A non-hydrostatic version of the NMC's regional Eta model

By WILLIAM A. GALLUS Jr. and MIODRAG RANČIĆ

National Meteorological Center, USA

(Received 25 March 1994; revised 12 June 1995)

Summary
A non-hydrostatic version of the regional Eta model used operationally at the National Meteorological Center (NMC) has been developed by implementing the ideas of Juang (1992) and Laprise (1992), who independently recommended a hydrostatically based, vertical coordinate for a fully compressible set of equations. The grid-point model dynamics is based on perturbation equations in the \( \eta \)-vertical coordinate; the base state may be taken from the operational NMC model, and thus updated with time.

The non-hydrostatic model uses a stepwise treatment of terrain present in the operational version, eliminating the pressure-gradient-term error associated with sigma-coordinate models over steep topography. The compressible equations are written in a form that allows conservation of energy in a horizontally closed domain with appropriate advective schemes.

A two-dimensional version of the model, without parametrizations of physical processes, has been used successfully to simulate ascending warm bubbles and collapsing cold bubbles at high resolutions.

Keywords: Convective-scale simulations Non-hydrostatic perturbation model Numerical weather prediction Step orography

1. Introduction

Most numerical models of the atmosphere that are run operationally for forecasting purposes use the hydrostatic approximation and a pressure-related coordinate system. There are only a few exceptions, such as the non-hydrostatic model of the Meteorological Office (e.g., Golding 1990, 1992), used during the early 1990s. The major reason for using the hydrostatic approximation is computational economy. The hydrostatic approximation eliminates vertically propagating sound waves, which may impose a severe linear stability condition since the vertical resolution in the models is typically much higher than the horizontal. In addition, non-hydrostatic models require the solution of either two more prognostic equations (typically for pressure and vertical velocity), or an elliptic equation if the anelastic approximation of Ogura and Phillips (1962) is made. Application of the hydrostatic approximation may be well justified for the horizontal resolutions of 100 km in the 1980s and 40 km in the early 1990s. With a rapid evolution of computing technology and implementation of massively parallel computers in forecasting applications, it is only reasonable to expect that a goal of running a regional model at a resolution of 5 to 10 km may be attainable in the near future. There is probably general agreement that, with these horizontal resolutions, the hydrostatic models should be replaced by non-hydrostatic ones.

A difference between the hydrostatic and non-hydrostatic atmosphere, according to Eckart's (1960) linear analysis of atmospheric waves, concerns the dispersive behaviour of gravity waves. There is a range of high wave-numbers that cannot be described adequately within the hydrostatic approximation. As the frequency of these gravity waves approaches the Brunt–Väisälä frequency, some of the energy must be transferred to vertical kinetic energy, and this process is excluded in the hydrostatic approximation. With horizontal resolutions that may resolve convective scales, the characteristic strong vertical flows cannot appear unless the non-hydrostatic effects are properly incorporated into the forecasting model.

Early non-hydrostatic models have been used mainly for studying small-scale phenomena (e.g., Klemp and Wilhelmson 1978; Cotton and Tripoli 1978; Wilhelmson and

* Corresponding author: National Meteorological Center, UCAR Visiting Research Program, W/NMC2, Room 204, 5200 Auth Road, Camp Springs, MD 20746, USA.
Chen 1982). Developed for research purposes rather than for operational applications, these models typically used geometric height as the vertical coordinate. They generally did not conserve the important integral properties of the continuum equations, based on a belief that typically open boundaries, small time-scale integrations and a predominant influence of parametrized physical processes at these scales do not require such a constraint. In contrast, Taylor (1984) advocated implementation of the conservation principles of Arakawa and Lamb (1977) in non-hydrostatic modelling and derived an energy-conserving finite-difference scheme for a switchable hydrostatic/non-hydrostatic model.

The advantages of pressure as a vertical coordinate in non-hydrostatic models were recognized by Miller (1974). Miller and White (1984) gave a strict mathematical derivation of this approach which consists of approximations that yield an anelastic-type system. As an extension of this method, Xue and Thorpe (1991) presented a non-hydrostatic model that used the Phillips (1957) pressure-related terrain-following vertical coordinate.

Laprise (1992) recommended a hydrostatically based, vertical coordinate for a fully-compressible non-hydrostatic model and Juang (1992) independently developed a non-hydrostatic spectral model with a hydrostatic-sigma vertical coordinate using a slightly different approach. Both methods offer an attractive opportunity to add non-hydrostatic effects within an existing operational hydrostatic model framework.

There are several groups that are currently engaged in this kind of research. For example, Dudhia (1993) presented a non-hydrostatic version of the Penn State–NCAR mesoscale model using the hydrostatic-sigma vertical coordinate, and Bubnova et al. (1995) presented the results of a similar project with a Météo-France numerical model.

