq x \leq x_2$, ($M_{sl}$, $M_{sl}$, $i = 2,3; M_{sl} = M_{sl} = 0$), and

$$
\begin{align*}
&g_{33} \frac{\partial^2}{\partial x^2} M_{11}^* - M_{11}^* P_3 f = 2\alpha f v_3^*
\end{align*}
\hfill (A5)
$$

must be solved for $x \geq x_2$. Homogeneous solutions are of the type $\exp(\pm \lambda x)$, where the roots $\lambda$ are identical to those of the related potential vorticity equations.

It is straightforward to obtain steady-state volume fluxes $M_{sl}$ from (A3) for the one-, two- and three-layer version of the problem. Evaluation of $M_i^*$ involves terms of the type $x \exp \lambda x$, which are not given here. Finally, we have to match three-layer and two-layer solutions at $x = x_1$ as well as a two-layer solution and a one-layer solution at $x = x_2$. We require continuity of $M_2$, $M_3$ at $x = x_1$ and of $M_3$ at $x = x_2$. However, as pointed
out by Egger (1992), we cannot impose continuity of $\partial M_{\alpha}/\partial x$ since that would violate the Galilean invariance of the model. In particular, we must have

$$M_i = U(h_i - h_{i-1}) \quad (A6)$$

in the absence of orography, and for $\alpha = 0$. Since, for example, $h_1$ is not differentiable at $x = x_1$, it follows that there is no derivative of $M_2 = U(h_2 - h_1)$ at $x = x_1$. However, we may satisfy (A6) by imposing continuity of $\partial / \partial x (M_1 + M_2)$, $\partial / \partial x (M_1 + M_2 + M_3)$ at $x = x_1$ and continuity of $\partial / \partial x (M_2 + M_3)$ at $x = x_2$. Matching and boundary conditions for heights and mass fluxes are summarized in Fig. 2. Another approach to satisfy (A6), given by Egger (1992; option I), is to have differentiable volume fluxes but to admit kinks in the height profiles. This approach is not acceptable here since the terms $\alpha v_i$ in (6) and (4) would not be defined above the surface fronts.

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