Gravity-wave drag on two mountains

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SUMMARY

The generation of linear internal gravity waves by the flow over two idealized mountains, whose summits are aligned in the same direction as the mean wind, is studied. In particular, the sensitivity of the wave drag to the downwind distance between the mountains is examined by using a three-dimensional numerical model based on the linearized steady-state Boussinesq equations of motion. Both two- and three-dimensional orography is considered. When the upstream conditions are such that the wind and buoyancy-frequency do not vary with height, and the mountains are wide enough for non-hydrostatic effects to be unimportant, the model predicts the same drag dependence on two-dimensional ridge spacing as was derived analytically by Grisogono et al. The corresponding three-dimensional numerical result predicted by the model is qualitatively very similar to the two-dimensional result. When the mountains are narrow and the waves are non-hydrostatic the drag dependence on mountain spacing is more complicated; maxima and minima in the drag can occur when the distance between the two mountains is such that waves generated by the mountain furthest upstream are in phase (maxima in the drag), or out of phase (minima in the drag), with the waves generated by the downstream mountain. It is shown that this is an important effect when the waves are trapped by upstream profiles whose wind speed varies with height.

KEYWORDS: Complex orography  Gravity-wave drag  Hydrostatic and non-hydrostatic effects  Trapped lee waves

1. INTRODUCTION

In this work the effect of complex orography on the generation of internal gravity waves is investigated. The work by Grisogono et al. (1993) shows how the drag exerted by linear hydrostatic gravity waves on two ridges is dependent on the spacing between the ridges. In the current paper some numerical results for linear non-hydrostatic gravity-wave drag exerted on two mountains will be presented. Both two- and three-dimensional orography will be considered.

2. THE NUMERICAL MODEL

The numerical model used in this study is a three-dimensional model for flow over arbitrary orography based on the linearized, steady-state equations of motion for an inviscid adiabatic fluid. The shallow-convection form of the Boussinesq approximation is made. Full details of the model are provided by Vosper (1995), and Vosper and Mobbs (personal communication) have compared the model predictions with observations of gravity waves. Therefore, we shall describe the model only briefly here.

Provided the Froude number, $U/Nh$, (where $U$ is the upstream wind speed, $N$ is the buoyancy frequency and $h$ is the orographic height) is greater than about 1, and the slope of the orography is small, the linearization of the equations of motion will generally be valid. The question of linearization is discussed in great detail by Smith (1980, 1988); according to Smith’s analysis, the model runs shown in this paper would all come within the linear regime (strictly speaking, this is true only when the upstream wind and stability are independent of height and this is discussed further in section 5).

Applying two-dimensional discrete Fourier transforms to the linearized equations of motion, and eliminating all dependent variables except the vertical velocity, results in the familiar vertical-structure equation which applies for each wave mode, namely,

$$\frac{d^2\tilde{w}}{dz^2} + [\ell^2 - (k^2 + l^2)]\tilde{w} = 0,$$

where $\ell^2(z) = \frac{N^2}{U^2} - \frac{1}{U} \frac{d^2U}{dz^2}$. \hspace{1cm} (1)

Here $\tilde{w}(z)$ is the Fourier transformed vertical velocity, at height $z$, for the wave mode with horizontal wavenumbers $k$ and $l$ in the $x$ and $y$ directions respectively. Note that in Eq. (1) we have considered upstream conditions for which the wind is in the $x$ direction only.

Wave energy will be partially reflected back towards the ground below levels where $\ell^2 - (k^2 + l^2)$ becomes negative. Wave modes for which this occurs are known as ‘trapped’ modes, and these modes

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will propagate horizontally through the model domain as well as vertically. The application of discrete Fourier transforms in the numerical model implies that the model domain is periodic in the horizontal. This causes strongly trapped modes to reappear upstream. A special case of a trapped mode is one which is completely reflected back towards the ground. These are known as resonant or lee-wave modes and they have zero amplitude at the ground. In order to prevent the resonant modes re-appearing upstream the model implements a quasi-analytical solution for the resonant modes, along the lines of Sawyer (1960 and 1962), Danielsen and Block (1970) and Vergeiner (1971), which is valid downstream of the orography only. The appropriate upstream solution for the resonant modes is one of zero amplitude.

