On the concept of pseudo-energy of T. G. Shepherd

By PASCAL MARQUET*
Météo-France, France
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1. INTRODUCTION

In a recent paper, Shepherd (1993) derived a general expression for the available potential energy for compressible, hydrostatic flow, where the sum of this available energy and the kinetic energy is called pseudo-energy. He demonstrated that for the special choice of a basic state defined by $\theta_b(p) = \tilde{\theta}(p)$ where the potential temperature $\tilde{\theta}(p)$ is the average on an isobaric surface, the small-amplitude limit of the generalized available potential energy reduces to the well-known approximate form of Lorenz (1955) expressed in a pressure vertical coordinate.

But other forms of available energies exist in atmospheric energetics and in thermodynamics, where the name exergy has been coined by Rant (1956) to denote the maximum work that can be extracted from any system when it is subject to some constraints (adiabatic transformations or constant total energy for instance).

The purpose of this note is to show that the specific available enthalpy function $a_h = (c_p T_0) - T(s - s_f)$ which is the flow energy of a fluid—see Marquet (1991)—can be obtained from the generalized approach of Shepherd if a constant basic state at temperature $T_0$ and pressure $p_0$ is considered. This special form of pseudo-energy also leads to the global hydrostatic concepts of Dutton (1973) and Pichler (1977). The function $a_h$ only depends on the specific enthalpy $h$ and entropy $s$ at any point, the values $h_r$ and $s_r$ refer to the special dead state at temperature $T_r$ and pressure $p_r$.

It is also explained that for a real isothermal basic state made up of an atmosphere at constant temperature $T_0$ but with a variable pressure, the generalized expression of Shepherd reduces with a good accuracy to the approximate functions introduced by Pearce (1978) or Blackburn (1983) in meteorology, it is moreover exactly the primary result obtained by Thomson (1853) in thermodynamics.

2. THE APPROACH OF SHEPHERD

The theory of Shepherd will not be described in detail, only the main steps will be sketched. We start with the non-canonical Hamiltonian representation of the system

$$\frac{\partial v_i}{\partial t} = l_{ij} \frac{\partial \mathcal{H}}{\partial v_j}$$

where the evolution of the dynamical variables $v_i$ depends on the skew-symmetric Poisson tensor $l_{ij}$ and on the functional derivative of the Hamiltonian $\mathcal{H}$.

Euler's equations for a compressible, hydrostatic, perfect gas correspond to the following non-canonical Hamiltonian

$$\mathcal{H} = E_k + H = \iint_{\mathcal{M}} \left\{ \frac{(u_i)^2}{2} + c_p T \right\} dm$$

$$\mathcal{H} = \iint_{\mathcal{M}} \left\{ \frac{(u_i)^2}{2} + c_p \theta \Pi(p) \right\} dm$$

where $m$ is an element of mass of the atmosphere and $\mathcal{M}$ is the mass integrating domain of the atmosphere. The specific kinetic energy of the horizontal wind $(u_i)^2/2$ is used because of the hydrostatic hypothesis. The integral of $(u_i)^2/2$ is the total kinetic energy which is denoted by $E_k$. The specific enthalpy for the dry perfect gas is supposed to be $c_p T = c_p \theta \Pi(p)$, where $c_p$ is the specific heat of dry air at constant pressure, $\Pi(p) = (p/p_0)^\kappa$ with $p_0 = 1000$ hPa and $\kappa = R/c_p$, where $R$ is the gas constant of dry air. The global enthalpy is denoted by $H$.

* Corresponding address: SCEM/PREVI/COMPAS, 42 avenue G. Coriolis, 31057 Toulouse, France.
The pseudo-energy is defined by Shepherd as
\[ \mathcal{A}(\mathbf{v}) = \mathcal{H}(\mathbf{v}) - \mathcal{H}(\mathbf{V}) + \mathcal{H}(\mathbf{v}) - \mathcal{H}(\mathbf{V}) \]  
(3)
where \( \mathbf{V} \) is a resting basic state in terms of the state vector \( \mathbf{v} \) of the Hamiltonian \( \mathcal{H} \) of system (1). Even if the Hamiltonian \( \mathcal{H}(\mathbf{V}) \) corresponds to an equilibrium state, \( \mathcal{H}(\mathbf{v}) - \mathcal{H}(\mathbf{V}) \) is only linear with respect to the perturbation amplitude \( \delta \mathbf{v} = \mathbf{v} - \mathbf{V} \), and it was necessary to introduce some Casimir invariant \( \mathcal{K} \) in order to make \( \delta \mathcal{A} \) quadratic with respect to this disturbance \( \delta \mathbf{v} \).

