A generalized Ekman layer profile with gradually varying eddy diffusivities

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SUMMARY

The classical analytical solution for the Ekman layer flow has been generalized by introducing an assigned non-rapidly-varying eddy diffusivity $K(z)$. Asymptotic expressions for the wind components $u$, $v$ obtained via the WKB method retain the form and, to a large extent, the simplicity of the classical case in which $K$ is kept constant. Three examples of idealized $K$-profiles are used to compare the numerical, WKB and constant-$K$ solutions for Ekman profiles.

1. INTRODUCTION

In general, textbooks on geophysical fluid dynamics, and dynamic meteorology in particular, contain derivations for the Ekman layer with a constant eddy diffusivity, $K$, (Holton 1979; Haltiner and Williams 1980; Pedlosky 1987). In reality $K$ does not stay constant with height; it is a complicated, nonlinear function of the flow structure, and there is no explicit relation between the boundary-layer profiles and $K$. Consequently the problem had to be tackled numerically (Wippermann 1974; Pielke 1984). Between the extremes of an analytical solution, given a constant $K$-profile, and a numerical solution, given a real-life $K$-profile, there exists a solution for which $K$ takes the form of an estimated, non-rapidly-varying function of space and/or time (i.e. it varies only gradually); then it is possible to combine the analytical and numerical techniques efficiently. Besides the need for state-of-the-art mesoscale numerical models, there is also a need for models with assigned $K$-profiles, for example, in theoretical studies of various boundary-layer pumping effects (Mason and Sykes 1978; Valdes and Hoskins 1988; Staley 1993), in analytical boundary-layer analyses (Singh et al. 1993) and also in applied studies (Scheffe and Morris 1993).

It is assumed that in the steady-state Ekman layer there is a three-way balance of forces, between the Coriolis force, the pressure-gradient force, and the eddy-viscosity force which is parametrized via the $K$-theory (Holton 1979). Assuming the $K$-theory to be valid, then, following Wippermann (1974), one may conclude that the $K(z)$ profiles will not generally vary rapidly with height and may even have a 'common form' (i.e. a similar shape in the vertical). The governing equation for the Ekman layer remains linear if $K$ is a specified function that does not depend on the flow structure; therefore, it is likely to be analytically tractable (asymptotically, if not exactly). Given the assumption that $K$ is a gradually varying function of height in the Ekman layer, the WKB theory can be applied. This is a powerful and elegant singular perturbation method which can be applied to linear ordinary differential equations of any order (Bender and Orszag 1978).

A search through the literature revealed that there have been successful attempts at deriving, analytically, a profile for the Ekman layer for various $K(z)$ profiles. Ertel (1938) reviewed a method based on the theory of integral equations. Monin and Yaglom (1965) indicated analytic solutions of the problem, while Sutton (1953) and Panchev (1985) reviewed solutions for the cases $K(z) \propto z^m$ and $K(z) \propto e^{\alpha z}$, and multi-layer approaches. One of Miles's (1994) solutions in his more general treatment of the Ekman layer, given in terms of modified Bessel functions, resembles the one obtained here (cf. his Eqs. C.1(a, b)). Our purpose is to obtain a global, approximate solution that has an elegant form and yet would be valid for any given, gradually-varying function $K(z)$ that tends to zero near the upper boundary. The WKB method provides such a solution. From a mathematical point of view, when the highest derivative of the unknown function is multiplied by a small quantity, a singular method is required.

Compared with other methods, the WKB approach is convenient because it is simple and the computational cost is small. Application of the WKB method is justified if the length-scale of the medium is much larger than that of the calculated quantities. In the case of the Ekman layer this means that the length-scale, $L$, for $(K/f)^{0.5}$ variations has to be much larger than the length-scale, $\lambda$, over which $u$ and $v$ vary appreciably ($f$ is the Coriolis parameter; $u$ and $v$ are the velocity

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components). In the atmosphere conditions frequently diverge from this WKB requirement, nevertheless the method has been shown to yield good agreement with numerical simulations so long as the properties of the medium vary at least slightly slower than the calculated quantities; in this case one may expect the WKB method to be applicable provided $L \gg \lambda$. In other situations, such as mountain wave drag, the agreement with numerical simulations is within a factor of two (Laprise 1993). Furthermore, Grisogono (1994) has used the WKB method in a study of the atmospheric boundary layer.

Numerical solutions for $u$ and $v$ are compared with solutions derived using the WKB method, given a selection of $K(z)$ profiles. For the sake of simplicity and brevity no further refinements have been introduced, for example there is no matching with the surface layer and no thermal wind inclusion, etc. Hence, the WKB solution, although presented as applying over the range $0 \leq z \leq h$, is actually applicable only from a height $z_s$, which could be the top of the surface layer. (The height $z$ in this analysis truly begins a few tens of metres above the surface where the neutral, mixed layer represented by the Ekman layer starts.)

