Short-range predictability of the atmosphere: Mechanisms for superexponential error growth

By C. NICOLIS*1, S. VANNITSEM2 and J.-F. ROYER2

1Institut Royal Météorologique de Belgique, Belgium
2Centre National de Recherches Météorologiques, France

(Received 27 May 1993; revised 4 August 1994)

SUMMARY

Recently it has been established, using simple mathematical models of chaos, low-order models, and large numerical models of the atmosphere, that small errors grow in the mean in a superexponential manner. In this paper the mechanisms behind this behaviour are examined with special emphasis on the non-orthogonality of the eigenvectors of the linearized evolution operator and the variability of the local Lyapunov exponents on the attractor. The study reveals a picture that is far more complex and system-dependent than what has been advanced so far in the literature. The general ideas are illustrated by Lorenz's low-order atmospheric model for which a simple phenomenological model of error growth is developed and tested successfully against the simulations.

1. INTRODUCTION

A characteristic of atmospheric complexity is the growth of small initial errors which owe their origin to the finite precision of experimental data and to the round-offs or truncations in the numerical solution of the model equations. The presence of this characteristic which to a large extent conditions the very possibility of predicting the future states of the atmosphere suggests that the relevant variables are controlled by an unstable dynamics and that they exhibit a sensitivity to initial conditions—a state for which deterministic chaos provides a universally accepted prototype (Lorenz 1984).

In chaos theory, it is usually stated that the existence of a positive Lyapunov exponent gives rise to an exponential growth of small initial errors. Recent studies using large numerical models of the atmosphere have revealed that, in fact, the small initial errors grow faster than the rate predicted by the most unstable eigenvalue of the system—a behaviour which is referred to as superexponential (Lacarra and Talagrand 1988; Schubert and Suarez 1989). This fact has been corroborated by low-order atmospheric models (Trevisan 1993; Krishnamurty 1993) or even by simple mathematical models of chaos (Nicolis and Nicolis 1993). In this paper we have analysed the generic mechanisms that may lie behind this behaviour and have carried out a numerical study and qualitative modelling of the fine structure of the error for short times in representative case-studies. More specifically we have tried to disentangle the roles of the two principal mechanisms which have, so far, been proposed in the literature, namely:

(i) Non-orthogonality of the eigenvectors of the linearized operator of the system, as a result of which certain linear combinations of the eigenvectors can grow faster than perturbations along a particular eigendirection (Lacarra and Talagrand 1988). The transient phase is dependent on the orientation of the perturbations in phase space and it is possible to identify those initial perturbations that will have the largest amplification over a given integration period (Molteni and Palmer 1993).

(ii) The variability of the local Lyapunov exponents on the attractor, reflecting the highly inhomogeneous character of this attractor. This variability is manifested by

* Corresponding author: Institut Royal Météorologique de Belgique, Avenue Circulaire 3, B-1180Bruxelles, Belgium.
appreciable fluctuations of the Lyapunov exponents around their mean values and can be evaluated quantitatively thanks to by now well-established algorithms (Abarbanel et al. 1991). Mechanism (ii) is the only one operating in systems amenable to one-dimensional recurrences like the logistic or cusp maps. Notice that certain low-order atmospheric models reduce to such mappings (Lorenz 1963), owing to the strong contraction induced by the fast timescales associated with motions along the stable directions.

In section 2 the general formulation of error growth is briefly described, with special emphasis on the probabilistic aspects and, in particular, on the role of the particular averaging (e.g. arithmetic or geometric) chosen. Section 3 deals with the role of non-orthogonality of eigenvectors on error growth. The study, which accounts fully for the statistical distribution of initial errors, reveals a picture that is much more complex and system-dependent than the one advanced by Lacarra and Talagrand (1988). In section 4 the role of the variability of Lyapunov exponents is explored. It is shown that the existence of a finite variance of the first Lyapunov exponent suffices to explain the superexponential behaviour of the arithmetic mean, whereas superexponential behaviour of the geometric mean requires, in general, more stringent conditions involving the variance and the entire hierarchy of time correlation functions of the first few leading Lyapunov exponents. These considerations are illustrated in section 5 with Lorenz's 1984 low-order atmospheric model, for which a simple phenomenological model of error growth is developed and compared successfully to the simulations. Final conclusions are drawn in section 6.

2. ERROR GROWTH: GENERAL FORMULATION

Let \( x_0 \) be an initial state of a dynamical system like the atmosphere, and let \( y_0 = x_0 + \varepsilon \) be the state to which \( x_0 \) is displaced as a result of an initial error vector \( \varepsilon \) of magnitude \( \varepsilon \). By definition, the instantaneous error, \( E_t \), is given by the expression

\[
E_t = |y(t; y_0) - x(t; x_0)|
\]  

(1)

where \( y(t; y_0) \) and \( x(t; x_0) \) are obtained by integrating according to the evolution laws starting at \( t = 0 \) with the initial conditions \( x_0 \) and \( y_0 \), and where the notation \( | \cdot | \) stands for the Euclidean norm in the phase space of the system.

