A new conservation law of the shallow-water equations

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SUMMARY

A new conservation law is derived from the inviscid shallow-water equations. This law governs the spreading of a fixed quantity of fluid. The symmetries of the Lagrangian of shallow-water flow are determined. The links of these symmetries to the new law and to other conservation laws are discussed.

1. INTRODUCTION

It is likely that the most important conservation laws of fluid dynamics have been found in the past. One may even expect that any new conservation law will be rather complex and, therefore, difficult to apply. In this note a conservation law for shallow-water flow is presented which is about as complicated as the law of energy conservation, but more difficult to interpret.

The shallow-water equations in Lagrangian coordinates \((a, b)\) are:

\[
x_{at} - 2\Omega y_{a} = -gh_x + \Omega^2 x
\]

\[
y_{at} + 2\Omega x_{a} = -gh_y + \Omega^2 y
\]

\[
h = \frac{\partial(a, b)}{\partial(x, y)}
\]

where \(\Omega\) is the earth’s angle of velocity, \(g\) is the acceleration due to gravity and \(h\) is height of the free surface; \(x(a, b, \tau), y(a, b, \tau), t = \tau\) are Eulerian coordinates and, for example,

\[
h_x = \frac{\partial(h, y)}{\partial(a, b)} h
\]

is the derivative of fluid depth \(h\) with respect to \(x\). Note that the gradients of the centrifugal potential are retained. In (3) a convenient labelling of the Lagrangian coordinates (e.g. Salmon 1983) is adopted.

A conservation law of (1)–(3) must be of the form

\[
\frac{\partial P_1}{\partial \tau} + \frac{\partial P_2}{\partial a} + \frac{\partial P_3}{\partial b} = 0
\]

where \(P_i = P_i(a, b, \tau, x^{(n)}, y^{(n)})\) and where the symbol \(x^{(n)}\) denotes \(x\) and derivatives with respect to \(a, b, \tau\) up to order \(n\). Material conservation laws with \(P_2 = P_3 = 0\) are of particular interest. However, it is known that potential vorticity \(q = (f + y_{x} - x_{y})h^{-1}\) and functions thereof are the only non-trivial materially conserved quantities of shallow-water flow (Egger 1989; Shepherd 1990). Of course a conservation law (5) based on (1)–(3) does not contain additional information with respect to the set of shallow-water equations. However, (5) can be integrated easily over a fluid domain, and it is mainly this integrated form of (5) that turns out to be useful.

2. THE CONSERVATION LAW

It is proposed that

\[
P_1 = \frac{1}{2} (x^2 + y^2)_{\tau} - 2\tau \mathcal{E}
\]

\[
P_2 = \frac{g}{2} h^2 (Q^{x} y_{b} - Q^{y} x_{b})
\]

\[
P_3 = -\frac{g}{2} h^2 (Q^{y} x_{a} - Q^{x} x_{a})
\]

\[
\mathcal{E} = \int_{\mathcal{D}} x_{a} \text{d}a \text{d}y
\]
where
\[ Q = (Q^x, Q^y) = (x + 2\Omega ry - 2\tau x, y - 2\Omega rx - 2\tau y) \] (9)
and
\[ E = \frac{1}{2} [(x_\tau - \Omega)^2 + (y_\tau + \Omega x)^2 + gh] \] (10)
is the energy of the flow in the 'absolute' nonrotating system. The law (5)–(8) is somewhat unusual in that it contains time \( \tau \) explicitly.