2. ANALYTICAL SOLUTION

The governing equation for the Ekman layer, incorporating $K(z)$, is

$$K \Phi'' + K' \Phi' - i \Phi = 0$$  \hspace{1cm} (1)

where $\Phi = (u - u_g) + i (v - v_g)$ is the complex wind velocity function, $u_g$ and $v_g$ refer to the constant geostrophic wind ($v_g = 0$, i.e. the geostrophic flow is aligned with the x-axis); primes denote z-derivatives (in the vertical). It is assumed that $K$ and $f$ do not change their signs and so, together with the previous assumptions, this allows a straightforward application of the WKB method.

The first and third terms in (1) represent the controlling behaviour which is the basis of the WKB analysis, while the term $K' \Phi'$ is identified with the modifications due to departures of $K$ from constant. The WKB expansion for $\Phi$ can be expressed in the form

$$\Phi \propto \exp \{ (S_0 + S_1 + S_2 \delta^2 + \ldots) / \delta \}.$$  \hspace{1cm} (2a)

Substituting (2a) into (1), we obtain a set of equations in terms of powers of a presumably small parameter $\delta$; $\delta$ has been introduced on account of the above-mentioned balance between the terms; at a later stage it will be equated to unity. The $S_i$ functions in (2a) now have to be determined. They must satisfy the conditions: $|S_i / \delta| \gg |S_1|$, $|S_1| \gg |S_2 \delta|$, etc. Otherwise (2a) cannot provide a good approximate solution of (1) and so the length-scale problem referred to in the introduction may arise. The balancing equations are

$$S_0'' = if / K \quad \text{(for } \delta^{-1})$$  \hspace{1cm} (2b)

$$2S_0' S_1 = -K'' / K S_0' - S_0'' \quad \text{(for } \delta^{-1})$$  \hspace{1cm} (2c)

and so on. The WKB approximation for $\Phi$ to a first order of accuracy ($S_0$ and $S_1$ included), is given by the equation

$$\Phi(z) = \Phi_0(K_0 / K(z))^{1/4} \exp \left\{ -(1 + i) (f/2)^{1/2} \int_0^z K(\zeta)^{-1/2} \, d\zeta \right\}$$  \hspace{1cm} (2d)

where $K_0 = K(z = 0)$, $\Phi_0$ is determined from the no-slip, lower boundary condition, i.e. $u(z = 0) = v(z = 0) = 0 \Rightarrow \Phi(z = 0) = -u_g$. The upper boundary condition is that the flow aloft must be geostrophic; this requires the sign in the exponential to be negative. More information about the validity of the WKB method can be found in the book by Bender and Orszag (1978). Gill (1982) has given a summary of the application of the WKB expansion in the case of atmospheric waves.

Separating the real and imaginary parts in (2d), we obtain for $u$ and $v$ the equations

$$u(z) = u_g [1 - \exp(-F(z)) (K_0 / K(z))^{1/4} \cos F(z)]$$  \hspace{1cm} (3a)

$$v(z) = u_g \exp(-F(z)) (K_0 / K(z))^{1/4} \sin F(z)$$  \hspace{1cm} (3b)

with

$$F(z) = (f/2)^{1/2} \int_0^z K(\zeta)^{-1/2} \, d\zeta.$$

\hspace{1cm} (3c)
Here there is an obvious resemblance to the classical Ekman layer \((u, v)\) profile for which \(K\) is a constant. Equations (3a) to (3c) are valid only when \(K(z) \sim 1/2\) varies gradually. If this condition is not satisfied (i.e. if \(K(z)\) were varying rapidly), then the WKB assumption will be violated. This would be noticed first of all in the \(v\)-component (\(v\) usually varies more than \(u\)).

3. Assigned \(K(z)\) and the WKB Application

To assess the accuracy and utility of (3), Ekman layer velocity profiles were derived in the cases of three idealized \(K\)-profiles shown in Figs. 1(a), 2(a), and 3(a), henceforth called cases 1, 2, and 3, respectively. Since \(K(z)\) normally decreases with height after passing its maximum in the boundary layer (O’Brien 1970; Peterson 1971; Wippermann 1974; Stull 1988), all these three idealized profiles are seen to tend to zero as \(z\) tends to infinity. In all three cases, the components of the geostrophic wind are given the values \(u_0 = 10\) m s\(^{-1}\), \(v_0 = 0\). In each case both components are calculated in three ways:

(i) the numerical method of finite centred differences for a semi-implicit, second-order scheme with the Gaussian elimination procedure. The numerical calculation includes time variations and is completed when steady-state Ekman profiles are obtained.

(ii) The WKB method based on (3).

(iii) Classical analytical Ekman profiles for a constant average \(K\).