Because of the complexity of the dynamics of typical systems of interest in the atmosphere \( E_t \), as defined by Eq. (1), fluctuates considerably, both over time and the state space. To relate the error dynamics intrinsically to the properties of the underlying system and, more particularly, to the structure of its attractor we resort to a probabilistic viewpoint: specifically, the operation defined by Eq. (1) is repeated for the initial conditions \( x_0 \) running all over the attractor. The arithmetic mean of the quadratic error over all these realizations, chosen for convenience over the mean of the error itself, is then defined by the equation

\[
\langle E_t^2 \rangle = \int dx_0 \, \rho(x_0) [y(t; x_0 + \varepsilon) - x(t; x_0)]^2
\]  

(2)

where \( \rho(x_0) \) is the invariant probability distribution over the attractor. Additional averaging over \( \varepsilon \) can also be performed if necessary.

In addition to the arithmetic mean featured in Eq. (2) other means can be considered (Royer et al. 1994). Of special relevance is the mean of the logarithm of the instantaneous error, viz.
\[ \langle \ln E_i^2 \rangle = \int \, dx_0 \, \rho(x_0) \ln \left| y(t; y_0) - x(t; x_0) \right|^2 \]

\[ = 2 \int \, dx_0 \, \rho(x_0) \ln |f'(x_0 + \varepsilon) - f'(x_0)| \]  \hspace{1cm} (3)

where \( f' \) is the resolvent operator of the system.

In chaos theory, expressions (1) to (3) are usually considered in the double limit of (in the indicated order) \( \varepsilon \to 0 \) and \( t \to \infty \). This confines the dynamics to the linearized regime (tangent space) and allows the identification of the largest positive Lyapunov exponent \( \sigma_{\text{max}} \) as

\[ \sigma_{\text{max}} = \lim_{t \to \infty} \lim_{\varepsilon \to 0} \frac{1}{t} \ln \left( \frac{E_i}{\varepsilon} \right). \]  \hspace{1cm} (4)

Actually, in most real-world situations one is interested in the time behaviour of small, but finite errors over a finite (and usually small) time interval. In recent work (Nicolis and Nicolis 1991; Nicolis 1992) we have established, using both simple mathematical models of chaos and low-order atmospheric models, that, under these conditions, the dynamics of growth of the arithmetic mean, Eq. (2), in a system exhibiting unstable dynamics follows three principal stages: a short-time regime during which errors remain small and can be adequately described by linearized theory; an intermediate regime displaying linear time dependence around an inflexion point during which errors attain appreciable values in a time, \( t^* \), of the order of \( \ln(1/\varepsilon) \); and a long-time regime where the mean error reaches a saturation level representative of the global structure of the system's attractor. In general the mean-error dynamics is not driven entirely by the largest Lyapunov exponent, even in the initial short time stage. This holds true not only for multivariable systems but also for chaos attractors generated by one-dimensional mappings. In this latter case this failure can be traced to the variability of the local expansion rates on the attractor (Nicolis and Nicolis 1993).

Consider now the logarithmic mean, Eq. (3). This expression takes a particularly simple form in a one-variable system in the form of discrete time recurrence, \( x_t = f(x_{t-1}) \). Indeed, dividing by the initial error \( \varepsilon \), using the definition of the derivative and applying the chain rule for differentiation we obtain for the short-time regime,

\[ \lim_{\varepsilon \to 0} \left( \ln \left( \frac{E_i}{\varepsilon} \right) \right)^2 = 2 \int \, dx_0 \, \rho(x_0) \sum_{t=0}^{t} \ln |f'(x_t)| \]

where \( f'(x_t) = df/dx_t \) and \( x_t \) are the points visited consecutively in the course of time. Using the definition of invariant probability this yields

\[ \lim_{\varepsilon \to 0} \left( \ln \left( \frac{E_i}{\varepsilon} \right) \right)^2 = 2t \int \, dx_0 \, \rho(x_0) \ln |f'(x_0)| \]

\[ = 2t \overline{\sigma} \]  \hspace{1cm} (5)

where the mean Lyapunov exponent \( \overline{\sigma} \) is identical to \( \sigma_{\text{max}} \) (Eq. (4)) if ergodicity holds. In other words, in this class of systems the variability of the attractor does not affect the mean logarithmic error even for short times.

The situation is considerably more involved in multivariate systems where Eq. (5) is now replaced by

\[ \lim_{\varepsilon \to 0} \left( \ln \left( \frac{E_i}{\varepsilon} \right) \right)^2 = 2 \int \, dx_0 \, \rho(x_0) \ln \left| \left( \frac{\partial f'(x_0)}{\partial x} \right) \cdot n \right| \]  \hspace{1cm} (6)
where \( \frac{\partial f(x_0)}{\partial x} \) is the Jacobian matrix of the resolvent evaluated at point \( x_0 \), and \( n = \varepsilon / \varepsilon \) defines the orientation of the initial error vector in phase space. For short times this expression depends in principle on the eigenvalues of the resolvent Jacobian matrix apart from the largest one and the associated eigenvectors, as well as on their variability on the attractor. One may therefore expect corrections to the result of Eq. (5), although at this stage their sign (and hence 'super' or 'sub' exponential behaviour) cannot be assessed. In the sequel we consider, successively, the role of the orientation of the eigenvectors and of the variability of the eigenvalues in the error dynamics in some representative case-studies.