3. The orography

The orography considered in this work is essentially the same as that used by Grisogono et al. (1993). In the latter paper, only the two-dimensional case was considered. Here, however, we will examine both two- and three-dimensional orography. We consider orography whose profile is described by \( h(x, y) = H_1 + H_2 \), where \( H_1 \) and \( H_2 \) are the heights of two bell-shaped mountains and \( H_1 \) (the mountain lying furthest upwind) is given by

\[
H_1 = \frac{h_1 b_1^2}{(x - x_1)^2 + y^2 + b_1^2}
\]  

\( \text{(2)} \)

and \( H_2 \) (the mountain furthest downwind) is obtained from Eq. (2) by simply replacing the subscript 1 by the subscript 2 everywhere. In Eq. (2) \( h_1 \) and \( b_1 \) are the height and half-width of the mountain lying furthest upwind, respectively, and \( h_2 \) and \( b_2 \) apply to the downwind mountain similarly. The upwind and downwind mountains are centred along the \( x \) axis at the points \( (x_1, 0) \) and \( (x_2, 0) \), respectively. The equations for the two-dimensional ridges are obtained by simply dropping the \( y \) dependence from Eq. (2).

4. The numerical experiments

(a) Constant upstream wind and buoyancy-frequency

Grisogono et al. (1993) derived an analytical solution for linear wave drag when the flow is over a double bell-shaped ridge (the two-dimensional form of Eq. (2)) and the waves generated are hydrostatic. In the notation used here, the drag is given by

\[
F = \pi \rho U N \left[ h_1^2 + h_2^2 + 8 b_1 h_2 \left( \frac{b_1 b_2}{(b_1 + b_2)^2} \right) \left( \frac{1 - d^2}{1 + d^2} \right) \right],
\]  

\( \text{(3)} \)

where \( d \) is the normalized ridge separation, \( d = (x_2 - x_1) / (b_1 + b_2) \) and \( \rho \) is the air density at the surface. In this derivation it is assumed that \( U \) and \( N \) are independent of height and that \( U / NL \ll 1 \), where \( L \) is a typical width of the orography (i.e., it is assumed that the waves are hydrostatic).

Three numerical experiments have been conducted with constant upstream values of \( U \) and \( N \). In the first (Exp1) the two mountains are both taken to be two-dimensional ridges and the ridge widths and upstream profile are chosen so that the waves are approximately hydrostatic. This experiment provides a validation of the model since the results can be compared with Eq. (3). The second experiment (Exp2) is a repeat of the first using three-dimensional mountains instead of the two-dimensional ridges. In the third experiment (Exp3) we consider two-dimensional ridges whose widths are such that the waves generated are non-hydrostatic. In all three experiments the model uses 128 vertical levels with the upper boundary placed at 20 km, giving a vertical resolution of 158 m. The constant value of \( N \) was set to be 0.01 s\(^{-1}\) and both the mountain heights were 300 m. Table 1 shows the mountain half-widths, the number of discrete points used in the \( x \) and \( y \) directions \( (n_x \) and \( n_y \), respectively), the horizontal resolutions, the values of

<table>
<thead>
<tr>
<th>TABLE 1. Parameters used for experiments Exp1, Exp2 and Exp3.</th>
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<tr>
<td>( h_1, h_2 ) (km)</td>
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<td>---------------------</td>
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<tr>
<td>Exp1 10</td>
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<td>Exp2 10</td>
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<td>Exp3 2</td>
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See text for an explanation of symbols.
Figure 1. (a) The relationship between drag and two-dimensional ridge separation, \( d \), given by Grisogono’s solution (dashed line, labelled GR), Exp1 (solid line) and Exp2 (dashed line); (b) \( D_1 \), \( D_{12} \) (solid line), \( D_{21} \) (dashed line) against \( d \), for Exp3; (c) the variation of \( \ell^2 \) with \( z \) given by Eq. (7) and (d) \( D_1 \), \( D_{12} \) (solid line) and \( D_{21} \) (dashed line) against \( d \), for the two-dimensional ridges when there is upstream wind shear.