We have to demonstrate that (6)–(8) satisfies (5). One finds immediately that
\[ \frac{\partial P_1}{\partial \tau} = (x_\tau - 2\Omega y_\tau - \Omega^2 x)Q^x + (y_\tau + 2\Omega x_\tau - \Omega^2 y)Q^y - gh - \tau gh_\tau \] (11)
so that
\[ \frac{\partial P_1}{\partial \tau} = -gh - \tau gh_\tau - Q^x \frac{\partial h}{\partial x} - Q^y \frac{\partial h}{\partial y}. \] (12)
Moreover
\[ \frac{\partial P_2}{\partial a} + \frac{\partial P_3}{\partial b} = \frac{g}{2} Q^x (\dot{h}_a^2 y_b - \dot{h}_b^2 y_a) - \frac{g}{2} Q^y (\dot{h}_a^2 x_b - \dot{h}_b^2 x_a) + \frac{g}{2} h^2 \{ (x_a + 2\Omega y_a - 2\tau x_a) y_a - (y_a - 2\Omega x_a - 2\tau y_a) x_a \} \]
\[ - (y_a - 2\Omega x_a - 2\tau y_a) x_b - (x_a + 2\Omega y_a - 2\tau x_a) y_b \]
It follows with (3), (4) that
\[ \frac{\partial P_2}{\partial a} + \frac{\partial P_3}{\partial b} = \frac{g}{2} Q^x \frac{\partial (\dot{h}_a^2 y_b - \dot{h}_b^2 y_a)}{\partial (a, b)} - \frac{g}{2} Q^y \frac{\partial (\dot{h}_a^2 x_b - \dot{h}_b^2 x_a)}{\partial (a, b)} + gh - \tau g\dot{h} x_a y_b - x_b y_a - x_b y_a + y_b x_a \]
\[ = g Q^x \frac{\partial h}{\partial x} + g Q^y \frac{\partial h}{\partial y} + gh + \tau gh_\tau. \] (13)
This completes the proof. The proof shows that the conservation law can be derived from the shallow-water equations by multiplying (1) by \( Q^x \) and (2) by \( Q^y \). By adding both expressions one obtains the characteristic form of the conservation law (e.g. Olver 1986). The law (5), (6)–(8) is then recovered by running backward through all manipulations.

The equation of continuity in Eulerian form
\[ \frac{\partial h}{\partial t} + \nabla \cdot (hv) = 0 \] (14)
where \( \mathbf{v} = (u, v) = (x_\tau, y_\tau) \) is the velocity, must be invoked to convert (5) to Eulerian form. One finds
\[ \frac{\partial}{\partial t} (P_1 h) + \frac{\partial}{\partial x} (P_1 u h + b_1 P_2 - a_1 P_3) + \frac{\partial}{\partial y} (P_1 v h + a_2 P_3 - b_2 P_2) = 0. \] (15)

3. SPREADING OF FLUID OF CONSTANT VOLUME

I wish to apply the new conservation law to the problem of spreading of a fixed quantity of fluid. Consider a simply connected blob \( B \) of fluid where \( h = 0 \) at the boundaries (see also Salmon 1983) and choosing the origin of the coordinate system such that
\[ \tilde{h} = \int_B \mathbf{x} \, da \, db = 0 \] (16)
\( (x = (x, y); \, \tilde{\cdot} \, \text{mean over} \, B) \). By integrating (5) over the domain of the blob one obtains
\[ \tilde{P}_{1\tau} = 0. \] (17)
A mean radius $\tilde{r}$ of the blob, where $r$ is the distance to the origin, is defined as:

$$\tilde{r}^2 = \int_B x^2 \, da \, db.$$  \hspace{1cm} (18)

Of course, $\tilde{E}$ is invariant. Then (6) and (17) yield

$$\frac{1}{2} \frac{d}{dt} \langle \tilde{r}^2 \rangle = 2 \tilde{E}.$$  \hspace{1cm} (19)

Obviously the blob will grow without limitation. Note, in particular, that (19) applies also to rotating systems. This unlimited growth of a blob appears to contradict the well-known fact that anticyclonic symmetric vortices can be maintained in cyclostrophic balance. However, these vortices remain stationary only in those rotating systems where the centripetal terms in (1), (2) can be removed. Let us assume a circular blob centred at the origin and rotating with rotation rate $\Omega^\prime$. The blob will not grow if we remove the terms $\Omega^2 x$ from the right-hand side of (1), (2) and if $\Omega^\prime$, $h(r)$ are chosen such that $\Omega^\prime \left( \Omega^\prime + 2\Omega \right) = g(\partial h/\partial r) < 0$ throughout the blob, i.e. if $\Omega^\prime < 0$, $\Omega^\prime + 2\Omega > 0$. However, the blob will grow according to (19) if these terms are retained. In other words a blob need not grow on the rotating earth.