3. ERROR GROWTH AND NON-ORTHOGONALITY OF THE EIGENVECTORS

The first non-trivial instance where the configuration of the eigenvectors of the linearized operator in phase space becomes relevant is a two-variable system of constant coefficients. This idealized model has been adopted previously by Lacarra and Talagrand (1988), but their results were dependent on the initial conditions since they considered errors along a particular direction. Our purpose in this section is to consider generic initial errors and to perform a probabilistic averaging over the ensemble of initial conditions. Notice that in a more general setting the two variables of the model may be thought of as spanning the two most unstable directions on the attractor. The analysis can also be extended straightforwardly to more than two variables.

The general form of a two-variable dynamical system of constant coefficients is given by the equations

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.
\]

(7)

Its general solution is of the form

\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = R \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
\]

(8a)

where the resolvent matrix \( R \) is given by

\[
R = \frac{1}{D} \begin{pmatrix} u_1 v_2 e^{\omega_1 t} - u_2 v_1 e^{\omega_2 t} & u_1 u_2 \left( e^{\omega_2 t} - e^{\omega_1 t} \right) \\ u_1 v_2 \left( e^{\omega_1 t} - e^{\omega_2 t} \right) & \frac{1}{D} \left( u_1 v_2 e^{\omega_1 t} - u_2 v_1 e^{\omega_2 t} \right) \end{pmatrix}.
\]

(8b)

Here \( \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \) and \( \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \) are the eigenvectors of \( A \) corresponding to the eigenvalues \( \omega_1, \omega_2 \), and

\[
D = u_1 v_2 - u_2 v_1.
\]

(8c)

Applying Eq. (8b) to the initial conditions \( \begin{pmatrix} x_0 + \varepsilon_1 \\ y_0 + \varepsilon_2 \end{pmatrix} \) and \( \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \) and subtracting we obtain the error dynamics in the form

...
or, using the Euclidean norm,

\[
E_i^2 = (R_{11}^2 + R_{22}^2)\varepsilon_1^2 + (R_{12}^2 + R_{21}^2)\varepsilon_2^2 + 2(R_{11}R_{12} + R_{21}R_{22})\varepsilon_1\varepsilon_2
\]

where \(R_{ij}\) are the matrix elements of \(R\). Averaging next over the distributions of \(\varepsilon_1\) and \(\varepsilon_2\), assumed to be statistically independent random variables of mean zero and variance \(\Delta^2\), we arrive, after some algebra, with the equation

\[
\langle E_i^2 \rangle = \frac{\Delta^2}{D^2} \{(K^2 + D^2)(e^{2\omega_1 t} + e^{2\omega_2 t}) - 2K^2 e^{(\omega_1 + \omega_2)t}\}
\]

(11a)

where \(D\) is defined in Eq. (8c) and \(K\) is the scalar product of the eigenvectors, viz.

\[
K = u_1 u_2 + v_1 v_2.
\]

(11b)

To assess the possibility of superexponential behaviour we need to compare Eq. (11a) with \(\langle E_0^2 \rangle e^{2\omega_1 t}\). The mean initial error follows directly from Eq. (11a), namely

\[
\langle E_0^2 \rangle = 2\Delta^2.
\]

We therefore inquire about the validity of the inequality

\[
\frac{\Delta^2}{D^2} \{(K^2 + D^2)(e^{2\omega_1 t} + e^{2\omega_2 t}) - 2K^2 e^{(\omega_1 + \omega_2)t}\} > 2\Delta^2 e^{2\omega_1 t}.
\]

Two cases are considered:

(a) **The eigenvalues are real and \(\omega_1 > \omega_2\) (no loss of generality)**

We find after some elementary algebra,

\[
\left\{1 + \left(\frac{K}{D}\right)^2\right\}w^2 - 2\left(\frac{K}{D}\right)^2 w + \left(\frac{K}{D}\right)^2 - 1 > 0
\]

(12a)

with

\[
w = \exp((\omega_2 - \omega_1)t).
\]

(12b)

The roots of the left-hand side of Eq. (12a) are

\[
w_\pm = \frac{\left(\frac{K}{D}\right)^2 \pm 1}{\left(\frac{K}{D}\right)^2 + 1}.
\]

(13)

The inequality is, therefore, valid if

\[
w > w_+ = 1, \quad \text{or} \quad w < w_- = \frac{\left(\frac{K}{D}\right)^2 - 1}{\left(\frac{K}{D}\right)^2 + 1}.
\]

(14)

The first condition is impossible in view of Eq. (12b). The second condition requires \((K/D)^2 > 1\), which may or may not be fulfilled depending on the coefficients of the linearized matrix \(A\). We notice that when the eigenvectors happen to be orthogonal \((K = 0)\) the condition is inevitably violated and superexponential behaviour is impossible; which
is in agreement with the conclusion reached by Lacarra and Talagrand in the more limiting setting of $\omega_2 = 0$. This statement must, however, be tempered by the following observations:

- When $K = 0$ (case of orthogonality) the behaviour does not reduce to a simple exponential but, rather, to a sum of two exponentials, implying that, in the more general case of $\omega_2 < 0$, the rate of increase of an initial perturbation will be less than the one given by the most unstable eigenvalue $\omega_1$—a behaviour that we may refer to as subexponential.
- When $K \neq 0$ (case of non-orthogonality) the behaviour need not always be superexponential: if $(K/D)^2 < 1$, an initial perturbation will actually evolve subexponentially—a behaviour similar to that considered above with $K = 0$ and $\omega_2 < 0$.
- Superexponential behaviour, whenever applicable, starts beyond a time $t_c$ given by (cf. Eqs. (12b) and (14))

$$ t > t_c = \frac{1}{|\omega_2 - \omega_1|} \ln \left( \frac{(K/D)^2 + 1}{(K/D)^2 - 1} \right). $$

