Stratospheric inertia–gravity waves generated in a numerical model of frontogenesis. II: Wave sources, generation mechanisms and momentum fluxes

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(Received 23 January 1995; revised 20 November 1995)

SUMMARY

Using a dry two-dimensional numerical model, we investigate the source regions and generation mechanisms for stratospheric inertia–gravity waves accompanying frontogenesis. Attention is focused on run 3 from Part I of this paper (Griffiths and Reeder 1996), here referred to as our 'standard calculation'. Ray-tracing techniques are used to identify the source regions of the waves. Although these results are strictly valid only in the far field away from the frontal region, they suggest that the waves originate in the frontal region. The ray equations also show that the horizontal wavelength decreases exponentially with time because of the action of the basic-state deformation, whereas the vertical wavelength increases linearly with time because of the shear of the basic-state cross-front wind.

It is argued that the cross-front circulation may be viewed, for the most part, as the unsteady source of stratospheric gravity-waves in the standard calculation. This position is supported by a linear forced-wave calculation, in which the forcing is provided by the cross-front circulation from the standard calculation. Comparison of the vertical-motion fields in the stratosphere for the standard and forced-wave calculations show good agreement. An estimate of the nonlinear interaction between the frontal circulation and the wave field is made. It is shown that the back-reaction of the waves on the forcing may be neglected, at least, qualitatively.

The time–space mean of the vertical flux of horizontal momentum in the stratosphere accompanying the modelled waves is approximately $1.4 \times 10^{-4}$ N m$^{-2}$ where the average is taken over $-2000 \text{ km} \leq x \leq 3000 \text{ km}$, $15 \text{ km} \leq z \leq 29 \text{ km}$ and $24 \text{ h} \leq t \leq 48 \text{ h}$.

Keywords: Frontogenesis Gravity-wave drag Inertia–gravity waves Jet streams Ray tracing Stratosphere

1. INTRODUCTION

Although generally weak in the troposphere, gravity-wave activity is a universally observed feature of the whole atmosphere. A number of possibilities have been proposed in the scientific literature to explain the wave-like oscillations accompanying jets and fronts. These possibilities include: convection (e.g. Pierce and Coroniti 1966; Curry and Murty 1974; Fovell et al. 1992; Pfister et al. 1993), shear instability (e.g. Lalas and Einaudi 1976; Mastrantonio et al. 1976; Stobie et al. 1983; Pecnick and Young 1984), symmetric instability (Ciesielski et al. 1989), geostrophic adjustment (e.g. Kaplan and Paine 1977; Ley and Peltier 1978; Van Tuyl and Young 1982; Uccellini et al. 1984; Uccellini and Koch 1987; Fritts and Luo 1992; Luo and Fritts 1993), a piston-like acceleration of the frontal boundary (Tepper 1950) and enhanced nocturnal convergence associated with the cold front (Smith et al. 1995). It is probably fair to say that the most widely studied of these mechanisms has been shear instability.

The present paper explores an alternative description of how some low-frequency waves may be generated, although it is related to the idea of geostrophic adjustment (especially the work of Ley and Peltier (1978) and Synder et al. (1993)). It will be shown that, in essence, the evolution of the vertical circulation accompanying a developing jet/front system is responsible for the generation of low-frequency gravity-waves in a simple two-dimensional numerical model without moist processes. The ideas discussed here are also related to the work by Ford (1994) concerning the non-existence of the slow manifold and emission of inertia–gravity waves by vortical motions, which in turn is closely connected to Lighthill's (1952) theory for the spontaneous emission of sound waves by unstratified turbulent flows.

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There are relatively few previous theoretical or modelling studies of wave generation associated with frontogenesis. Ley and Peltier (1978) were the first to examine the problem theoretically. The basic assumption underpinning their work is that waves are generated by cross-front parcel accelerations neglected in the balanced confluent-deformation frontogenesis theory of Hoskins and Bretherton (1972). More recently, Gall et al. (1988), Garner (1989), and Snyder et al. (1993) have investigated the behaviour of surface fronts as they collapse to form a discontinuity, and have examined the gravity waves generated during this process. These numerical studies are based on primitive-equation extensions of Hoskins and Bretherton’s (1972) confluent-deformation model, although Snyder et al. (1993) also examine a primitive-equation version of Eady’s treatment of baroclinic instability. Several recent studies have dealt with gravity-wave generation associated with the geostrophic adjustment of upper-level jets (Fritts and Luo 1992; Luo and Fritts 1993) but have not explicitly taken into account the evolution of the jet/front system.