Assume now a nonrotating axisymmetric blob of radius $R$ in a nonrotating system ($\Omega = 0$). For the sake of simplicity, suppose a triangular profile $h = h_0(r)(1 - r/R)$ and radial speed $r_t = \alpha(r)r/R$ with $\alpha(0) = 0$. Integration of (19) yields

$$\alpha R = \frac{20\tilde{E}t}{3V},$$  \hspace{1cm} (20)

where $V = R^2 h_0 \pi / 3$ is the volume of the blob. Since $R_t = \alpha$ in the model it is immediately found that

$$R^2 = R'(0) + 20\tilde{E}t^2 / 3V.$$  \hspace{1cm} (21)

The blob expands with almost constant speed in the long-term limit, i.e. for $R \gg R(0)$. This prediction is based on the assumption of a triangular profile and can be tested by performing a numerical integration of the model equations (1)–(4). A brief description of the numerical scheme is given in the appendix. Figure 1 shows the ratio $\left( R^2 - 1 \right) / \tau^2$ as a function of nondimensional time $\tau' = \left[ g h_0(0) \right]^{1/2} / R(0)$, where $\tau' = R/R(0)$ for an axisymmetric blob with an initially triangular height profile. This ratio would equal $20\tilde{E} / 3V$ (dashed), if the assumptions on $h$ and $r_t$ were correct.

![Figure 1. Nondimensional ratio $(R^2 - 1)/\tau^2$ as a function of nondimensional time $\tau'$ (full line) as obtained in a numerical integration of the shallow-water equations (1)–(4) for an axisymmetric blob, where $R'$ is the nondimensional radius. The ratio is not defined at $\tau' = 0$. The initial height profile is triangular and $r_t = 0$ at $\tau' = 0$. Also given is the quantity $20\tilde{E}/3V$ (dashed line; nondimensional).](image-url)
There is a first phase where the blob expands faster than predicted by (21), but the agreement with (21) is surprisingly good for $r' > 5$, say. Then the growth is almost linear in time and (21) provides a reasonably accurate guess of the growth rate. This example shows that the new law allows us to arrive at simple estimates of the growth rate of a blob in the long-term limit.

When looking for observational verification of these results one has to keep in mind that although (17) is an exact result, observations will disagree with (17) if the underlying assumptions are not satisfied, i.e. if inviscid shallow-water flow is not a good model of the observed flow. Instantaneous release of a finite volume of fluid into another fluid in so-called lock-exchange experiments (e.g. Simpson 1987), through oil spreading on the sea (Houtl 1972) or during dispersion of dense gases (Britter 1989) provide observational tests for (17), (21). An experimental result is $R_i = 1.2 \left(\frac{g\bar{h}}{r'}\right)^{1/2}$, where $\bar{h}$ is the depth of the fluid just behind the head of the density current, which forms soon after the release (Huppert and Simpson 1980). However, (21) yields in the long-term limit $R_i = \left(\frac{5gh_0(0)}{3}\right)^{1/2}$. Because $h_0(0) \gg \bar{h}$, the shallow-water theory predicts considerably larger speeds of the front than observed. This has been known for a long time. Whitham (1955), for example, invokes friction to explain this discrepancy, while Harlow and Welch (1965) claim that this is a nonhydrostatic effect. A more detailed discussion of this point is beyond the scope of this note, but it is clear that inviscid shallow-water theory is not capable of explaining the observations. There is, however, even the contradicting theoretical result of Houtl (1972), who predicts $R \sim r'^{3/2}$ for nonviscous expansion of an axisymmetric blob. Houtl (1972) proposes a similarity solution of the form

$$r_t = \frac{r}{t} s(\eta)$$  \hspace{1cm} (22)

$$h = H(\eta)/r^2$$ \hspace{1cm} (23)

($\eta = r/\sqrt{r'}$; see also Britter 1989), which satisfies the shallow-water equations for axisymmetric flow. Houtl imposes the additional condition that the Froude number at the leading edge is constant and arrives at analytic expressions for the similarity functions $s, H$. Houtl’s solution clearly contradicts the new law, which is based on the same equations. This paradox is resolved easily: A straightforward calculation shows that the blob’s energy decreases in Houtl’s solution. Moreover, the similarity solution (22), (23) as evaluated by Houtl does not satisfy the boundary condition $h(R) = 0$. Thus, (22), (23) do not solve the shallow-water equations for a symmetric blob.