The primary operational regional model at the National Meteorological Center (NMC) of the United States is the grid-point Eta model (e.g. Mesinger et al. 1988; Black 1994). NMC has recently begun the development of a non-hydrostatic version of the Eta model. One goal of the project is a comparison of hydrostatic and non-hydrostatic effects to determine the resolution above which a non-hydrostatic model should be used to improve forecasts. In addition, the NMC desires to provide high-resolution operational forecasts for the 1996 summer Olympics in Atlanta, and non-hydrostatic effects may be important in models with resolutions of 10 km or less. The non-hydrostatic version of the Eta model has been developed by using the technique suggested by Juang (1992, 1994) in the NMC’s Regional Spectral Model. This technique uses perturbation equations from an arbitrary hydrostatic base-state that might be the atmosphere provided by the run of an operational hydrostatic model. A one-way nesting can be used to feed time-dependent hydrostatic variables into the non-hydrostatic model. The final goal of this project is to use hydrostatic data from the 40 km or 29 km mesoscale hydrostatic Eta model (Black 1994) in a roughly 5 km version of the non-hydrostatic model.

With this goal in mind, it seems appropriate to keep the framework of the non-hydrostatic model as close to that of the hydrostatic model as possible. One of the basic constraints built into the formulation of the original Eta-model numerics is strict conservation of energy. So far as possible, therefore, the non-hydrostatic version was developed following the same principle. The divergence term in the pressure equation is written so as to facilitate conservation of energy with appropriate advective schemes. However, in the fine resolution (100 m) simulations of collapsing cold bubbles and ascending warm bubbles made so far, the centred-difference vertical advection scheme and the Arakawa-type energy/enstrophy-conserving horizontal advection scheme (Janjić 1984) in the Eta model were found to be inadequate, especially when large vertical velocities existed close to the lower boundary. This was particularly true in cold-bubble simulations, where spurious noise developed in the vertical direction. Therefore, in the high-resolution simulations, the Eta advection schemes were replaced with the two-step Takacs scheme (1985). Strictly, this
scheme does not conserve energy but, being third-order accurate in both space and time, it yielded significantly more realistic results in cold-bubble simulations where the Eta advective schemes were least adequate. Additionally, the total energy within the model domain was found to be very nearly conserved even over periods as long as 40,000 time-steps.

The present paper describes the underlying dynamics of the non-hydrostatic Eta model and summarizes the development effort. Section 2 reviews the continuous equations used in the model and section 3 discusses the spatial and time differencing. Results of high-resolution warm- and cold-bubble simulations are given in section 4.

2. CONTINUOUS EQUATIONS

The system of equations describing a non-hydrostatic, dry, adiabatic atmosphere without rotation (e.g. Laprise 1992, Eqs. (1) to (7)) can be written as

\[ \frac{dV}{dt} = -\alpha \nabla_z p \]  

\[ \frac{dw}{dt} = -\alpha \frac{\partial p}{\partial z} - g \]  

\[ \frac{dT}{dt} = -\frac{RT}{c_v} D_3 \]  

\[ \frac{d\alpha}{dt} = \alpha D_3, \]  

where

\[ D_3 = \nabla \cdot V + \frac{\partial w}{\partial z} \]  

and

\[ p = \frac{RT}{\alpha}. \]  

In these equations, standard notation is used. Vector \( V \) is the horizontal velocity with components \( (u, v) \), \( w \) is the vertical velocity, \( \dot{z} = dz/dt \), where \( z \) is the geometric height, \( g \) is gravity, \( T \) is temperature, \( p \) is pressure, \( \alpha \) is specific volume, \( R \) is the gas constant, and \( c_p \) and \( c_v \) are the specific heats of dry air at constant pressure and constant volume respectively. The total derivative of any variable \( A \) is defined as

\[ \frac{dA}{dt} = \frac{\partial A}{\partial t} + V \cdot \nabla A + w \frac{\partial A}{\partial z}. \]  

It is more convenient to use a prognostic equation for pressure than for specific volume. Using (3) and (6) the equation for the total derivative of pressure is

\[ \frac{dp}{dt} = -\frac{c_p}{c_v} p D_3, \]  

which together with (1) to (3) and the definition of \( \alpha \) from (6) completes the system.

Kasahara (1974) formulated the following rules for transformation between a system with geometric height as the vertical coordinate and one using a general vertical coordinate \( s \), where \( s \) is a monotonic function of \( z \), and \( z = z(x, y, s, t) \):

\[ \left( \frac{\partial A}{\partial t} \right)_z = \left( \frac{\partial A}{\partial t} \right)_s - \left( \frac{\partial s}{\partial z} \right)_t \left( \frac{\partial z}{\partial t} \right) \left( \frac{\partial A}{\partial s} \right)_s \]  

(9)
\[
\n\nabla_z A = \nabla_z A - \left( \frac{\partial s}{\partial z} \right) \left( \frac{\partial A}{\partial s} \right) \nabla_z z
\]
\[
\frac{\partial A}{\partial z} = \left( \frac{\partial s}{\partial z} \right) \frac{\partial A}{\partial s}.
\]

Using these rules, the total derivative of a variable \( A \) may be written as
\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + \mathbf{V} \cdot \nabla_z A + \hat{s} \frac{\partial A}{\partial s},
\]

where \( \hat{s} \) is the vertical velocity in the new system;
\[
\hat{s} \equiv \frac{ds}{dt} = \left( \frac{\partial s}{\partial z} \right) \left( w - \frac{\partial s}{\partial t} - \mathbf{V} \cdot \nabla_z z \right).
\]