In Part I of this paper (Griffiths and Reeder 1996), the generation of inertia–gravity waves in a dry two-dimensional numerical model of frontogenesis was examined. A basic state which included both horizontal deformation and vertical shear was used, and the model was found to predict long-wavelength, low-frequency waves in the stratosphere. It is the source of these waves and the vertical flux of momentum that accompanies them that are examined further in the present paper.

The paper is organized as follows. Section 2 summarizes the governing equations and the model run which forms the basis of the study. In section 3 the source region of the stratospheric inertia–gravity waves is determined using ray tracing. A simple forced-wave calculation is presented in section 4. Conceptually, the approach is to treat the cross-front ageostrophic circulation as an unsteady vortex whose attendant circulation spans the depth of the troposphere. Our hypothesis is that inertia–gravity waves are excited as a result of the unsteady nature of the cross-front ageostrophic circulation. The nonlinear interaction of the generated waves with the forcing (i.e. the back-reaction on the frontal circulation) is assessed in section 5. This is done with the aid of a balanced version of the model which precludes inertia–gravity waves. In section 6, calculations of the vertical flux of horizontal momentum are presented. Finally, our main conclusions are summarized in section 7.

2. THE GOVERNING EQUATIONS AND STANDARD CALCULATION

Keyser and Pecnick (1985) were the first to formulate a two-dimensional anelastic frontogenesis-model incorporating the effects of large-scale confluent-deformation and horizontal shear. Using much the same model equations as Keyser and Pecnick (1985), the generation of inertia–gravity waves during frontogenesis was investigated in Part I. It will prove convenient to rewrite the model equations from Part I with the nonlinear terms involving, inter alia, the frontal circulation transposed to the right-hand side. This results in a set of linear equations with nonlinear forcing terms. The equations governing the evolution of finite-amplitude perturbations from the basic state (Part I Eqs. (7) to (11)) may be rewritten as

\[
\begin{align*}
\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + U_u u + U_v w - f v + \frac{\partial \phi}{\partial x} - \mathcal{D}(u) &= F_u, \\
\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + V_u u + V_v w + f u - \mathcal{D}(v) &= F_v,
\end{align*}
\]
\[
\frac{\partial \phi}{\partial z} - b = 0, \\
(3)
\]
\[
\rho_a \frac{\partial u}{\partial x} + \frac{\partial (\rho_a w)}{\partial z} = 0, \\
(4)
\]
\[
\frac{\partial b}{\partial t} + U \frac{\partial b}{\partial x} + B_v v + N_a^2 w - \mathcal{D}(b) = F_b, \\
(5)
\]

where

\[
F_u = -\frac{1}{\rho_a} \left\{ \frac{\partial}{\partial x} (\rho_a u^2) + \frac{\partial}{\partial z} (\rho_a wu) \right\}, \\
(6)
\]
\[
F_v = -\frac{1}{\rho_a} \left\{ \frac{\partial}{\partial x} (\rho_a uv) + \frac{\partial}{\partial z} (\rho_a wv) \right\}, \\
(7)
\]
and

\[
F_b = -\frac{1}{\rho_a} \left\{ \frac{\partial}{\partial x} (\rho_a ub) + \frac{\partial}{\partial z} (\rho_a wb) \right\}. \\
(8)
\]