4. SYMMETRIES OF THE SHALLOW-WATER EQUATIONS

Conservation laws can be derived immediately (see (31)) if the symmetries of the Lagrangian $L$ are known. This motivates one to search for the symmetries of the Lagrangian of shallow-water flow:

$$L = \frac{1}{2} \left( x_t^2 + y_t^2 \right) - \frac{1}{2} gh + \frac{1}{2} \Omega^2 (x^2 + y^2) - \Omega x y_t + \Omega y x_t,$$  \hspace{1cm} (24)

(e.g. Salmon 1983). In particular, the symmetry related to the new law needs to be found. An operator (‘infinitesimal generator’)

$$p = \xi_t \frac{\partial}{\partial t} + \xi_a \frac{\partial}{\partial a} + \xi_b \frac{\partial}{\partial b} + \phi_t \frac{\partial}{\partial x} + \phi_y \frac{\partial}{\partial y}$$ \hspace{1cm} (25)

is said to generate a symmetry of (24) if

$$pr^{\langle 1 \rangle} p(L) + L \Div \xi = 0$$ \hspace{1cm} (26)

with

$$\Div \xi = \xi_t + \xi_a + \xi_b + \xi_x x_t + \xi_y y_t + \xi_x x_a + \xi_y y_b$$ \hspace{1cm} (27)

(e.g. Olver 1986, Theorem 4.12), where the coefficients $\xi$ and $\phi$ in (25) may depend on $a, b, r, x, y$. It is straightforward to evaluate the prolongation $pr^{\langle 1 \rangle} p$ of $p$ in (26) (Olver 1986; (2.38), (2.39)), but the related formula is too involved to be written down here.

In particular, (26) reduces to a simple differentiation of the Lagrangian if the coefficients $\xi$ and $\phi$ are constants. Then a symmetry exists if $pL = 0$ and we see immediately from (24) that the infinitesimal generators $p = \delta/\delta t$ and $\bar{p} = \delta/\delta a$ generate symmetries of $L$ (see also (31), II, III). In
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general, symmetries of the Lagrangian (24) are found by solving (26) for \( \xi^r, \xi^a, \xi^b, \phi^r, \phi^a \). The related procedures are cumbersome but straightforward and are, therefore, not presented (see also Albert (1992) for a discussion of the symmetries of the Euler equations).

For the sake of simplicity, results are presented for the nonrotating case only. In that case (26) is satisfied if \( \phi^r = \phi^a(x, y), \phi^b = \phi^b(x, y), \xi^r = \xi^b(t), \xi^a = \xi^a(a, b), \xi^b(a, b) \), and if

\[
\begin{align*}
\phi^r &= \phi^a \\
\phi^b &= \phi^a \\
\xi^r &= \xi^b = 0 \\
\xi^a + \xi^b &= 0 \\
\phi^r + \phi^b &= 0.
\end{align*}
\]  

The solution to (28) is

\[
\begin{align*}
\phi^r &= c_1x + c_2y + c_3 \\
\phi^b &= c_1y - c_2x + c_4 \\
\xi^r &= 2c_3t + c_5 \\
\xi^a &= \frac{\partial S(a, b)}{\partial b} + Q(b) \\
\xi^b &= -\frac{\partial S(a, b)}{\partial a} + K(a)
\end{align*}
\]  

where \( c_1 - c_4 \) are arbitrary constants and \( S, Q, K \) are smooth functions. Although (29) implies that there is an infinite number of conservation laws, closer inspection reveals that there are only six symmetries, which must be taken into account in a search for conservation laws. These are

\[
\begin{align*}
(\text{I}) & \quad \phi^r = 1 \ (\text{or} \ \phi^b = 1) \\
(\text{II}) & \quad \xi^r = 1 \\
(\text{III}) & \quad \xi^a = 1 \ (\text{or} \ \xi^b = 1) \\
(\text{IV}) & \quad \phi^r = x, \ \phi^b = y, \ \xi^r = 2t \\
(\text{V}) & \quad \xi^a = -a, \ \xi^b = b \\
(\text{VI}) & \quad \phi^r = y, \ \phi^b = -x
\end{align*}
\]  

where all other unspecified coefficients \( \phi, \xi \) vanish for each symmetry. The related conservation laws are given by:

\[
P_i = \left( \phi^r - \sum_{j=1}^{3} \xi^j x_j \right) \frac{\partial L}{\partial x_i} + \left( \phi^b - \sum_{j=1}^{3} \xi^j y_j \right) \frac{\partial L}{\partial y_i} + \xi^i \xi^j \frac{\partial L}{\partial \xi^j} \]  

where \( i = 1, 2, 3 \) corresponds with \( r, a, b \) as in (5) (e.g. Olver 1986; (4.41)). The symmetry (I) yields (1), but in a form which is compatible with (5). One obtains, of course, energy conservation from (II). The symmetry (III) is a relabelling symmetry with the conservation law

\[
P_1 = -c_1 x_t - c_2 y_t, \quad P_2 = c_3 \left( x_t^2 + y_t^2 \right) - gh, \quad P_3 = 0.
\]  