The Casimir invariants somehow span the tangent manifold of \( \mathcal{K} \) at point \( \mathbf{V} \). They are the solutions of
\[ \mathbf{l}_{ij} \frac{\partial \mathcal{K}}{\partial u_i} = 0, \quad \text{with} \quad \frac{\partial \mathcal{K}}{\partial u_i} \bigg|_{\mathbf{v}} = -\frac{\partial \mathcal{K}}{\partial u_i} \bigg|_{\mathbf{V}}. \]  
(4)
Solutions of \( \delta \mathcal{K}/\delta u_i \bigg|_{\mathbf{V}} \neq 0 \) exist because the Hamiltonian is non-canonical, i.e. \( \mathbf{l}_{ij} \) is non-invertible.

Following Shepherd (1993) the Casimir invariant of Eq. (2) is the integral over the mass \( \mathcal{M} \) of a function \( C(\theta) \) which only depends on the potential temperature \( \theta \). The function \( C(\theta) \) is determined from Eq. (2) by the functional derivative of the Hamiltonian \( \delta \mathcal{K}/\delta \theta = c_p \Pi(\theta) \) together with the second part of Eq. (4) applied to the resting basic state \( \mathbf{V} \) which will be denoted by a subscript 0.

It can be inferred that \( dC(\theta)/d\theta = -\delta \mathcal{K}/\delta \theta = -c_p \Pi(\theta) \) and that
\[ \mathcal{K} = -\int_{\mathcal{M}} \left( \int_0^\theta \left[ c_p \Pi(\theta') \right] d\theta' \right) dm. \]  
(5)
It is assumed that \( d\theta_0/d\phi < 0 \) so that the inverse function \( \mathcal{P}[\theta(\phi)] = \theta(\phi)^{-1} \) exists. After some manipulation it results from Eq. (3) with the use of Eqs. (2) and (5) that
\[ \mathcal{A} = (\mathcal{K} - \mathcal{K}_0) + (\mathcal{K} - \mathcal{K}_0) = \int_{\mathcal{M}} \int_{\theta_0}^{\mathcal{P}(\theta_0)} \left[ \Pi(\theta' + \theta_0) \right] d\theta' \]  
(6)
where the variable \( \theta'' = \theta' - \theta_0 \) is used.

Equation (7) gives the pseudo-energy as expressed by Shepherd. According to Eq. (6) the integrand \( (u_0)^2/2 + a \) is the specific form of the pseudo-energy. It is the sum of the specific kinetic energy \( c_k = (u_0)^2/2 \) and the specific available potential energy
\[ a = -c_p \int_{\theta_0}^{\theta - \theta_0} \left[ \Pi(\mathcal{P}(\theta'' + \theta_0)) - \Pi(\mathcal{P}(\theta_0)) \right] d\theta''. \]

Shepherd mentions that both \( c_k \) and \( a \) obey local conservation laws, they can be interpreted locally as real forms of energy attached to a parcel of the fluid.

3. The Exergetics

According to Shepherd’s paper, Lorenz’s available potential energy corresponds to the leading-order approximation of Eq. (7). Indeed the term \( \Pi(\mathcal{P}(\theta'' + \theta_0)) - \Pi(\mathcal{P}(\theta_0)) \) can be approximated by the quadratic function \( R p^{x-1} \theta_0^{x-1} (-d\theta_0/d\phi)^{-1} \left( \theta - \theta_0 \right)^2/2 \). The special choice \( \theta(\phi) = \theta(\phi) \) gives the available potential energy of Lorenz.

But if the other exergy functions of Thomson, Dutton, Pichler, Pearce, Blackburn or Marquet are relevant to atmospheric energetics, they must be similarly derived as special cases of the general pseudo-energy of Shepherd. These results have not been derived by Shepherd.