\( U \) and the corresponding values of \( U/Nb_1 \) which give an indication of how important non-hydrostatic effects are.

Figure 1(a) shows how the drag depends on the separation \( d \) for Exp1 (solid line), Exp2 (dashed line) and the two-dimensional analytical solution due to Grisogono et al. (1993) (dashed line, labelled GR). Note that the numerical model solution for Exp1 is in such close agreement with Grisogono’s solution that the curves partially hide each other. The numerical solutions for the drag for each single isolated ridge \( H_1 \) and \( H_2 \). Grisogono’s solution has been normalized by the equivalent quantity \( F_1 + F_2 \), where \( F_1 = \pi \rho U Nh_1^2 / 4 \) and \( F_2 \) is expressed similarly, by replacing \( h_1 \) by \( h_2 \).

Grisogono et al. (1993) explained the dependence of the normalized drag on \( d \) in the following way. For values of \( d \) close to zero the orography is nearly a single ridge. The drag is then high, due to the quadratic dependence on height. When \( d = 3^{1/2} \) the drag has a minimum. This is because the overall height of the orography is reduced, relative to \( d = 0 \), whilst the width is increased. For increasingly large values of \( d \) the normalized drag approaches 1, but is always less than 1 since the ‘effective height’ of the downstream ridge is reduced, due to the airflow being elevated as it passes over the upstream ridge. The effect of the upstream orography becomes smaller as \( d \to \infty \) because air passing over it is allowed to return to its original level before encountering the downstream ridge. The numerical solution for Exp2, the three-dimensional case, is qualitatively very similar to the two-dimensional result. This result can be explained in the same way.
As an aid to interpreting the results for experiment Exp3 (and a following experiment) we notice that the surface drag for each single isolated mountain can be written as \( D_i \) and \( D_2 \) where

\[
D_i = \int \int p_i \nabla_h H_i \, dx \, dy \quad \text{for } i = 1, 2, \tag{4}
\]

where \( p_i \) and \( p_2 \) are the pressure perturbations at the ground induced by mountains \( H_1 \) and \( H_2 \) respectively, \( \nabla_h = (\partial / \partial x, \partial / \partial y) \) and the integration is over the entire horizontal extent of the model domain. The total normalized drag, \( D_i \), can then be expressed as

\[
D_i = \frac{1}{D_1 + D_2} \int \int p_i \nabla_h (H_1 + H_2) \, dx \, dy = D_{i1} + D_{i2} + 1 \tag{5}
\]

where \( p_i \) is the total surface pressure perturbation due to both mountains and

\[
D_{i12} = \frac{1}{D_1 + D_2} \left[ \int \int p_i \nabla_h H_2 \, dx \, dy - D_2 \right] \tag{6}
\]

and \( D_{i21} \) is obtained by simply interchanging the subscripts 1 and 2 in Eq. (6). \( D_{i12} \) can be thought of as the effect that the upstream mountain has on the downstream mountain, since it is the difference between the drag on the downstream mountain in the presence of upstream orography and the drag in the absence of upstream orography. Similarly, \( D_{i21} \) is the effect of the downstream orography on the upstream orography. Note, however, that since the orography is formed by simply adding together two mountain shapes, when the mountain separation is small the orography may not consist of two clearly separated shapes. Thus for small values of \( d \) little significance can be attached to the quantities \( D_{i12} \) and \( D_{i21} \).