Physically, the terms on the right-hand side of \((1), (2) \text{ and } (5)\), (written in flux form in \((6), (7) \text{ and } (8))\), represent the advection of the perturbation velocity and the perturbation buoyancy by the perturbation wind fields. We note that the perturbation fields will, in general, include both the frontal circulation and inertia–gravity waves (among other possibilities), and that, strictly speaking, it is not possible to extract the frontal circulation and the gravity-wave field from the flow uniquely. The basic-state wind \(U = (U, V)\), the basic-state buoyancy \(B\) and the basic-state Brunt–Väisälä frequency \(N_a^2\) are defined by

\[
U(x, z, t) = -\alpha x + \Lambda e^{-\alpha t} Z, \\
(9)
\]
\[
V(y, z, t) = \alpha y - 2\alpha_f \Lambda e^{-\alpha t} Z, \\
(10)
\]
\[
W = 0, \\
(11)
\]
\[
B(y, z, t) = \frac{g}{\theta_0} \theta_s(z) - \Lambda^2 (1 - e^{-2\alpha t}) Z - f \Lambda e^{-\alpha t} y, \\
(12)
\]
\[
\Phi(x, y, z, t) = f \alpha xy - \frac{\alpha^2}{2} (x^2 + y^2) - f \Lambda e^{-\alpha t} yZ - \frac{\Lambda^2}{2} (1 - e^{-2\alpha t}) Z^2 \\
+ \frac{g}{\theta_0} \int_0^z \theta_s(z) dz, \\
(13)
\]
\[
N_a^2 = \frac{\partial B}{\partial z}, \\
(14)
\]

where \(Z = z - 8.2 \text{ km}\). The notation throughout this paper follows that of Part I.

The present work focuses on run 3 from Part I, which will be referred to as the ‘standard calculation’. In this calculation, the model domain is \(7000 \times 70 \text{ km}^2\), with uniform horizontal and vertical grid spacing of 40 km and 175 km respectively. Sponge regions occupy the uppermost 40 km of the model domain and the outermost 1000 km adjacent to the lateral boundaries. The basic-state shear \(\Lambda\) is \(-6 \times 10^{-3}\text{s}^{-1}\) and the basic-state deformation \(\alpha\) is \(10^{-5}\text{s}^{-1}\). The initial conditions comprise opposing upper and lower along-front jets in thermal-wind balance with the potential-temperature field, together with a balanced cross-front ageostrophic circulation. The along-front gradient of potential temperature, defined by \(\Lambda\), is chosen to give cold-air advection at upper levels and warm-air advection at low levels. The along-front wind, potential temperature and tropopause after 36 hours
Figure 1. The standard calculation at $t = 36$ hours. $\alpha = 1 \times 10^{-5}$ s$^{-1}$ and $\Lambda = -6 \times 10^{-3}$ s$^{-1}$. (a) Isotachs of along-front velocity $v$ (thick lines) and isentropes of total potential temperature $\Theta$ (thin lines); contour interval 5 m s$^{-1}$ for $v$, 15 K for $\Theta$. (b) Isotachs of vertical velocity $w$; contour interval 0.6 cm s$^{-1}$. (c) Isotachs of total cross-front velocity $U + u$; contour interval 5 m s$^{-1}$; the shaded areas are where the Richardson number is less than 0.25. Dashed lines denote negative values. The thick horizontally-oriented line represents the tropopause, defined to be the region wherein the potential vorticity lies in the interval $[1 \times 10^{-8}, 2 \times 10^{-8}]$ m$^2$ s$^{-1}$ K kg$^{-1}$. The thick vertically-oriented lines (in (a) and (b)) are rays (see Table 1), and are numbered from left to right.
of model integration are shown in Fig. 1(a). Briefly, the along-front velocity and potential-temperature fields have developed pronounced upper-level and lower-level fronts, together with a deep tropopause-fold. Figure 1(b, c) shows isotachs of the vertical velocity and the total cross-front velocity \((U + u)\). In the troposphere, the frontal circulation is characterized by ascent in the warm air ahead of the front and subsidence behind it. A prominent fan-like pattern of waves is evident in the stratosphere with the phase lines tilting into the wind (cf. Fig. 1(b, c)). A detailed discussion of this is given in Part I.