To demonstrate them, it is easier to start with an equation different from (7) in order to avoid the hypothesis \( d\theta_0/d\phi < 0 \) and the use of any inverse function of \( \mathcal{P} \) of \( \theta_0(\phi) \). The following expression of the pseudo-energy is directly expressed in terms of the temperatures \( T = \theta \Pi(\phi) \) and \( T_0 = \theta_0 \Pi(\phi) \), and in terms of the potential temperatures \( \theta \) and \( \theta_0 \):
\[ \mathcal{A} = E_k + \int_{\mathcal{M}} \int_{\theta_0}^{\theta} \left( c_p (T - T_0) - c_p \frac{T_0(\theta')}{\theta} \right) d\theta' \right) dm. \]  
(8)
This equation is an intermediate step in Shepherd’s paper, and \( \mathcal{A} \) as defined by Eq. (8) can be computed as far as the resting basic state defined by \( T_0(\lambda, \phi, \theta) \), and \( \theta_0(\lambda, \phi, \theta) \) is known at every point located at \( (\lambda, \phi, \theta) \) where \( \lambda \) and \( \phi \) are latitude and longitude respectively.
Starting from Eq. (8) the special case \( T_0 = T_i = c_T^{\text{st}} \) and \( \theta_0 = \theta_i = T_i \ (p_i/p_0)^{\kappa} = c_T^{\text{st}} \) can be studied, whereas it is not possible starting from Eq. (7) since the potential temperature is constant and the inverse function \( \hat{\theta}_i = \theta_i^{-1} \) does not exist. The values of the two constants, \( c_T^{\text{st}} \) and \( c_T^{\text{st}} \), will not be determined. This basic state is not a reference atmosphere, it is made up of a fixed parcel of fluid at constant temperature \( T_i \) and pressure \( p_i \). It is the one used in the specific available enthalpy approach (Marquet 1991). The pseudo-energy can then be computed with Eq. (8), and as expected the result is \( \mathcal{A} = E_x + A_H \), where \( A_H = A_T + A_p \) is the global available enthalpy:

\[
A_H = \iint_M c_T((T - T_i) - T_i \ln(T/T_i)) \, dm + \iint_M RT_i \ln(p/p_i) \, dm. \tag{9}
\]

The first integral is the temperature component \( A_T \) and the second one is the pressure component \( A_p \).

Moreover it can be shown that Eq. (9) corresponds to the global hydrostatic version of the static entropic energy of Dutton (1973), denoted by \( T_0 \Sigma \), or to the integral of the local associated version of Pichler (1977), denoted by \( T_0 \alpha \). In these studies the basic state is an isothermal stratified atmosphere at \( T_0 \) with a density \( \rho_0(z) = \rho_0(0) \exp(-gz/(RT_0)) \), where \( g \) is gravity and \( z \) is height. The function \( T_0 \Sigma \) is the integral over the mass \( M \) of \( c_T((T - T_0) - T_0 \ln(T/T_0)) - RT_0 \ln[\rho_0(z)/\rho] \), where \( c_T \) is the specific heat at constant volume, which can be transformed, with the exponential form of \( \rho_0(z) \) and with the specific internal energy \( e_p = \text{geopotential} \Phi = g z \), into the integral of \( c_T((T - T_0) - T_0 \ln(T/T_0)) + RT_0 \ln(p/p_0) \). The hydrostatic equality between the global potential energy, internal energy and enthalpy, that is to say \( E_T + E_i = H \), together with \( p_0(0) = p_{00} = e_p \), have been used to obtain \( T_0 \Sigma = A_H \). The hydrostatic static entropic energy of Dutton is therefore closely related to the global available enthalpy \( A_H \), thus to the pseudo-energy of Shepherd.

It is also interesting to examine the case of a basic state made of an isothermal and stratified resting atmosphere where the temperature \( T_0 \) is a constant and where the potential temperature \( \theta_0(p) = T_0(p/p_{00})^{\kappa} \) varies with \( p \). The value of \( T_0 \) will not have been determined. It ensues from Eq. (8) that the pseudo-energy becomes \( \mathcal{A} = E_x + A_T \), where

\[
A_T = \iint_M c_T((T - T_0) - T_0 \ln(T/T_0)) \, dm = \iint_M c_T T_0 \hat{\varphi}(T/T_0 - 1) \, dm. \tag{10}
\]

It is the temperature component of the global available enthalpy \( A_H = A_T + A_p \), where \( \hat{\varphi}(X) \equiv X - \ln(1 + X) \).

Equation (10) is the primary expression obtained by Thomson (1853) in thermodynamics when he studied the maximum work obtainable from an unequally heated space. This equation can be approximated using \( \hat{\varphi}(X) \approx X^2/2 \) in order to obtain the two meteorological formulations of Pearce (1978) and Blackburn (1983): \( a \approx c_T(T - T_0)^2/(2T_0) \). Therefore the pseudo-energy of Shepherd also leads to these two meteorological results.

The other formulation of McHall (1990) is not so easy to derive from Eqs. (7) or (8). One of the reasons could be the fact that he makes use of two conservation laws whereas other studies involve only one conservation law. McHall takes the total entropy and the integral of the potential-temperature constant and he searches for states of minimum possible enthalpy. Lorenz and Dutton, for instance, are rather concerned with conservation of mass (between two isentropes or the global mass, respectively), and they search for states of minimum enthalpy (Lorenz) and maximum entropy (Dutton).