Equations (1), (2), (3) and (7) from the initial non-hydrostatic system may be rewritten in the system with an arbitrary general coordinate as
\[
\frac{\partial \mathbf{V}}{\partial t} = -\mathbf{V} \cdot \nabla_z \mathbf{V} - \hat{s} \frac{\partial \mathbf{V}}{\partial s} - \alpha \nabla_z p + \alpha \left( \frac{\partial s}{\partial z} \right) \frac{\partial p}{\partial s} \nabla_z z
\]
\[
\frac{\partial w}{\partial t} = -\mathbf{V} \cdot \nabla_z w - \hat{s} \frac{\partial w}{\partial s} - \alpha \left( \frac{\partial s}{\partial z} \right) \frac{\partial p}{\partial s} - g
\]
\[
\frac{\partial T}{\partial t} = -\mathbf{V} \cdot \nabla_z T - \hat{s} \frac{\partial T}{\partial s} - \frac{RT}{c_v} D_3
\]
\[
\frac{\partial p}{\partial t} = -\mathbf{V} \cdot \nabla_z p - \hat{s} \frac{\partial p}{\partial s} - \frac{c_p}{c_v} p D_3.
\]
The system (14) to (17) is completed with
\[
\alpha = \frac{RT}{p}.
\]

The general form of Eqs. (3) and (7) remains unchanged, but the three-dimensional divergence, \( D_3 \), is now defined as
\[
D_3 = \nabla_z \cdot \mathbf{V} - \left( \frac{\partial s}{\partial z} \right) \frac{\partial \mathbf{V}}{\partial s} \nabla_z z + \left( \frac{\partial s}{\partial z} \right) \frac{\partial w}{\partial z},
\]
as one may verify by inspection.

At this point, we will specify the general vertical coordinate \( s \) by assuming that the non-hydrostatic model operates over the same domain and at the same time as the operational hydrostatic model. In our case, the operational hydrostatic model is the major NMC regional Eta model, whose vertical coordinate is defined as
\[
\eta = \frac{\bar{p} - \bar{p}_T}{\bar{p}_S - \bar{p}_T} \eta_S,
\]
where
\[
\eta_S = \frac{\bar{p}_{ref}(\zeta_S) - \bar{p}_T}{\bar{p}_{ref}(0) - \bar{p}_T}.
\]
Here, $\tilde{\rho}$ is the hydrostatic pressure from the operational model, and subscripts 'T' and 'S' stand for the values at the top and surface respectively. The reference hydrostatic pressure, $\tilde{p}_{\text{ref}}(z)$, is defined as a suitable function of geometric height. The surface heights $z_S$ are allowed to take only a discrete set of values, so that terrain is represented as a series of steps (Mesinger 1984). Note that by setting $\eta_S$ equal to 1, Eq. (20) defines the standard terrain-following $\sigma$ coordinate introduced by Phillips (1957) and used commonly ever since in numerical forecasting. The operational model is governed by the following set of hydrostatic equations in the case of a dry, adiabatic atmosphere without rotation:

$$\frac{\partial \tilde{V}}{\partial t} = -\tilde{V} \cdot \nabla \eta \tilde{V} - \tilde{\eta} \frac{\partial \tilde{V}}{\partial \eta} - \tilde{\alpha} \nabla \eta \tilde{\rho} - \nabla \phi \tag{22}$$

$$\frac{\partial \tilde{T}}{\partial t} = -\tilde{V} \cdot \nabla \eta \tilde{T} - \tilde{\eta} \frac{\partial \tilde{T}}{\partial \eta} + \tilde{\alpha} \frac{\partial \tilde{w}}{\partial \eta} \tag{23}$$

$$\frac{\partial}{\partial t} \left( \frac{\partial \tilde{p}}{\partial \eta} \right) = -\nabla \eta \cdot \left( \tilde{V} \frac{\partial \tilde{p}}{\partial \eta} \right) - \frac{\partial}{\partial \eta} \left( \tilde{\eta} \frac{\partial \tilde{p}}{\partial \eta} \right) \tag{24}$$

$$\frac{\partial \phi}{\partial \eta} = -\tilde{\alpha} \frac{\partial \tilde{p}}{\partial \eta} \tag{25}$$

$$\tilde{\alpha} = \frac{\bar{R} \tilde{T}}{\tilde{\rho}} \tag{26}$$

Hydrostatic variables are denoted with overbars. Geopotential $gz$ is $\phi$, and $\tilde{w}$ stands for

$$\tilde{w} \equiv \frac{d \tilde{p}}{dt} = \frac{\partial \tilde{p}}{\partial t} + \tilde{V} \cdot \nabla \eta \tilde{p} + \tilde{\eta} \frac{\partial \tilde{p}}{\partial \eta}. \tag{27}$$

The major difference between the two systems is that the continuity equation for hydrostatic pressure (24) is used in the hydrostatic model both to diagnose vertical velocity $\tilde{\eta}$ and to determine surface pressure $p_S$. Prognostic equations are used in the non-hydrostatic model to determine pressure $p$ and vertical velocity $w$.