In the initial time regime the behaviour is always subexponential, as can be seen (see also Fig. 1) by evaluating the initial rate of change of $\langle E_t^2 \rangle^2$, namely

$$ \left( \frac{d}{dt} \ln \langle E_t^2 \rangle \right)_{t=0} = \omega_1 + \omega_2 < 2\omega_1. \quad (15) $$

![Figure 1. Time evolution of $\langle E_t^2 \rangle$ normalized with $\Delta^2$, (Eq. (11a)), with $(K/D)^2 = 1.5$, $\omega_1 = 0.2$ and $\omega_2 = -0.3$ (full line) as compared to a purely exponential function whose rate is $2\omega_1$ (dotted line).](image)

(b) The eigenvalues are complex conjugate, $\omega_{1,2} = \Lambda \pm i\Omega$

As the eigenvectors are also complex conjugate, $K$ remains real (Eq. (11b)) whereas $D$ becomes purely imaginary (Eq. (8e)). Equation (11a) remains valid and the condition for superexponential behaviour now becomes

$$ \left( \frac{K^2}{D^2} + 1 \right) \cos 2\Omega t - 1 > 0. \quad (16) $$
Since the second factor is negative or zero this condition implies that
\[
\left( \frac{K^2}{D^2} + 1 \right) < 0.
\]
Substituting the values for \( K \) and \( D \) (notice that the eigenvectors in this case can never be orthogonal, \( K > 0 \), and that \( D^2 \) is negative) one can easily verify that Eq. (16) is always greater or equal to zero.

In summary, it appears that when the eigenvalues are real, non-orthogonality of the eigenvectors of the linearized operator does not entail universal behaviour of the error. This result reflects the statistical behaviour of generic errors distributed isotropically over all phase-space directions. It is always possible to identify, as did Lacarra and Talagrand (1988), perturbations along a specific direction behaving superexponentially from the initial time (see also Molteni and Palmer 1993). On the other hand, the behaviour is universally superexponential when the eigenvalues are complex conjugate.

4. **Error growth and variability of local Lyapunov exponents**

We now turn to the role of the variability of the Lyapunov exponent in the superexponential behaviour of small errors. We again adopt an idealized model limited to the two dominant directions on the attractor. So as to focus specifically on the effect of the variability of the Lyapunov exponent versus the effect of configuration of the instantaneous eigenvectors (which was the object of section 3) we take the latter to be uncoupled and fixed in phase-space. The model allows therefore for a variable rate of exponential expansion along each of these directions, but discards the rotation of the Lyapunov vectors.

The linearized equations for the error vector \( \mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \) can be written as
\[
\frac{d}{dt} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} \bar{\sigma}_1 + S_1(t) & 0 \\ 0 & \bar{\sigma}_2 + S_2(t) \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}
\]

(17)

where \( \bar{\sigma}_1 = \sigma_{\text{max}} \) and \( S_1(t), S_2(t) \) describe the variability of the Lyapunov exponents around their mean. By definition the time average of \( S_1(t), S_2(t) \) should be zero or, assuming that ergodicity applies, we have
\[
\langle S_1(t) \rangle = \langle S_2(t) \rangle = 0
\]

(18)

where the angular brackets now denote the average over the probability distribution of the corresponding quantity.

Equations (17) can be integrated straightforwardly to give
\[
\begin{align*}
E_{1t} &= e_1 \exp \left\{ \bar{\sigma}_1 t + \int_0^t dt' S_1(t') \right\} \\
E_{2t} &= e_2 \exp \left\{ \bar{\sigma}_2 t + \int_0^t dt' S_2(t') \right\}
\end{align*}
\]

(19)

From these expressions the arithmetic mean of the total quadratic error can be expressed formally as
\[ \langle E_p^2 \rangle = \epsilon_1^2 \exp(2\overline{\sigma}_1 t) \left\langle \exp\left\{ 2 \int_0^t dt' S_1(t') \right\} \right\rangle + \epsilon_2^2 \exp(2\overline{\sigma}_2 t) \left\langle \exp\left\{ 2 \int_0^t dt' S_2(t') \right\} \right\rangle \]  

(20)

where it is understood that an additional averaging over the initial errors \( \epsilon_1 \) and \( \epsilon_2 \) has been performed.

Equation (20) features the average of the exponential functions

\[ G(\phi) = \langle \exp \phi \rangle. \]

Expanding formally one gets

\[ G(\phi) = \sum_{n=0}^{\infty} \frac{\langle \phi^n \rangle}{n!}. \]  

(21a)

We define the cumulants, \( C_n \), as the coefficients of the Taylor expansion of the logarithm of \( G \) (Van Kampen 1981), viz.

\[ \ln G(\phi) = \sum_{n=1}^{\infty} \frac{C_n}{n!} \]  

(21b)

Grouping equal powers of \( n \) one obtains

\[ C_1 = \langle \phi \rangle \]
\[ C_2 = \langle \phi^2 \rangle - \langle \phi \rangle^2 \]
\[ C_3 = \langle \phi^3 \rangle - 3\langle \phi \rangle^2 \langle \phi \rangle + 2\langle \phi \rangle^3 \]

etc.