3. Ray Tracing

In this section, the source and propagation characteristics of the gravity waves generated in the standard calculation are clarified using ray tracing. The method was discussed by Jones (1969) and Lighthill (1978); an application of it relevant to the present study was used by Steffens (1990) to examine the front-generated waves modelled by Gall et al. (1988). Ray tracing has also been employed successfully by Dunkerton (1984) to examine the propagation and refraction of stationary inertia–gravity waves in the winter stratosphere, and by Eckermann (1992) and Marks and Eckermann (1995) in an investigation of the global propagation and refraction of inertia–gravity waves with wavelengths similar to those found in the present study.

The analysis outlined in this section closely follows Jones (1969), the principal differences being the inclusion of rotation here and the choice of a specific background-flow defined by (9) to (14). We proceed in the standard way by linearizing the governing equations (i.e. setting \(F_u = F_v = F_b = 0\) in (1) to (5)) and then assuming travelling wave solutions of the form

\[
r = Re \left[ \hat{r} \exp \left\{ i(kx + mx - \omega t) + \frac{z}{2H_s} \right\} \right].
\]

Here, \(r\) is any perturbation variable, \(\omega\) is the frequency as measured by an observer in the coordinate system of the model and \(H_s = -\rho_s/(\partial \rho_s/\partial z)\) is the density scale-height. In the analysis that follows, it is assumed that \(H_s\) is a constant, although in fact it varies from 5 km at the tropopause to 8 km at the top of the model domain. It follows that

\[
(-i\omega_s + U_x)\hat{u} - f\hat{v} + U_z\hat{w} + ik\hat{\phi} = 0,
\]

\[
f\hat{u} + (-i\omega_s + V_x)\hat{v} + V_z\hat{w} = 0,
\]

\[
B_y\hat{v} + N_a^2\hat{w} - i\omega_s\hat{b} = 0,
\]

\[
(im + \frac{1}{2H_s})\hat{\phi} - \hat{b} = 0,
\]

\[
(ik\hat{u} + (im - \frac{1}{2H_s})\hat{w} = 0,
\]

where \(\omega_s \equiv \omega - kU\) is the intrinsic frequency of the wave, that being the frequency measured by an observer moving with the local background flow speed \(U\). The hydrostatic dispersion relation derived from (15) to (19) takes the usual form

\[
\omega = U k + \omega_s = Uk + \left[ f^2 + \frac{N_a^2 k^2}{m^2 + 1/4H_s^2} \right]^{1/2}.
\]
provided that

\[ |U_z|, |V_z| \ll |\omega_z|, \]  

\[ |U_z| \ll \left| \frac{\omega_z}{k} \left( \frac{im - \frac{1}{2H_z}}{} \right) \right|, \]  

\[ |V_z| \ll \left| \frac{f}{k} \left( \frac{im - \frac{1}{2H_z}}{\omega_z} \right) \right|, \]  

and

\[ |B_z| \ll \left| \frac{N_z^2k\omega_z}{f \left( \frac{im - \frac{1}{2H_z}}{\omega_z} \right)} \right|. \]  

Note that the inequalities (22) and (24) are not independent since \( U_z = -B_z/f \). Inequalities (21) to (24) have been checked a posteriori and found to be very well satisfied.

Equations (15) to (19) describe the propagation and refraction of inertia–gravity waves by the basic state only; in other words, the effect of the front and its attendant circulation is ignored. This approximation leads to very simple expressions for the equations governing the rays which are easily integrated numerically and which do not require a knowledge of the model grid-point fields. While the results are strictly valid only in the far field away from the frontal region, Steffens (1990) has shown that the wind and temperature distributions associated with the front itself have only a minor effect on the rays. Moreover, the focus of the present paper is on waves in the stratosphere, a region in which the far-field approximation is reasonably well satisfied.

The equations governing the propagation of a wave packet may be shown to be

\[ \frac{dx}{dt} = \frac{\partial \omega}{\partial \omega}, \]  

\[ \frac{dz}{dt} = \frac{\partial \omega}{\partial \omega}, \]  

\[ \frac{dk}{dt} = -\frac{\partial \omega}{\partial x}, \]  

\[ \frac{dm}{dt} = -\frac{\partial \omega}{\partial k}, \]  

where \( \frac{d}{dt} \) denotes the rate of change following the group motion (see, for example, Lighthill 1978). Physically, (25) and (26) state that the velocity of the wave packet is equal to the group velocity relative to the ground, \( c_g = \frac{\partial \omega}{\partial k} \), while (27) and (28) describe the changes in wave number caused by refraction.