The vertical $\eta$ coordinate, according to definitions (20) and (21), is, like the general coordinate $s$, a monotonic function of $z$, and can therefore be chosen as a new vertical coordinate of the non-hydrostatic model without any reference whatsoever to relations between the non-hydrostatic atmosphere and the hydrostatic atmosphere generated by the operational model. This approach was originally introduced by Juang (1992, 1994) for applications in the regional spectral model.

This approach was directed at several objectives. For instance, the non-hydrostatic model variables are defined at the same vertical levels as the variables of the hydrostatic model; as will be shown below, this conveniently enables one to take the results of the operational model into account within the non-hydrostatic run from time to time. In addition, the underlying numerical framework of the non-hydrostatic model may be defined as a consistent extension of the existing numerics of the operational model. Finally, in the specific case of the $\eta$ coordinate, errors in the pressure-gradient force, inherent to terrain-following coordinate systems, should be eliminated.

Now, the system (14) to (18), together with definition (19), can be rewritten in terms of the operational-model $\eta$ coordinate. First, it will be assumed that the non-hydrostatic variables may be expressed as a sum of the hydrostatic portion and the non-hydrostatic perturbation. That is,

$$V = \tilde{V} + V'$$
\[ \frac{\partial V'}{\partial t} = -V \cdot \nabla V - \eta \frac{\partial V}{\partial \eta} - \alpha \nabla \phi - \frac{(\alpha \partial p / \partial \eta)}{(\tilde{\alpha} \partial \tilde{p} / \partial \eta)} \nabla \phi - \left( \frac{\partial \tilde{V}}{\partial t} \right) \]

(28)

\[ \frac{\partial w'}{\partial t} = -V \cdot \nabla w - \eta \frac{\partial w}{\partial \eta} - g \left\{ 1 - \frac{(\alpha \partial p / \partial \eta)}{(\tilde{\alpha} \partial \tilde{p} / \partial \eta)} \right\} - \left( \frac{\partial \tilde{w}}{\partial t} \right) \]

(29)

\[ \frac{\partial T'}{\partial t} = -V \cdot \nabla T - \frac{\partial T}{\partial \eta} + \frac{\alpha}{c_p} \omega - \left( \frac{\partial \tilde{T}}{\partial t} \right) \]

(30)

\[ \frac{\partial p'}{\partial t} = -V \cdot \nabla p - \frac{\partial p}{\partial \eta} - \frac{c_p}{c_v} PD_3 - \left( \frac{\partial \tilde{p}}{\partial t} \right) \]

(31)

\[ \frac{\partial \phi}{\partial \eta} = -\tilde{\alpha} \frac{\partial \tilde{p}}{\partial \eta} \]

(32)

\[ \tilde{\alpha} = \frac{RT}{\rho} \]

(33)

\[ \tilde{\eta} = \left( \alpha \frac{\partial \tilde{p}}{\partial \eta} \right)^{-1} \left( -gw + \frac{\partial \phi}{\partial t} + V \cdot \nabla \phi \right) \]

(34)

Here, \( \omega \) and \( D_3 \) are defined respectively as

\[ \omega = \frac{dp}{dt} = \frac{\partial p}{\partial t} + V \cdot \nabla p + \eta \frac{\partial p}{\partial \eta} \]

(35)

and

\[ D_3 = \left( \alpha \frac{\partial \tilde{p}}{\partial \eta} \right)^{-1} \left\{ V \cdot \left( \tilde{\alpha} \frac{\partial \tilde{p}}{\partial \eta} \right) V + \frac{\partial}{\partial \eta} (V \cdot \nabla \phi) - g \frac{\partial w}{\partial \eta} \right\} \]

(36)

The hydrostatic model is typically run on a lower resolution and with a larger time-step, while the non-hydrostatic model may be considered as the one with the transient nature. Thus, in principle, if both models are run simultaneously, the time-tendecies of the hydrostatic variables may be assimilated from the hydrostatic run and added from time to time to the right-hand side of Eqs. (28) to (31). The divergence term \( (D_3) \) in the pressure equation has been written in a form in which discretization straightforwardly provides proper conversion between kinetic and available potential energy; (see the appendix).
Idealized test experiments have been made: the hydrostatic state was defined as a time-independent, resting, horizontally-uniform atmosphere, which significantly simplified the non-hydrostatic system. With this base state, primes can be dropped and the equations written as

$$\frac{\partial \mathbf{V}}{\partial t} = -\mathbf{V} \cdot \nabla_\eta \mathbf{V} - \hat{\eta} \frac{\partial \mathbf{V}}{\partial \eta} - \alpha \nabla_\eta p \tag{37}$$

$$\frac{\partial w}{\partial t} = -\mathbf{V} \cdot \nabla_\eta w - \hat{\eta} \frac{\partial w}{\partial \eta} - g \left\{ 1 - \frac{(\alpha \partial p / \partial \eta)}{(\hat{\alpha} \partial \hat{p} / \partial \eta)} \right\} \tag{38}$$

$$\frac{\partial T}{\partial t} = -\mathbf{V} \cdot \nabla_\eta T - \hat{\eta} \frac{\partial T}{\partial \eta} + \frac{\alpha}{c_p} \omega \tag{39}$$

$$\frac{\partial p}{\partial t} = -\mathbf{V} \cdot \nabla_\eta p - \hat{\eta} \frac{\partial p}{\partial \eta} - \frac{c_p}{c_v} p D_3 \tag{40}$$