(22)

Substituting for \( \phi \) its value in Eq. (20), and taking Eq. (18) into account, we obtain

\[ \left\langle \exp\left\{ 2 \int_0^t dt' S_i(t') \right\} \right\rangle = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n!} C_n^{(i)} \right\} \quad (i = 1, 2) \]  

(23a)

with

\[ C_1^{(i)} = 0 \]
\[ C_2^{(i)} = 2 \int_0^t dt_1 \int_0^t dt_2 \langle S_i(t_1) S_i(t_2) \rangle \]  

(23b)

where \( \langle S_i(t_1) S_i(t_2) \rangle \) is the two-time autocorrelation function of \( S_i \), and where the higher-order cumulants \( C_n \) are suitable combinations of \( n \)-time correlation functions and also of those of lower-order. As the process is stationary, only time differences will matter. We can therefore write

\[ \langle S_i(t_1) S_i(t_2) \rangle = q_i^2 \psi_i(|t_1 - t_2|) \]  

(24)

where henceforth we normalize \( \psi_i \) in such a way that \( \psi_i(0) = 1 \), in which case \( q_i^2 \) is the variance of the \( i \)th Lyapunov exponent.

It is a characteristic of a Gaussian random process (Benzi and Carnevale 1989) that only the second-order cumulants survive. For a process of this kind, after substituting from Eqs. (23a) and (23b) into Eq. (20), we obtain

\[ \langle E_p^2 \rangle = \epsilon_1^2 \exp(2\overline{\sigma}_1 t) \exp\left\{ q_1^2 \int_0^t dt_1 \int_0^t dt_2 \psi_1|t_1 - t_2| \right\} + \]

\[ + \epsilon_2^2 \exp(2\overline{\sigma}_2 t) \exp\left\{ q_2^2 \int_0^t dt_1 \int_0^t dt_2 \psi_2|t_1 - t_2| \right\}. \]  

(25)
SUPEREXPONENTIAL ERROR GROWTH

Since $\bar{\sigma}_1 > \bar{\sigma}_2$, the condition for superexponential behaviour concerns, essentially, the first term and can be expressed by the condition

$$\int_0^t dt_1 \int_0^t dt_2 \psi_1(|t_1 - t_2|) > 0.$$ 

For any non-vanishing $\bar{\sigma}_1^2$ this inequality is always satisfied for short times, since $\psi_1(0) = 1$. Notice that for short times this contribution depends on $t$ as $t^2$ provided that $\psi_1$ is a smooth function of time such that the double integral can be expanded in a Taylor series in terms of $t$. It is only in the limit of white noise, i.e. $\psi_1(|t_1 - t_2|) = \delta(t_1 - t_2)$, that the behaviour becomes linear with $t$.

Turning now to the logarithmic mean we have, from Eqs. (19)

$$\langle \ln E_i^2 \rangle = \langle \ln \left( \varepsilon_1^2 \exp \left[ 2\bar{\sigma}_1 t + 2 \int_0^t dt' S_1(t') \right] + \varepsilon_2^2 \exp \left[ 2\bar{\sigma}_2 t + 2 \int_0^t dt' S_2(t') \right] \right) \rangle$$

$$= \left\langle \ln \left( (\varepsilon_1^2 + \varepsilon_2^2) \exp \left[ 2\bar{\sigma}_1 t + 2 \int_0^t dt' S_1(t') \right] \right) \left[ 1 + \frac{\varepsilon_2^2}{\varepsilon_1^2 + \varepsilon_2^2} \exp \left( 2(\bar{\sigma}_2 - \bar{\sigma}_1) t + 2 \int_0^t dt' (S_2(t') - S_1(t')) \right) - 1 \right] \right\rangle.$$

Replacing the logarithm of the product by a sum of logarithms, setting $\varepsilon^2 = \varepsilon_1^2 + \varepsilon_2^2$ and performing the averaging, keeping Eq. (18) in mind, we obtain

$$\left\langle \ln \left( \frac{E_i}{e} \right)^2 \right\rangle = 2\bar{\sigma}_1 t + \left\langle \ln \left( 1 + \frac{\varepsilon_2^2}{\varepsilon^2} \exp \left[ 2(\bar{\sigma}_2 - \bar{\sigma}_1) t + 2 \int_0^t dt' (S_2(t') - S_1(t')) \right] - 1 \right) \right\rangle.$$ 

(26)

This expression can again be handled by the cumulant expansion. For a Gaussian process it takes the explicit form (to be compared with Eq. (25)):

$$\left\langle \ln \left( \frac{E_i}{e} \right)^2 \right\rangle = 2\bar{\sigma}_1 t + \ln \left( 1 + \frac{\varepsilon_2^2}{\varepsilon^2} \exp \left[ 2(\bar{\sigma}_2 - \bar{\sigma}_1) t + 2 \int_0^t dt' (S_2(t') - S_1(t')) \right] - 1 \right)$$

$$+ \int_0^t dt_1 \int_0^t dt_2 \psi_2(|t_2 - t_1|) - \bar{\sigma}_1^2 \int_0^t dt_1 \int_0^t dt_2 \psi_1(|t_1 - t_2|) - 1 \right\rangle.$$ 

(27)