Figure 1(b) shows the variation of \( D_i, D_{i12} \) and \( D_{i21} \) with \( d \) for the non-hydrostatic experiment Exp3. It is clear that the variation of the drag with \( d \) is different to the hydrostatic case. \( D_i \) decreases from 2 at \( d = 0 \) in the same way as it does for the hydrostatic solution and then reaches a minimum at about \( d = 2.5 \). As \( d \) is increased further \( D_i \) tends to 1 but with a superimposed oscillation. Examining the variation of \( D_{i12} \) and \( D_{i21} \) with \( d \) reveals that this oscillation is due to \( D_{i21} \). In other words, the upstream ridge affects the drag on the downstream ridge even when the two ridges are spaced far apart. This is due to the generation of non-hydrostatic waves by the upstream ridge which propagate downstream towards the downstream ridge, as well as propagating vertically. The effect of the downstream ridge on the drag exerted on the upstream ridge is negligible for values of \( d \) greater than about 2. This is shown by the fact that the variation of \( D_{i21} \) contains no oscillations and tends to zero with increasing values of \( d \).

(b) Upstream wind dependent on height

We now examine how the drag is affected by the separation of two two-dimensional ridges when the upstream wind is dependent on height. The mountains chosen for this study are again of the form given by Eq. (2) with \( h_1 = h_2 = 300 \) m and \( b_1 = b_2 = 2 \) km. The upstream conditions are taken to be

\[
U(z) = U_0 + \tilde{U} \sin \left( \frac{\pi z}{z_u} \right), \tag{7}
\]

where \( U_0 = 5 \) m s\(^{-1}\), \( \tilde{U} = 12.5 \) m s\(^{-1}\) and \( z_u = 20 \) km is the height of the upper boundary. \( N \) is again taken to be independent of height, at 0.01 s\(^{-1}\). With such a profile, we would expect some wave modes to be trapped in the lower layers of the atmosphere and to propagate horizontally. This is illustrated by Fig. 1(c), which shows the dependence of the function \( \tilde{U}^2 \) on \( z \). A minimum in \( \tilde{U}^2 \) occurs at \( z = z_u / 2 \) and at levels for which \( \tilde{U}^2 < k^2 + F^2 \) the wave amplitude will decay exponentially with height. Upward propagating wave energy will be partially reflected back towards the surface beneath this level, and from Fig. 1(c) we can see that for all wavelengths with \( k^2 + F^2 > 3.4 \times 10^{-7} \) m\(^{-2}\) (i.e. for all horizontal wavelengths less than 10.7 km) the waves will be trapped below some level lower than \( z_u / 2 \). This \( \tilde{U}^2 \) profile is fairly realistic since \( \tilde{U}^2 \) will usually decrease with height in the troposphere (due to a wind speed which increases with height) and will have larger values in the stratosphere (due to lower wind speeds above the tropospheric jet and to higher values of \( N \)).

The horizontal and vertical resolutions used for this model run were 400 m and 197 m respectively. In order to allow the partially trapped wave modes to leak sufficient energy upwards, so that on passing through the downstream boundary and re-appearing upstream they were of small amplitude, a very long
domain consisting of 4096 points along the x axis was required. As before, the model used 128 vertical levels. Figure 1(d) shows $D_1$, $D_{12}$ and $D_{31}$ against $d$. We can see that the upstream profile has altered the non-hydrostatic solution for the constant wind and buoyancy-frequency case significantly (compare with Fig. 1(b)). The total drag, $D_1$, decreases from 2 when $d = 0$ and reaches a minimum value at about $d = 1.7$. For larger values of $d$, we can see that $D_1$ exhibits a quasi-periodic nature, with a separation between successive maxima and minima in the drag of about $d = 1.4$. For values of $d$ greater than about 2 the variation of $D_{12}$ reveals that the fluctuations in $D_1$ are almost entirely due to the effect of the upstream mountain on the downstream one. Oscillations in the quantity $D_{31}$ appear also. However, these are of small amplitude compared with those in $D_1$ and $D_{12}$, and are caused by the ‘wrap-around’ behaviour of strongly trapped (but non-resonant) wave modes.