Using the expressions for \( \omega \) (20) and \( U \) (9), (25) to (28) may be expressed as

\[ \frac{dx}{dt} = -\alpha x + \Lambda Z e^{-\alpha t} + \frac{N_z^2k}{(m^2 + \frac{1}{4H_z^2})^{\frac{1}{2}}} \left[ f^2(m^2 + \frac{1}{4H_z^2}) + N_z^2k^2 \right]^{\frac{1}{2}}, \]  

\[ \frac{dz}{dt} = \frac{-N_z^2k^2m}{\left[ f^2(m^2 + \frac{1}{4H_z^2}) + N_z^2k^2 \right]^{\frac{1}{2}}} (m^2 + \frac{1}{4H_z^2})^{\frac{1}{2}}, \]  

\[ k = k_0 e^{\alpha(t-t_0)}, \]  

\[ m = m_0 - \Lambda k_0 e^{-\alpha t}(t-t_0), \]
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TABLE 1. HORIZONTAL AND VERTICAL WAVELENGTHS

<table>
<thead>
<tr>
<th>Ray</th>
<th>$t_0$ (hours)</th>
<th>$\lambda_x (t_0)$ (km)</th>
<th>$\lambda_x (36$ hours$)$ (km)</th>
<th>$\lambda_z (t_0)$ (km)</th>
<th>$\lambda_z (36$ hours$)$ (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>1000</td>
<td>470</td>
<td>2.7</td>
<td>9.5</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>1000</td>
<td>470</td>
<td>2.8</td>
<td>11.5</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>1000</td>
<td>470</td>
<td>2.9</td>
<td>12.0</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>1000</td>
<td>470</td>
<td>2.9</td>
<td>12.0</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>1064</td>
<td>500</td>
<td>2.8</td>
<td>9.5</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>1064</td>
<td>500</td>
<td>2.9</td>
<td>11.0</td>
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</tbody>
</table>

Horizontal and vertical wavelengths, $\lambda_x = \frac{2\pi}{|k|}$ and $\lambda_z = \frac{2\pi}{|m|}$ respectively, at the beginning ($t = t_0$) and at the end point ($t = 36$ hours) of each ray depicted in Fig. 1. Rays in Fig. 1 are numbered from left to right.

where $k_0$ and $m_0$ are the horizontal and vertical wavenumbers at a given starting time, $t_0$. Equation (31) indicates that the horizontal wave-number increases exponentially with time along a ray because of the action of the deformation component of the basic-state field of wind. Setting $\alpha = 0$ results in a constant horizontal wave-number following the wave motion. In a similar way, the vertical shear of the cross-front geostrophic wind linearly decreases the magnitude of vertical wave-number by tilting the phase lines towards the vertical; when $\Lambda = 0$ the vertical wave-number is constant along a ray (Eq. (32)). It can be shown that the frequency, as measured along a ray, changes according to

$$\frac{d\omega}{dz} = -\alpha \Lambda (z(t) - 8.2) e^{-\alpha t} k(t)$$

where $z(t)$ is expressed in km. The frequency is not constant because the background flow is time dependent. Although this equation is redundant, it provides a useful check on the integration of (29) to (32); in the calculations shown, it is satisfied to within a few percent. Note that $1/H_a \ll m$ in the calculations presented.