The condition for superexponential behaviour is now very different, and more stringent, than for the arithmetic mean. Since $\bar{\sigma}_2 - \bar{\sigma}_1$ is by necessity negative and since the part depending on the correlation functions starts as $t^2$ (except in the unlikely case of white noise) the term inside the logarithm is less than unity for short times, and therefore the contribution of the logarithm itself is negative. In other words, if the process happens to be Gaussian then the behaviour of the logarithmic mean will be subexponential. Conversely, a superexponential behaviour of the logarithmic mean would imply that the Gaussian assumption (Eq. (27)) fails and that one must resort to the full Eq. (26). This means that higher-order correlation functions of the leading local Lyapunov exponents play an essential part in the error dynamics. To reach a quantitative understanding, the knowledge of higher-order correlation functions of these local exponents would be necessary.
5. A case-study: Lorenz's low-order atmospheric model

Lorenz's (1984) low-order atmospheric circulation model reads

\[
\begin{align*}
\frac{dx}{dt} &= -y^2 - z^2 - ax + aF \\
\frac{dy}{dt} &= xy - bxz - y + G \\
\frac{dz}{dt} &= bxy + xz - z
\end{align*}
\]

where \( x \) is the amplitude of the westerly wind, \( y \) and \( z \) the phases of large-scale atmospheric waves affecting \( x \), and \( a, F \) and \( G \) are the thermal forcings. The variables are non-dimensional and the unit of time is 5 days, which is the synoptic timescale of the atmosphere.

Figure 2 depicts the system's attractor given \( a = 0.25, b = 6, F = 16 \) and \( G = 3 \). For these values of the parameters the mean Lyapunov exponents of the system have been computed using the Shimada and Nagashima algorithm (Shimada and Nagashima 1979). One finds that convergence is attained for an integration time of about 3000 time units and a time-step, \( \delta t \), of 0.01. The values of the three Lyapunov exponents found are

\[
\begin{align*}
\bar{\sigma}_1 &= \sigma_{\text{max}} = 0.56 \\
\bar{\sigma}_2 &= \sigma_0 = 0 \\
\bar{\sigma}_3 &= \sigma_{\text{min}} = -1.41
\end{align*}
\]

(29a)

Using the same algorithm (see also Abarbanel et al. 1991) one can also evaluate the variability of the local Lyapunov exponents along the corresponding eigendirections. This is achieved by monitoring the instantaneous values at each integration step. One finds, in this way, the following values for the variances:

\[
\begin{align*}
q_1^2 &= \langle \delta \sigma_{\text{max}}^2 \rangle = 7.3 \\
q_2^2 &= \langle \delta \sigma_0^2 \rangle = 6.8 \\
q_3^2 &= \langle \delta \sigma_{\text{min}}^2 \rangle = 3.2
\end{align*}
\]

(29b)

Figure 2. Two-dimensional projection of the attractor of Eqs. (28). Crosses indicate two of the three fixed points of the system. Parameter values are \( a = 0.25, b = 6, F = 16 \) and \( G = 3 \).
which show the high inhomogeneity of the attractor. Figure 3 depicts the time auto-correlation functions, \(\psi_i(t), (i = 1, 2, 3)\) of the three exponents. We notice that they all feature a decaying envelope modulated by aperiodic oscillations.

We now turn to error growth, following the general setting of section 2. To capture the fine structure of short-time error evolution we register at every 0.01 time unit the values obtained from the integration of the model equations (Eqs. (28)) and compare them with a perturbed run in which small errors distributed uniformly around zero have been introduced initially in each of the equations. The full line of Fig. 4(a) summarizes the results of the simulation, averaged over 100,000 realizations, for the arithmetic mean, \(\langle E_1^2 \rangle\), whereas in the dotted line the effective amplification rate, viz.

\[
\sigma_{\text{eff}} = \frac{1}{2t} \ln \left( \frac{\langle E_1^2 \rangle}{\varepsilon^2} \right)
\]

(30)

is plotted against time. Notice that Eq. (30) is independent of time if \(\langle E_1^2 \rangle\) is purely exponential. For short times we observe clearly superexponential behaviour since \(\sigma_{\text{eff}}\) attains a maximum of about \(4\sigma_1\) for \(t = 0.4\) time units. This behaviour is insensitive to the choice of the magnitude of the initial errors provided that the latter are sufficiently small (\(10^{-3}\) and less). For longer times the rate decreases steadily, but for very small initial errors (at least down to \(10^{-3}\)) it turns out that there is no time interval for which \(\sigma_{\text{eff}}\) attains a value close to \(\sigma_1\), since in the meantime \((t = 12\) time units for \(\langle \varepsilon \rangle \approx 10^{-5}\), Fig. 4(a)) the error has already increased sufficiently and tends toward its saturation level. We also notice from Fig. 4(b), which describes the short-time behaviour of the error, the existence of slight oscillations in agreement with the results on the correlation functions of the local Lyapunov exponents shown in Fig. 3.

Figure 5 summarizes the results of the time behaviour of the logarithmic mean \((\ln E_1^2)\) (full line) together with its effective amplification rate (dotted line). Again the behaviour is clearly superexponential although its deviation from a purely exponential function is less pronounced than in the case of the arithmetic mean (Fig. 4(a)).