$$\alpha = \frac{RT}{\rho} \tag{41}$$

$$\hat{\eta} = -\left( \hat{\alpha} \frac{\partial \hat{p}}{\partial \eta} \right)^{-1} g w, \tag{42}$$

where

$$\omega = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla_\eta p + \eta \frac{\partial p}{\partial \eta} \tag{43}$$

$$D_3 = \left( \hat{\alpha} \frac{\partial \hat{p}}{\partial \eta} \right)^{-1} \left\{ \nabla_\eta \cdot \left( \hat{\alpha} \frac{\partial \hat{p}}{\partial \eta} \mathbf{V} \right) - g \frac{\partial w}{\partial \eta} \right\}. \tag{44}$$

It should be noted that this special case with the relatively simple set of equations (37) to (44) is not possible with a terrain-following \( \sigma \) coordinate in the presence of topography, since \( \nabla_\eta \phi \) cannot equal zero. In this particular case, with a horizontally-uniform, time-independent base-state, \( \eta \) becomes a simple function of height,

$$\eta = -gz \left( \hat{\alpha} \frac{\partial \hat{p}}{\partial \eta} \right)^{-1},$$

and this set of equations can be obtained directly from the \( z \) coordinate equations.

The primary difference between the fully-compressible non-hydrostatic equations in the \( \eta \) system shown above (Eqs. (28) to (36)) and the system suggested by Laprise (1992) is that hydrostatic pressure in our system is defined outside the non-hydrostatic model (i.e. time-independent state, or updated in a hydrostatic model). We also tested the approach used by Laprise (1992) where hydrostatic pressure is updated within the non-hydrostatic model by using the full winds in the continuity equation. For our purposes, the technique of Juang (1992, 1994) seems more appropriate as a means of developing a non-hydrostatic model closely paralleling the hydrostatic model, and taking advantage of existing hydrostatic fields. In addition, we encountered serious numerical instabilities in the non-hydrostatic Eta model when updating hydrostatic pressure using the continuity equation (following Laprise (1992)). It appeared that sound waves influenced the hydrostatic pressure, and thus interfered with the coordinate surfaces. The instability was removed when the hydrostatic pressure was determined using velocities in the continuity equation from a hydrostatic model, or was assumed to be time-independent. Though it is beyond the scope of this paper to explain this instability in any detail, we note in passing that extra care is perhaps necessary when discretizing the equations presented by Laprise (1992).
3. DISCRETIZATION

The distribution of variables in the vertical cross-section of the non-hydrostatic model's domain is shown in Fig. 1. Horizontal velocity and thermodynamic variables are located in the middle of so-called eta layers, at what we call the 'velocity' and 'scalar' points hereafter. All other variables, that is $\omega$, $\eta$, $\bar{\rho}$ and $\phi$, are defined on the interfaces, on the same vertical line as the scalar points. The vertical boundary conditions are that $\eta$ and $\omega$ equal zero at the top and bottom interfaces of the domain.

Other arrangements of variables are also possible. Tokioka (1978) has considered the effect of different vertical arrangements of variables on dispersive characteristics of vertically propagating gravity-waves. According to his classification, the distribution presented in Fig. 1 may be referred to as the Tokioka vertical A-grid. This is a consistent non-hydrostatic extension of the vertical distribution, suggested by Lorenz (1960) for hydrostatic models, that is currently used in the operational Eta model.

\[ \cdots \]

\[ \omega, \eta = 0 \]

\[ \cdots \]

\[ \cdots \]

\[ \omega, \eta, \Pi, \phi \]

\[ \psi, p, T \]

\[ \Delta \eta_k \]

\[ \omega, \eta, \Pi, \phi \]

\[ \cdots \]

\[ \cdots \]

\[ \cdots \]

\[ \omega, \bar{\eta} = 0 \]

\[ \cdots \]

Figure 1. Distribution of variables in a vertical cross-section of the model domain. Tokioka A-grid.

The hydrostatic Eta model uses a semi-staggered horizontal distribution of variables placed in the middle of the eta layers, that, following Arakawa (1972) and Arakawa and Lamb (1977), is usually referred to as the Arakawa E-grid. In accordance with the principle of building a non-hydrostatic extension to the hydrostatic Eta model, the same horizontal arrangement of variables has been adopted, with $p$ and $T$ located at the same grid points (Fig. 2).

In general, the spatial and time-differencing schemes present in the operational Eta model are also used here. A simple additive split-explicit technique is applied for advection and sound-wave terms (pressure-gradient terms in the momentum equations, and the divergence term in the pressure equation). For the simulations discussed in this paper, the advection time-step is chosen to be twice the sound-wave time-step. However, sensitivity tests have found that similar results can be obtained with advective time-steps as large as 16 times the sound-wave time-step. Efficiency is increased by nearly an order of magnitude.
when the 1:16 ratio is used; however, diffusion or filtering must be increased, resulting in some small damping of the meteorologically significant features.

The sound-wave terms are treated with a modified forward–backward time-differencing scheme (similar to that of Mesinger et al. 1988). With this scheme, the velocity values are updated using forward time-differencing and then the new values of the pressure field are obtained using backward time-differencing.