The average \(K\), \(K_{\text{mean}}\) is a constant in each case and is calculated throughout the range \(0 \leq z \leq H_0\) provided \(K(z) \geq 0.15\) m\(^2\) s\(^{-1}\), or \(z = H_0 > \pi(2K_0/f)^{1/2}\), which is an estimation of the Ekman layer thickness. The numerical method is the most accurate. Solutions from these three methods are compared, respectively, in the figures corresponding to the case number (\(u\)-component comparisons in (b), and \(v\)-component comparisons in (c)). A discussion regarding these \(K\)-profiles and solutions, in each case, now follows.

Case 1. \(K(z)\) decreases monotonically and nonlinearly (Fig. 1). While the WKB result closely approximates the numerical solution, the solution with \(K\) constant exhibits a different form.

Case 2. \(K(z)\) in Fig. 2 roughly corresponds to an O’Brien’s profile without having its surface value for \(K\) (where \(K_0 = 0\)). As in the previous cases, the \(v_{\text{WKB}}\) profile approximates the numerical solution more closely than the classical (constant) Ekman profile. However, due to the comparatively rapid change of \(K(z)\), \(v_{\text{WKB}}\) underestimates the extreme values of the \(v\) profile (note that \(L = \lambda\)).

Case 3. This is the most complicated \(K(z)\) in this analysis. The \(K(z)\) profile contains a secondary maximum at the top of the layer that models the shear between the free atmosphere and the layer; this hypothetical \(K(z)\) may correspond to a case with a capping inversion overlying a neutral mixed layer. Qualitatively similar discrete \(K\) values, based on aircraft measurements, have appeared in a paper by Tjernström and Smedman (1993; cf. their Fig. 8).

The constant-\(K\) solution, although somewhat improved, compared to cases 1 and 2, intrinsically cannot ‘see’ any structure within the Ekman layer. In Table 1 is given a summary of the information from Figs. 1 to 3, and also the maximum departures of WKB and constant-\(K\) solutions from the numerical results (\(\Delta_{\text{WKB}}\), and \(\Delta_{\text{C}}\), respectively). In terms of phase accuracy, the WKB method provides high fidelity (a usual WKB characteristic), which cannot be said for the constant-\(K\) solutions in any of the three cases.

The characteristic depth of the Ekman layer, \(H\), if defined by the formula

\[
(f/2)^{1/2} \int_0^H \{K(\zeta)^{-1/2} \, d\zeta = \pi,
\]

differs from that of the constant-\(K\) case where

\[
H_0 = \pi(2K_{\text{mean}}/f)^{1/2}.
\]

(Note that \(H_0\) is a special case of \(H\) when \(K = \text{const.} = K_{\text{mean}}\). This is also apparent from Figs. 1 to 3 which indicate that constant-\(K\) solutions approach geostrophic flow at different heights from the WKB and numerical solutions. Moreover, the mass flux towards lower pressure that is associated with the \(v\)-component may differ between the constant-\(K\), WKB and numerical approaches.)
Figure 1. (a) The given $K(z)$ profile, (b) The $u$-component. (c) The $v$-component. Ekman $u$ and $v$ components are shown for numerical (solid), WKB (dashed) and constant-$K$ (dotted) calculations. The constant-$K$ solution is based on a $K_{\text{cpr}}$ (1(a), dashed line). Notice differences in phase and amplitude of $u$ and $v$. While the WKB solution closely approximates the numerical (most accurate) result, the constant-$K$ solution departs considerably.
Figure 1. Continued.

Figure 2. As Fig. 1, but that $K(z)$ decreases non-monotonically with height. This $K(z)$ corresponds to that due to O'Brien (1970).
Figure 2. Continued.
Figure 3. As Fig. 2, but with an additional $K(z)$ maximum around the layer’s top that models the shear between the mixed layer and the free atmosphere.
Figure 3. Continued.

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<th>Case</th>
<th>C.1, u</th>
<th>C.1, v</th>
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4. Concluding Remarks

The analytical solution for the Ekman layer has been extended to include the case of any given, gradually varying eddy diffusivity \(K(z)\). Wind components \((u, v)\) were approximated using the WKB theory; thus, the form and overall simplicity of the classical case (\(K = \text{constant}\)) were preserved. WKB solutions were compared with numerical and constant-\(K\) results using three chosen \(K(z)\) profiles (one of the profiles approximates to that due to O'Brien). A satisfactory agreement between the WKB and numerical solutions for \(u\) and \(v\) was found. The WKB solution that has been derived for the Ekman layer could be employed in theoretical analyses, or for a fast estimation of the Ekman layer profiles in applied research when a near-neutral stratification is encountered, or for a quick initialization of mesoscale models. Matching with the surface layer where \(K \propto z\) is straightforward (Holton 1979).

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