This law appears to be new but hard to interpret. It corresponds with a law derived by Albert (1992) for the three-dimensional Euler equations ((3.7) of Albert 1992). Inserting (IV) in (31) we find that (IV) is the symmetry of the conservation law (6)–(8).

Although the full machinery of the theory of symmetries had not yet been applied to the shallow-water equations, it has been suggested strongly (e.g. Salmon 1983) that potential-vorticity conservation is related to a relabelling symmetry. Symmetry (V) yields with (31)

\[
P_1 = a(x_a x_t + y_a y_t) - b(x_b x_t + y_b y_t).
\]
Simple manipulations transform this term so that one obtains

$$ P_1 = abq + \frac{\partial}{\partial b} \{ab(x_ay_r + y_ax_r)\} - \frac{\partial}{\partial a} \{ab(x_by_r + y_bx_r)\} \] $$

$$ P_2 = -\frac{a}{2}(x_1^2 + y_1^2 - 2gh) $$

$$ P_3 = \frac{b}{2}(x_1^2 + y_1^2 - 2gh) \] $$

(34)

With (32), we have indeed $q_x = 0$.

Not surprisingly, (VI) corresponds with the conservation of angular momentum.

Symmetry calculations were performed for the one-dimensional shallow-water equations as well. No symmetry was found that is sufficiently close to (IV) to be of interest here.

5. DISCUSSION

It has been demonstrated that the new conservation law governs the spreading of a fixed quantity of nonviscous shallow water. Using simple estimates for the shape of the blob and for the velocity distribution an almost constant growth rate in the long-term limit is predicted.

So far only the integrated version of (5)–(8) has been discussed, where $\vec{P}_{1r} = 0$. One could think of applying the new law to situations where the fluid is confined by walls. However, the expressions $P_2$ and $P_3$ in (7), (8) are so involved (as is (15)) that useful information is hard to extract from the new law. On the other hand, the law may be of help in the evaluation of numerical models with tracers. After all, (17) provides a global constraint for a blob which ought to be satisfied by a good numerical scheme that predicts the position of tracers.

A set of symmetries of the Lagrangian of the shallow-water equations has been determined. In particular, the symmetry related to the new law has been found. As a by-product, another new conservation law (32) has been derived and it has been established that potential-vorticity conservation is related to a relabelling symmetry. As has been pointed out, even more conservation laws could have been derived. If, however, the general solution (29) is inserted in (31), one finds that the resulting laws can be derived from those presented above. Nevertheless, (26) with the generator (25) is not the most general condition for symmetries. Therefore, the set of conservation laws as presented is presumably not complete.

Finally, I wish to stress again that the new law (6)–(8) is not valid for the shallow-water equations in a rotating system if they are written without centripetal terms. In particular, the centripetal potential is supposed to be part of the gravitational potential in most applications of the shallow-water equations to atmospheric synoptic-scale flow. In that case, the terms $\Omega^2x$ in (1), (2) are dropped. This assumption destroys the symmetry (IV) in the rotating system.

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APPENDIX

Numerical integrations of (1)–(4) with $\Omega = 0$ are performed for the case of an axisymmetric blob. The equations are transformed to polar coordinates. Variables $h, r, r$, are non-dimensionalized using $h(0, 0) = h_0(0)$, $R(0)$ and $(gh_0(0))^{-1/2}R(0)$ as scales. The model equations are discretized using 100 grid points. The variables $r, r, r$, are defined at the grid points, whereas heights are evaluated at intermediate points. Differences are centred in space and forward in time. The nondimensional time step is $dx' = 0.1$. As can be seen from Fig. 1, energy $E$ is conserved quite well by this model. The same is true for $P_1$.

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