We come now to the origin of the superexponential behaviour seen in Figs. 4 and 5. Numerical evaluation of the instantaneous eigenvectors of the linearized matrix shows that they are, typically, non-orthogonal. Furthermore, two of them correspond to a pair of complex conjugate eigenvalues. On the other hand the probability density of the local Lyapunov exponents, shown in Fig. 6, reveal pronounced skewness and kurtosis,
Figure 4. (a) Time dependence of the arithmetic mean of error, $\langle E_t^2 \rangle$, (full line) and $\sigma_{eff}$ (dotted line) for model (28) averaged over 100,000 realizations; (b) short time evolution of $\ln(\langle E_t^2 \rangle)$ for a mean initial error $\varepsilon = 10^{-5}$. Parameter values as in Fig. 2.

Figure 5. As in Fig. 4, but for the geometric mean of error, $\langle \ln E_t^2 \rangle$. 
suggesting that their fluctuations around the mean is a non-Gaussian process. On the grounds of the above observations we expect that in the Lorenz model both sources of superexponential behaviour—non-orthogonality of the eigenvectors and variability of Lyapunov exponents—are present.

It is interesting to realize that the variability of Lyapunov exponents alone, together with the assumption of a Gaussian distribution already provides a reasonably good picture of the behaviour of the arithmetic mean. This is illustrated in Fig. 7 where we have
plotted the time evolution of the logarithm of the first term in Eq. (25), normalized with \( \varepsilon_1^2 \), and with \( q_1^2 \) as given by Eq. (29a) (full line), and also when the fluctuations of \( \sigma_1 \) are not taken into account (dotted line). To reach a more quantitative understanding we need to model the correlation function \( \psi_t \) of the leading Lyapunov exponent. The simplest Gaussian stochastic process compatible with the structure shown in Fig. 3 is a damped oscillator driven by Gaussian white noise, \( W(t) \), viz.

\[
\frac{d^2 S}{dt^2} + 2\omega_0 \zeta \frac{dS}{dt} + \omega_0^2 S = W(t)
\]  

(31)

where \( \omega_0 \) is the oscillator frequency in the absence of damping and \( 2\omega_0 \zeta \) is the friction coefficient (Nicolis and Nicolis 1986). These parameters account, in a qualitative manner, for the winding motion of the trajectory away from the unstable fixed point and the subsequent reorientation toward this point, both of which constitute typical ingredients of chaotic dynamics. Other modelling possibilities which at first sight seemed natural have been explored, such as an Ornstein–Uhlenbeck process with additive periodic forcing or periodically modulated noise. Both proved unsuccessful.

Equation (31) can easily be solved by Fourier transform methods. The resulting power spectrum, \( P(\omega) \), is given by the equation

\[
P(\omega) = \frac{Q^2}{(\omega_0^2 - \omega^2)^2 + 4\omega_0^2 \omega^2 \xi^2}
\]  

(32)

where \( Q^2 \) is the variance of \( W(t) \). This equation gives rise to a maximum value of \( P(\omega) \) at \( \omega_{\text{max}} = \omega_0 (1 - 2\xi) \)\(^{1/2} \) subject to the condition \( \xi \leq (1/\sqrt{2}) \).

We now proceed to evaluate the first term of Eq. (25) which provides the leading contribution to \( \langle E_1^2 \rangle \).

\[
\langle E_1^2 \rangle = \varepsilon^2 \exp(2\sigma_1 t) \exp \left\{ \int_0^t dt_1 \int_0^{t_1} dt_2 \langle S(t_1) S(t_2) \rangle \right\}.
\]  

(33)

The correlation function of the \( S \)-process can in turn be computed as the Fourier transform of \( P(\omega) \), namely
\[ \langle S(t_1)S(t_2) \rangle = \frac{Q^2}{4\omega_0\zeta\lambda} \exp\left(-\frac{\omega_0 \zeta |t_1 - t_2|}{\lambda^2 + \omega_0^2 \zeta^2}\right) (\lambda \cos \lambda |t_1 - t_2| + \omega_0 \zeta \sin \lambda |t_1 - t_2|) \]  \hfill (34a)

where

\[ \lambda = \omega_0 (1 - \zeta^2)^{1/2}. \]  \hfill (34b)

The variance and minima of the correlation function (34a) are given, respectively, by

\[ q^2 = \langle \delta S^2 \rangle = \frac{Q^2}{4\omega_0^3 \zeta} \]  \hfill (34c)

and

\[ r_{\text{min}} = \frac{n \pi}{\lambda}. \]  \hfill (34d)

Substituting from Eq. (34a) into Eq. (33) and performing the double integration we find

\[ \langle E_i^2 \rangle = \varepsilon^2 \exp\left\{ 2\overline{\sigma}_1 t + \frac{Q^2}{2\omega_0^3} \left( 2t + \frac{\sin(3\theta - \lambda t) e^{-\omega_0 \zeta t} - \sin 3\theta}{\zeta \lambda} \right) \right\} \]  \hfill (35)

with \( \theta = \arccos(-\zeta) \).