The purpose of the following ray-tracing calculations is to identify the source of the stratospheric inertia–gravity waves generated in the standard run and evident in Fig. 1(b). (See also Part I, Figs. 6(c), 8 and 9). However, as indicated previously, these calculations do not use the numerical solution explicitly, except to estimate the wavelengths of the stratospheric waves as an initial condition for the ray-tracing calculation. Estimates of the horizontal and vertical wavelengths at 36 hours were made graphically from Fig. 1(b), and the ray equations ((29) to (32)) integrated numerically backwards in time using fourth-order Runge–Kutta. In these calculations, $m < 0$ as is required for upward energy radiation (see (30)). Six representative rays are plotted in Figs. 1(a) and (b). The most prominent feature of these figures is that the rays appear to have their origins in the frontal region. Table 1 lists the horizontal wavelength $\lambda_x$ and the vertical wavelength $\lambda_z$ at the beginning ($t = t_0$) and at the end points ($t = 36$ hours) of each ray. In each case, the horizontal wavelength is reduced and the vertical wavelength is increased along the ray, in accordance with (31) and (32). The wavelengths in the horizontal and vertical at 36 hours range between 470 km and 500 km, and 9.5 km and 12 km respectively.

The frequencies and intrinsic frequencies at the beginning and at the end of each ray are listed in Table 2, and the components of the group velocity ($c_{gr}$, $c_{gr}$) at these points are given in Table 3. As indicated earlier, the waves have frequencies relatively close to $f$. At this point, it is worth recalling from Part I that a transformation of the vertical coordinate has been made ($Z = z - 8.2$ km) so that the jet/front system remains almost stationary in the coordinate system of the model.
TABLE 2. Frequency and intrinsic frequency

<table>
<thead>
<tr>
<th>Ray</th>
<th>$\omega(n_0)$ ($\times 10^{-4}s^{-1}$)</th>
<th>$\omega$ (36 hours) ($\times 10^{-4}s^{-1}$)</th>
<th>$\omega_\ast(t_0)$ ($\times 10^{-4}s^{-1}$)</th>
<th>$\omega_\ast$ (36 hours) ($\times 10^{-4}s^{-1}$)</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1.71</td>
<td>2.21</td>
<td>1.27</td>
<td>5.94</td>
</tr>
<tr>
<td>2</td>
<td>0.82</td>
<td>1.88</td>
<td>1.30</td>
<td>7.09</td>
</tr>
<tr>
<td>3</td>
<td>1.07</td>
<td>1.98</td>
<td>1.30</td>
<td>7.38</td>
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<td>1.78</td>
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<td>1.72</td>
<td>1.26</td>
<td>5.59</td>
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<td>0.39</td>
<td>1.28</td>
<td>1.28</td>
<td>6.40</td>
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</table>

Frequency $\omega$ and intrinsic frequency $\omega_\ast$ at the beginning ($t = t_0$) and end point ($t = 36$ hours) of each ray depicted in Fig. 1. Rays in Fig. 1 are numbered from left to right.

TABLE 3. Horizontal and vertical group velocities

<table>
<thead>
<tr>
<th>Ray</th>
<th>Time of starting (hours)</th>
<th>$c_{gs}$ ($t_0$) (m s$^{-1}$)</th>
<th>$c_{gs}$ (36 hours) (m s$^{-1}$)</th>
<th>$c_{gs}$ ($t_0$) (m s$^{-1}$)</th>
<th>$c_{gs}$ (36 hours) (m s$^{-1}$)</th>
</tr>
</thead>
<tbody>
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<td>15.31</td>
<td>2.10 $\times 10^{-2}$</td>
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<tr>
<td>2</td>
<td>20</td>
<td>0.853</td>
<td>13.04</td>
<td>2.39 $\times 10^{-2}$</td>
<td>1.26</td>
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<tr>
<td>3</td>
<td>15</td>
<td>4.90</td>
<td>13.82</td>
<td>2.46 $\times 10^{-2}$</td>
<td>1.37</td>
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<td>12.32</td>
<td>2.45 $\times 10^{-2}$</td>
<td>1.37</td>
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<td>5</td>
<td>15</td>
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<td>2.13 $\times 10^{-2}$</td>
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<td>8.96</td>
<td>2.37 $\times 10^{-2}$</td>
<td>1.08</td>
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</table>

Horizontal and vertical group velocities, $c_{gs}$ and $c_{gs}$, respectively, at the beginning ($t_0$) and at the end point ($t = 36$ hours) of each ray depicted in Fig. 1. Rays in Fig. 1 are numbered from left to right.