By examining vertical cross-sections of vertical velocity we can understand why the trapping makes such a significant difference to the drag. Figure 2 shows contours of vertical velocity on a vertical slice along the x axis when the separation $d = 2.8$. This value of $d$ corresponds to a maximum in the drag. We can see a strongly trapped wave response with near-vertical phase-lines in the lowest 4 km and a horizontal wavelength of about 6 km. Above this there is a train of partially trapped waves with a phase-line tilt that increases with height, and between heights of 8 km and 12 km the horizontal wavelength is approximately 11 km. We now compare the wave field shown in Fig. 2 with the wave field when $d = 4.1$ (see Fig. 3). In this case there is a minimum in the drag. Again we can see a strongly trapped response in the lowest 4 km, however the amplitude of the 11 km wavelength mode downstream of the second ridge is greatly reduced. When $d = 2.8$ the mountains are separated by 11.2 km. This distance is approximately equal to the 11 km wavelength. It seems, therefore, that when maxima in the drag occur, the waves generated by the upstream mountain are ‘in phase’ with the waves generated by the downstream mountain and there is a ‘constructive interference’ effect. This has the effect of increasing the amplitude of the waves generated by the downstream mountain, thus increasing the quantity $D_{12}$ and the overall drag. When the mountains are moved further apart by another half a wavelength (this is the case when $d = 4.1$), the waves generated by the upstream mountain are out of phase with those generated by the downstream mountain. The effect is a decrease in the amplitude of the waves generated by the downstream mountain as the waves generated by the upstream mountain interfere with them ‘destructively’. The drag on the downstream mountain, and
hence the total drag, is therefore decreased. Examining the vertical velocities at all other values of \( d \) for which there are maxima and minima in \( D \), reveals that the maxima occur when \( d \) is close to an integer times the dominant horizontal wavelength, and the minima occur when \( d \) is close to half an integer times the wavelength. This is consistent with the above argument.

5. DISCUSSION AND CONCLUSIONS

It has been shown that the linear hydrostatic gravity-wave drag exerted on two three-dimensional mountains is qualitatively similar to the two-dimensional result due to Grisogono et al. (1993). Furthermore, in the two-dimensional case when the waves are non-hydrostatic, the drag can be sensitive to the distance between the two mountains even when this distance is large. This effect is caused by interference between the waves generated by the two mountains and appears to be most dramatic when the waves are trapped in the lower layers of the atmosphere.

It should be noted that although linear theory is able to predict wave amplitude and drag well when the upstream wind and stability are independent of height (Smith 1988), this is not always the case when the upstream profiles are more complicated and the waves are trapped. Both Smith (1976) and Durran (1986) have demonstrated that in some cases linear theory poorly predicts the amplitude of resonant waves even at large values of \( U/Nh \) (see Durran 1986 Table 1, for example), as finite amplitude effects can play an important role in the partial reflection of wave energy from aloft. Furthermore, the results of Durran (1986) seem to indicate that when the waves are trapped, nonlinear effects might cause the ridge separation for which maxima in the drag occur to be a function of the mountain height (see Durran 1986, Fig. 3) as well as of the horizontal wavelength (as demonstrated in this work).

This investigation highlights a problem associated with simple gravity-wave drag parametrization schemes. The interference effects that may occur when trapped waves are generated by a region of complex sub-grid scale orography cannot be accounted for by schemes which make the hydrostatic assumption or which do not contain details of the orography. There is also an implication for measurements made of the drag during field experiments such as those described by Smith (1978) and Vosper and Mobbs (personal communication). Namely, waves generated by any surrounding orography may significantly affect the drag on the mountain being studied. This could make the interpretation of surface pressure measurements made during field experiments very difficult.
NOTES AND CORRESPONDENCE

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