Actually, in view of the structure of Eq. (25), relations (33) to (35) can be extended straightforwardly to moments of arbitrary order \( k \), i.e.

\[ \langle E_i^k \rangle = \varepsilon^k \exp\left\{ k\overline{\sigma}_1 t + \frac{kQ^2}{4\omega_0^3} \left( 2t + \frac{\sin(3\theta - \lambda t) e^{-\omega_0 \zeta t} - \sin 3\theta}{\zeta \lambda} \right) \right\} \]  \hfill (36)

For very short times, \( t < 1/(\omega_0 \zeta) \), we can expand Eq. (35) keeping terms of the lowest order in \( t \). The result is

\[ \langle E_i^2 \rangle = \varepsilon^2 \exp\left( 2\overline{\sigma}_1 t + \frac{Q^2}{2\zeta \omega_0^3 \lambda} t \right) \]  \hfill (37)

which requires that the correction to the exponential growth driven by \( \overline{\sigma}_1 \) is always positive and proportional to \( t^2 \), in agreement with the comments made in section 4. This explains qualitatively the superexponential behaviour found in the simulation.

It is interesting to observe that in the range of times for which \( t > 1/(\omega_0 \zeta) \) the contribution of the exponential terms in the exponent of Eq. (35) becomes negligible, giving

\[ \langle E_i^2 \rangle \approx \varepsilon^2 \exp\left( 2\overline{\sigma}_1 + \frac{Q^2}{\omega_0^3 \lambda} t + \frac{Q^2}{2\omega_0^3 \lambda} \right). \]  \hfill (38)

We notice that in this time range the attractor variability leads to a renormalization of \( \sigma_{\text{max}} \) by a positive quantity directly proportional to the variance of the effective noise.

Figure 8 depicts the time dependence of \( \ln\langle E_i^2/\varepsilon^2 \rangle \) as obtained from the atmospheric model described by Eqs. (28) (full line) and from Eq. (31) with \( q^2 = 7.3 \), \( \omega_0 = 4 \) and with a friction parameter \( 2\omega_0 \zeta = 2 \) (dotted line). Curves (i) and (ii) of Fig. 8(b) refer to the evolution of \( \sigma_{\text{eff}} \), (Eq. (30)), evaluated from Eq. (35) with the same parameters and two different friction coefficients \( 2\omega_0 \zeta = 1.6 \) and \( 2.08 \), respectively, whereas curve (iii) is obtained as in curve (a) but with \( q^2 = 8.5 \). For comparison we show in the same figure (curve (iv)) the value of \( \sigma_{\text{max}} \). The agreement is reasonable considering the crudeness of the model. It reproduces the main qualitative features of the short-time behaviour of the arithmetic mean of error: superexponential evolution and transient oscillations.
6. Conclusions

We have analysed the two typical mechanisms responsible for the superexponential behaviour of small errors in a chaotic system which, so far, have been advanced in the literature: the non-orthogonality of the eigenvectors of the linearized evolution operator and the variability around their mean values of the local Lyapunov exponents on the attractor. Special emphasis has been placed on the statistical aspects of the problem, particularly on the role of sampling over an ensemble of initial errors, and on the type of averaging (arithmetic or logarithmic) chosen in the sampling process.

We have shown using a simplified model that, for real eigenvalues of the linearized operator, the non-orthogonality of the eigenvectors is a necessary condition for superexponential behaviour. This property is, however, far from being sufficient: in particular, superexponential behaviour sets in only after a finite lapse of time, whose value depends on the parameters built into the linearized evolution operator. On the other hand, when the eigenvalues of the linearized operator are complex the behaviour of the error is universally superexponential.
A second element in superexponential behaviour turned out to be the variability of the Lyapunov exponents. Specifically, the superexponential behaviour of the arithmetic mean is driven to a large extent by the variability of the largest exponent. In contrast, the behaviour of the logarithmic mean is specific to the system and depends on the variability of the system's Lyapunov exponents other than the maximum and also on the non-Gaussian character of the processes associated with this variability.

Our results suggest that the problem of prediction of a chaotic dynamical system in general, and of the atmosphere and climate in particular, is more complicated than is usually thought: given a mean initial error, knowledge of the leading Lyapunov exponent is, in general, not sufficient for the purpose of estimating the validity of prediction $n$ time-steps ahead. In actual fact the skill of the model will depend on the entire hierarchy of cumulants (or at least of the variance in the case of a Gaussian process) of the Lyapunov exponents as well as on the orientation of the associated Lyapunov vectors in phase space.

The analysis in sections 3 and 4 has been carried out on minimal mathematical models. For instance, in section 3 the structure of the perturbations was studied on the basis of a model linearized around a time-independent reference state. In the setting of section 4 the more realistic picture of a time-dependent reference state was adopted, but for the purpose of technical simplicity the coupling between eigenmodes was discarded. One obvious extension would be to introduce a random coupling between $E_1$ and $E_2$ (Eq. (17)) to account qualitatively for the rotation of the Lyapunov vectors along the attractor. An analysis of this kind is considerably more involved than the one that we have carried out, since the averaging would now bear on products of random functions; it would be interesting to consider it in a future investigation.

The study of Lorenz's atmospheric model (section 5) revealed that the superexponential character of the arithmetic mean is considerably more pronounced than that of the logarithmic mean. Furthermore, it turned out that the behaviour of the arithmetic mean could be modelled rather satisfactorily by assimilating the variability of the leading Lyapunov exponent to a Gaussian process. It would be interesting to test the generality of these conclusions on more representative meteorological models.

ACKNOWLEDGEMENTS

This work is supported, in part, by the Commission of the European Communities and by the Federal Office for Scientific, Technical and Cultural Affairs (OSTC).

REFERENCES


Lacarra, J. F. and Talagrand, O. 1988 Short-range evolution of small perturbations in a barotropic model. Tellus, 40A, 81-95