TABLE 4. Horizontal phase velocities

<table>
<thead>
<tr>
<th>Ray</th>
<th>Time of starting (hours)</th>
<th>$c_{ph}$ ($t_0$) (m s$^{-1}$)</th>
<th>$c_{ph}$ (36 hours) (m s$^{-1}$)</th>
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<tr>
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<td>6.60</td>
<td>10.19</td>
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</table>

Horizontal phase velocities $c_{ph}$ at the beginning ($t_0$) and at the end point ($t = 36$ hours) of each ray depicted in Fig. 1. Rays in Fig. 1 are numbered from left to right.

The horizontal phase velocities based on Tables 1 and 2, and hence relative to the jet/front system, are shown in Table 4, and lie between 6 and 28 m s$^{-1}$. Thus, unlike lee waves, these waves are propagating disturbances. For this reason, we do not believe that the waves are generated by shear flow over the undulating tropopause.

4. Wave generation mechanism

In this section, it is argued that the stratospheric inertia–gravity waves generated in the standard calculation, and analysed in the previous sections, are caused by rapid evolution of the frontal circulation. The theoretical basis for this mechanism is based on work by
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Ford (1994) and is closely related to Lighthill’s theory for the aerodynamic generation of sound (Lighthill 1952).

In Part I it was shown that the most intense upper-level and surface fronts developed in run 2, whereas the standard calculation (run 3 from Part I) showed the greatest wave-activity. It is worth recalling from Part I that the only difference in these model runs at the initial instant is the sign of the along-front gradient of potential temperature, i.e. the sign of $\Lambda$; all other parameters are the same*. It is significant that the frontogenetic forcing, and hence the cross-front circulation, evolves much more rapidly in the standard calculation than in run 2. For example, in the standard calculation the frontogenesis attains a maximum value after about 19 hours of integration and thereafter declines (see Part I, Fig. 5). On the other hand, run 2 takes almost 42 hours to attain its maximum frontogenetic forcing (which is about 1.8 times the maximum reached in the standard calculation).

An alternative way to state this is that if the Rossby number is defined as $Ro = (\partial v/\partial x)/f$, then the maximum value attained at low-levels in the standard calculation is 6.5 whereas the maximum low-level value in run 2 is 6.7. At upper levels, $Ro$ attains maximum values of 3.7 and 3.9 in the standard calculation and run 2 respectively. Consequently, it does not seem possible to use $Ro$ so-defined, to distinguish between occasions in which the wave activity is high (e.g. the standard calculation) and those in which little wave activity results (e.g. run 2). It does appear, however, that the degree of gravity-wave activity is related to the Rossby number when it is defined as the ratio of the magnitude of the maximum cross-front acceleration to the maximum Coriolis acceleration in the neighbourhood of the front, i.e. $Ro_L = \max(v_{xy})/\max(v)$. Figure 2 shows the time-series of $Ro_L$ for the three model runs described in Part I. The key point to note is that the rapidly evolving cross-front circulation in the standard calculation is characterized by relatively large cross-front parcel accelerations, and hence large values of $Ro_L$. At both upper and lower levels, $Ro_L$ in the standard calculation intensifies, reaches a maximum at about 12 hours, and then weakens. In contrast, at low-levels, $Ro_L$ in run 2 is large initially and weakens throughout the remainder of the integration.

Equations (1) to (5) can be combined to form a single complicated equation for $w$:

$$\begin{align*}
\frac{D^4}{Dt^4} \left\{ \frac{w_{zz}}{H_a} - \left( \frac{w}{H_a} \right)_z \right\} + 4U_x \frac{D^3}{Dt^3} \left\{ \frac{w_{zz}}{H_a} - \left( \frac{w}{H_a} \right)_z \right\} + \frac{1}{f} \frac{D^3}{Dt^3} B_y \left( \frac{w}{H_a} \right)_z \\
+ \frac{D^2}{Dt^2} \left\{ N_a^2 w_{xx} - f V_z w_{xz} + \left( f^2 + 4U_y^2 \right) \left( \frac{w}{H_a} \right)_z \right\} + \frac{2}{f} U_x B_y \left( \frac{w}{H_a} \right)_x \\
- 2 \frac{D}{Dt} \left\{ B_x V_x w_{xx} + f (U_x V_z - B_z) w_{xz} - f^2 U_x w_{zz} + \frac{f B_y}{H_a} \left( \frac{w}{H_a} \right)_x + f^2 U_x \left( \frac{w}{H_a} \right)_z \right\} \\
- 2 U_x B_y \left\{ V_z w_{xx} - f w_{xz} + f \left( \frac{w}{H_a} \right)_x \right\} = -\mathcal{F} \tag{33}
\end{align*}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x},$$