It is not easy either to discover the connection between the pseudo-energy of Shepherd and the other form of static entropic energy of Dutton, which is the specific exergy of Karlsson (1990). The basic states of Dutton and Karlsson are nevertheless the same stratified and isothermal atmosphere that have been considered above to find the results of Thomson, Pearce and Blackburn. The main difference is that the theories of Dutton and Karlsson can deal with a hydrostatic or non-hydrostatic atmosphere associated with a hydrostatic reference basic state where the temperature is constant at \( T_0 \). Karlsson defines the specific function \( c_T T_0 \hat{\varphi}(T/T_0 - 1) + RT_0 \hat{\varphi}(\rho_0(z)/\rho - 1) \) which is positive and doubly quadratic with respect to \( T/T_0 - 1 \) and \( \rho_0(z)/\rho - 1 \). It is the integrand of one of the global integrals introduced by Dutton, but it is not equal to the available enthalpy (9) which is the hydrostatic pseudo-energy for the isothermal reference state. It would be interesting to derive the pseudo-energy for a non-hydrostatic atmosphere and to make the comparison with the results of Dutton and Karlsson.

One of the remaining problems is that the available enthalpy (9) is not positive definite. Of course the first component \( A_T \) is positive and of quadratic order with respect to the disturbance amplitude \( T - T_0 \), this is necessary by virtue of the general method of Shepherd; but the integrand
of the pressure component $A_p$ is not of any definite sign. This problem is solved because the reference pressure $p_r$ is chosen in such a way that the global integral $A_p$ cancels out. The definition $p_r = p_{00}/e$ is coherent with the positive and quadratic global function $A_H = A_T$.

The two hydrostatic formulations (9) and (10) are thus the same on a global stage providing $p_r = p_{00}/e$, but they are associated with two different local available energies: $a_T + a_p$ for (9), only $a_p$ for (10). The specific value of $A_T$ and $A_p$ are respectively $c_f((T - T_0) - T_0 \ln(T/T_0))$ and $RT_0 \ln(p/p_r)$. In fact the term $a_p$ is important in local budgets in order to balance the conversion term $-\omega \sigma T_0/p$, where $\omega$ is the vertical velocity, between $a_T$ and $a_p$. With Eq. (10), the local budget of $a_T$ cannot be closed with the kinetic-energy equation and $-\omega \sigma T_0/p$ remains as an extra conversion term. This problem does not arise with the specific available enthalpy function $a_h = a_T + a_p$ which is at the same time consistent on local and global points of view.

4. CONCLUSION

This note establishes a close connection between the generalized available potential energy of Shepherd and almost all other forms of availability functions used in meteorology (Shepherd had already derived Lorenz’s approach as a small-amplitude limit). An alternative formulation is given by Eq. (8) which appears to be suitable in order to deal with various isothermal basic states. The global available enthalpy and the global hydrostatic results of Dutton and Pichler correspond to a basic state made of an isothermal and isobaric parcel at $T_0$ and $p_r$. As for the available energies of Thomson, Pearce and Blackburn, they correspond to the basic state made up of an isothermal and stratified resting atmosphere at $T_0$ with a variable pressure.

The available enthalpy formulation is different from Lorenz’s results, which are more familiar for meteorological purposes; it is also true for the theories of Dutton and Pearce. It was, therefore, important to demonstrate these connections via the general concept of pseudo-energy of Shepherd, and to prove that the two special cases of basic states described above generate the two families of concepts used in meteorology: the available potential energy of Lorenz on the one hand, and the various exergy functions of Dutton, Pearce and Marquet on the other hand.

The specific exergy functions such as $a_h$ or $T_0 \sigma$ have already been studied and applied on a local point of view. The meteorological properties of the specific static entropic energy $T_0 \sigma$ were first investigated by Pichler (1977). As for the properties of the specific available enthalpy $a_h$, they have been explored by Marquet (1991, 1994).

It is proven by Marquet (1994) that a hydrostatic available enthalpy cycle can be rigorously derived for an isobaric layer of a limited-area domain. It appears that a Lorenz-like cycle is embedded in a more general one where various boundary fluxes are associated with each energy reservoir. There are also three large terms all depending on the isobaric average of the vertical velocity $(\vec{\omega})$ which is equal to zero only for the global atmosphere. In fact there are two orders of magnitude between these large terms and the other one of the cycle, but it is important to notice that the Lorenz-like cycle is somewhat protected from these large terms because they only touch the static stability component $a_h$ (depending on the vertical variations of $T - T_0$), and the kinetic energy of the mean wind $(\bar{u}^2 + \bar{v}^2)/2$. It turns out also that the boundary fluxes are important when considering the energetic of a baroclinic wave on a limited area.

All these results show that it was indeed important to arrive at a local definition of the concept of availability since new properties can be demonstrated. The local pseudo-energy of Shepherd seems to generalize all previous thermodynamical and meteorological exergy-like functions.

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