The pressure-gradient force consists of two terms. One involves pressure gradients, and the other geopotential-height gradients. When a time-independent, horizontally homogeneous, resting, hydrostatic base-state is used with the \( \eta \) vertical coordinate, the \( \phi \) term disappears, even in the presence of topography. As stated earlier, a major feature of the vertical \( \eta \) coordinate is that the coordinate surfaces remain quasi-horizontal even in the presence of steep mountains. This guarantees that the two terms comprising the definition of the pressure-gradient force do not generate the error inherent to hydrostatic models that use sigma-type coordinates (e.g. Mesinger and Janjić 1985). One way this error may show up is by the generation of a flow in a resting atmosphere. Mesinger and Black (1992) and Mesinger, Black and M. E. Baldwin (1995, personal communication) found that Eta-model forecasts of precipitation are dramatically improved with the step-wise treatment of terrain. One might expect that the problems caused by formulation of the pressure-gradient force would still exist in non-hydrostatic models that use the hydrostatic pressure for definition of vertical coordinate. Therefore, a step-wise treatment of the terrain may be of benefit for such non-hydrostatic models, as it appears to be for the hydrostatic ones. A comparison of the performance of the sigma and eta vertical coordinates near steep mountains within the non-hydrostatic framework will be addressed in future work.

In high-resolution simulations with the model, the Arakawa energy/entropy-conserving scheme, derived for the horizontal E-grid by Janjić (1984) and used in the operational Eta model for horizontal advection, was found to resolve the evolution of cold bubbles inadequately. Similar problems occurred with the Eta vertical advection schemes used for momentum and temperature, which use simple centred differences. Therefore, for the high-resolution simulations, the two-step, third-order-accurate (in both time and space), advection scheme of Takacs (1985) is substituted for both horizontal and vertical advection.

The omega–alpha (energy conversion) term present in both the pressure and thermodynamic equations is computed using an Eulerian forward scheme.

4. Simulations of thermal bubbles

A simple hydrostatic, horizontally homogeneous, time-independent, resting base-state is assumed for several simulations of thermal bubbles. The three-dimensional (3-D) version of the model successfully simulated both cold and warm bubbles, but was restricted to a rather coarse resolution of 500 m in both the horizontal and vertical due to
computational costs. Comparison of results with other studies was complicated by the fact that most thermal bubble experiments in the literature are performed with two-dimensional (2-D) models. Therefore, a 2-D version of the model was developed, and the discussion of results will be with regard to it.

In both the cold- and warm-bubble experiments, the top of the model domain is defined as 10 mb, and the domain extends approximately 60 km in each horizontal direction. A dry adiabatic lapse rate is assumed throughout most of the domain, with increasing stability near the top boundary (where Newtonian damping is applied in the top four layers). Open radiative lateral boundaries are used, with a rigid top and bottom. A constant gradient of temperature is assumed near the top and bottom boundaries in the treatment of vertical advection. The modification of divergence (Mesinger 1977; Janjić 1979), usually used to prevent two-grid-interval noise due to grid separation on the semi-staggered grid, did not work well in the fine-resolution non-hydrostatic experiments. Instead, a weak filter is applied on the pressure variable in order to eliminate the small-scale noise.

(a) Cold-bubble simulations

The model is tested in the benchmark cold-bubble experiment suggested by Straka et al. (1993). The temperature perturbation is defined as

\[ \Delta T = \Delta T_0 \cos^2 \left( \frac{\pi \beta}{2} \right) \quad \text{for} \quad \beta < 1, \]

where \( \Delta T_0 = -15.0 \text{ K} \) and \( \beta \) is a shape profile defined in two dimensions as

\[ \beta^2 = \left( \frac{x - x_c}{x_t} \right)^2 + \left( \frac{z - z_c}{z_t} \right)^2, \]

where the subscripts ‘c’ and ‘r’ refer to the centre position and the radial dimension of the bubble respectively. The bubble radius is chosen to be 4.0 km, and the bubble is placed near the centre of the domain with \( z_c = 3.0 \text{ km} \). The simulations discussed below are run with a horizontal resolution of 100 m, and a roughly similar vertical resolution. This is far higher than the 5 to 10 km resolutions proposed for the model when used for forecasting-applications at NMC.

The potential-temperature perturbation simulated at 0, 300, 600 and 900 seconds in the Eta model is shown in Fig. 3. The corresponding figure from the University of Oklahoma Center for Analysis and Prediction of Storms (CAPS) model (Xue, personal communication) is shown in Fig. 4, and from the grid-converged 25 m simulation of Straka et al. (1993) in Fig. 5. The model simulates the evolution of the density current successfully, agreeing with 100 m results obtained using the CAPS model (Xue and Thorpe 1991; Xue, personal communication). The results also agree closely with the 25 m grid-converged reference solution of Straka et al. (1993); (note the different aspect ratios in Figs. 3 and 5). In this test, the original Eta advection scheme produces noise in the vertical direction; the Takaes (1985) scheme, by contrast, performs well. The noise in the original Eta advection scheme appears to arise as the cold bubble approaches the bottom boundary and strong vertical motions occur there. This noise slowly amplifies with time, so that small layers (roughly two \( \Delta \eta \) stops in depth) of negative perturbation alternate with positive perturbations in the region through which the bubble descends. In addition, total energy within the model domain is still found to be almost conserved, varying by less than 0.05% over as many as 40 000 iterations, in spite of open lateral-boundary conditions.

The density current evolves with three Kelvin–Helmholtz shear-instability rotors apparent along the top boundary of the cold-air outflow during the 900-second simulation.
Figure 3. Potential-temperature perturbation from the 100 m resolution non-hydrostatic Eta model: (a) at 0 sec; (b) after 300 sec; (c) after 600 sec and (d) after 900 sec. Contour interval of 1 K. Note displacement of horizontal scale in (d).

similar to those reported by Straka et al. (1993). The density current spreads outward at roughly 18 m s$^{-1}$.

The evolution of the circulation and pressure fields in the Eta simulations (Fig. 6) shows even more similarity to the CAPS results (Fig. 7). (Note that the domain for the Eta simulation has been shifted 4 km to the right to show the main area of interest.) At 900 seconds, both simulations show that the greatest horizontal winds near the surface lie around $x = 8$ to 9 km, with peak speeds of 38 m s$^{-1}$ in the Eta model, and around 35 m s$^{-1}$ in the CAPS model. The lack of diffusion in the Eta model simulation may explain the slightly greater intensity of the density current. The greatest positive velocities occur in an arc from near the front of the density current back to $x = 9$ km with four maxima in both simulations. Both simulations show three cores or enhanced areas of negative velocity above the cold pool, with comparable intensity.
Figure 4. As Fig. 3, except from the CAPS model (Xue, personal communication): (a) at 300 sec; (b) at 600 sec and (c) at 900 sec.
Vertical velocities are strikingly similar (Figs. 6(b) and 7(b)). Both models show the same number of updraughts and downdraughts, with reasonably similar magnitudes. The Eta model shows somewhat more intense vertical motions in the forward portion of the density current. This again may be attributed to the lack of diffusion in the Eta model, or perhaps to the third-order accuracy of the advection scheme. The pressure fields are also nearly identical (Figs. 6(c) and 7(c)). Sensitivity tests, in which the computational efficiency is increased by taking a larger advective time-step relative to the sound-wave time-step, show decreasing strength of the circulations, thus agreeing more with the other solutions where diffusion was present.

The results from the cold-bubble simulations indicate that the non-hydrostatic Eta model can simulate features far smaller than those that would be resolvable in an operational mesoscale model. In addition, total energy within the model domain is found to be almost conserved, varying by less than 0.05% over as many as 40 000 iterations (in spite of open lateral boundary conditions and the use of the Takacs (1985) advection schemes instead of the energy-conserving operational Eta model schemes).

(b) Warm-bubble simulations

The warm-bubble simulations described by Mendez-Nunez and Carroll (1994) have also been run with a resolution of 100 m in both the horizontal and vertical directions. The temperature perturbation is defined in a similar manner to that used in the cold-bubble simulations, except that the perturbation is made in \( \theta \) and not \( T \). A \( \Delta \theta_0 \) of 6.6 K is used with a bubble radius of 2.5 km and \( z_c = 2.75 \) km, as used by Mendez-Nunez and Carroll (1994).
The evolution of the potential-temperature perturbation can be seen in Fig. 8. The bubble shows the effect of strong vertical motions which are greatest near the center of the perturbation. The circulations and pressure field agree qualitatively with other warm-bubble simulations (run with different resolutions). Eventually some of the positive temperature-perturbation is caught up in the wake below the bubble (Figs. 8(b), (c)). The temperature field resembles that found by Robert (1993), who ran warm-bubble simulations with a much finer resolution (10 m) and smaller temperature perturbation. In the warm-bubble simulations, total energy within the model domain varies by less than 0.05% during the length of the simulation, a period with over 8000 model iterations. This again indicates that energy is highly conserved, even though the explicitly energy-conserving advection schemes of the operationalEta model have been replaced for these high-resolution simulations.
Figure 7. As Fig. 6, except for 100 m CAPS simulation (Xue, personal communication). Contour interval in (c) is 0.32 mb.
5. CONCLUDING REMARKS

A fully compressible non-hydrostatic version of the NMC's grid-point regional Eta model has been developed using perturbation equations from a hydrostatic base-state, as by Juang (1992, 1994). The model successfully simulates ascending warm bubbles and spreading density-currents with no explicit diffusion necessary. Energy is approximately conserved in these experiments though the advection scheme applied does not guarantee a formal conservation. The operational Eta model uses step-wise treatment of terrain that results in quasi-horizontal coordinate surfaces, eliminating the pressure-gradient-term error inherent to hydrostatic models that use sigma-type coordinates (e.g. Mesinger and Janjić 1985). The hydrostatic Eta model generally performs well in comparisons with other operational models at NMC (e.g. in tests of skill in forecasting precipitation, Mesinger et al. (1995)), and the non-hydrostatic model will be an important extension to the Eta model at higher resolutions.
NON-HYDROSTATIC REGIONAL MODEL

Although the simulations discussed in this paper were produced using a fully explicit version of the model, which placed a severe stability-condition on the time-step, computational efficiency has recently been improved through the use of several numerical techniques. First, the semi-implicit treatment of terms responsible for the vertical propagation of sound waves (Klemp and Wilhelmson 1978) has been adopted, allowing the small time-step to be increased to the value required by the hydrostatic Eta model for the same horizontal resolution. In addition, the slowing down of the sound wave (Browning and Kreiss 1986) as in the CAPS model (Xue and Thorpe 1991), has also been found to work well and permit an even larger time-step. A reduction of the sound-wave speed by 50% results in little change in the simulation, while allowing a doubling of the smallest time-step. These techniques increase efficiency and will assist in operational implementation of the non-hydrostatic Eta model. The final goal for the implementation is to update the hydrostatic base-state using the operational Eta model. Prior to this, however, 3-D real-data simulations will be performed using a simple static base-state as in the 2-D test simulations. The reference state will use the standard atmosphere, from which terrain heights are currently computed in the hydrostatic Eta model. Because of results from other non-hydrostatic modelling studies (e.g. Dudhia 1993), however, we do not expect serious problems with this simplified interim approach.

ACKNOWLEDGEMENTS

This research would not have been possible without the extraordinary research environment of the Development Division of the U.S. National Meteorological Center, Camp Springs, Maryland. The support of Dr Ronald McPherson, Director, and Dr Eugenia Kalnay, Chief of the Development Division of the Center, is gratefully acknowledged. The initiative and encouragement for the work on developing the non-hydrostatic version of the Eta model came from the Head of the Regional and Mesoscale Modeling branch of the Division, Dr Geoffrey DiMego. Dr Fedor Mesinger suggested the development of a conservative scheme for a set of fully-compressible equations that employs the hydrostatic-pressure-based vertical coordinate. The authors benefited from discussions with Drs Fedor Mesinger, Tom Black, Hann-Ming Henry Juang and James Purser. Dr Rančić would like to acknowledge the Center for Analysis and Prediction of Storms, where his initial work on development of the non-hydrostatic Eta model began. Lastly, the constructive comments of Dr Rene Laprise and an anonymous reviewer were appreciated.

APPENDIX

Conservation of energy in transformation between kinetic and available potential energy

In the non-perturbed system of non-hydrostatic equations, consider only those terms that are responsible for the transformation of energy:

\[
\frac{\partial \mathbf{v}}{\partial t} = \cdots - \alpha \nabla \eta p - \frac{\alpha \partial p/\partial \eta}{\alpha \partial \mathbf{p}/\partial \eta} \nabla \phi \tag{A.1}
\]

\[
\frac{\partial w}{\partial t} = \cdots + g \frac{\alpha \partial p/\partial \eta}{\alpha \partial \mathbf{p}/\partial \eta} \tag{A.2}
\]

\[
\frac{\partial T}{\partial t} = \cdots - \frac{RT}{c_v} D_3. \tag{A.3}
\]
The $\omega \alpha$ term in the thermodynamic equation (A.3) is replaced using the pressure equation (17). An energy equation can be formed by first multiplying (A.1) by $u$, (A.2) by $w$, and (A.3) by $c_v$, and then summing these equations and finally multiplying the resulting equation by

$$\frac{\partial E}{\partial \eta} = \cdots - \hat{\alpha} \frac{\partial \hat{p}}{\partial \eta} \frac{\partial \hat{V}_p}{\partial \eta} + \frac{\partial p}{\partial \eta} \frac{\partial \hat{V}_p}{\partial \eta} + g w \frac{\partial \hat{p}}{\partial \eta} - \hat{\alpha} \frac{\partial \hat{p}}{\partial \eta} \rho D_3.$$

This yields

$$\frac{\partial E}{\partial t} = \cdots - \nabla \cdot \left( \rho \nabla \right) \left( \frac{\partial \hat{V}_p}{\partial \eta} \right)^2 + g \frac{\partial (p w)}{\partial \eta}.$$

Here, the sum of kinetic and internal energy, $E$, is defined as

$$E = \frac{\alpha}{\hat{\alpha}} \frac{\partial \hat{p}}{\partial \eta} \left( \frac{\nabla^2 + w^2}{2} + c_v T \right).$$

By replacing $D_3$ from (44), (A.4) may immediately be written in a flux form as

$$\frac{\partial E}{\partial t} = \cdots - \nabla \cdot \left( \rho \nabla \right) \left( \frac{\partial \hat{V}_p}{\partial \eta} \right)^2 + g \frac{\partial (p w)}{\partial \eta}.$$

Vertical boundary conditions, such as $V = 0$ and $w = 0$, at the top and bottom of the model’s atmosphere, provide conservation of energy in a horizontally closed domain.

The discretization for the pressure gradient in (A.1) in the non-hydrostatic Eta model has been defined consistently with the original formulation in the operational version of the model. The buoyancy terms in (A.2) and $D_3$ in (A.3) are discretized so that energy conservation as demonstrated in (A.5) and (A.6) is maintained in the algebraic equations.

**References**

- Eckart, C. 1960 *Hydrodynamics of Oceans and Atmospheres*. Pergamon


Phillips, N. A. 1957 A coordinate system having some special advantage for numerical forecasting. *J. Meteorol.*, 14, 184–185