and

$$\mathcal{F} = \left( \frac{D}{Dt} + 2U_x \right) \left( \frac{D^2}{Dt^2} F_{uxz} + f \frac{D}{Dt} F_{uxz} - f U_z F_{usz} \right) - \frac{D^2}{Dt^2} F_{bxt}.$$

* A change in the sign of $\Lambda$ does, however, imply a change in the cross-front ageostrophic circulation derived from the Sawyer–Eliassen equation (Part I Eq. (13)).
Although it was not necessary to do so, for simplicity the diffusion terms have been neglected. Apart from this, no further approximation has been made in the derivation of (33).

Suppose for the moment that the products of perturbation quantities are neglected. From (6) to (8), $F_u = F_v = F_b = 0$, and hence $\mathcal{F} = 0$, whereupon (33) is reduced to a linear homogeneous equation for $w$. The resulting equation, together with appropriate boundary-conditions, provides an account of linear two-dimensional disturbances in a background environment defined by (9) to (14). Among the phenomena it can describe are inertia–gravity waves and baroclinic instability. Furthermore, setting $\alpha = \Lambda = 0$ (with $\mathcal{F} = 0$)
yields
\[
\left( \frac{\partial^2}{\partial t^2} + f^2 \right) \left\{ w_{zz} - \left( \frac{w}{H_0} \right)_z \right\} + N'_a^2 w_{xx} = 0,
\]
which is the usual inertia–gravity wave equation, for which (20) is the dispersion relation.

Returning to (33), it appears that the generation of inertia–gravity waves may be associated with the right-hand-side term $\mathcal{F}$, which involves temporal and spatial derivatives of $F_u$, $F_v$ and $F_b$. Accordingly, we choose to interpret (33) as a linear partial differential (wave) equation for $w$ with a nonlinear forcing term on the right-hand side. Such an interpretation only makes sense if the forcing term $\mathcal{F}$ is dominated by the frontal circulation and the back-reaction of the emitted waves on the frontal circulation is negligible. These ideas will be justified more carefully in the remainder of the paper.

Note that the forcing term $\mathcal{F}$ is essentially confined to the troposphere. In the frontal region, the perturbation variables $u$, $v$ and $w$ are relatively large, on account of the frontal circulation, and hence $F_u$, $F_v$ and $F_b$, which are quadratic in these quantities, are large also. Conversely, away from the frontal region where $u$, $v$ and $w$ are small, the forcing is negligible. Moreover, $\mathcal{F}$ presumably forces a broad spectrum of modes at frequencies both higher and lower than $f$.

We present now a numerical solution of (33) that helps in understanding the wave-generation mechanism in the standard calculation, and which will be referred to as the ‘forced-wave calculation’. In the calculation that follows, (33) is not solved explicitly; instead we choose an equivalent approach and integrate (1) to (5). These equations are integrated numerically from an initially unperturbed state, i.e. a state in which $u = v = w = b = 0$. The tropopause is initially horizontal and set at a height of about 8 km. Thus, the front and its attendant circulation are explicitly excluded from the initial conditions. However, $F_u$, $F_v$ and $F_b$ (and hence $\mathcal{F}$) are determined from the standard calculation, the idea being to test whether or not the generation of inertia–gravity waves can be attributed to the cross-front circulation as characterized by $\mathcal{F}$. In the calculations presented here $\Lambda = -6 \times 10^{-3} \text{s}^{-1}$ as in the standard calculation. Isotachs of $w$ for the forced-wave calculation after 36 hours of integration are shown in Fig. 3. Comparison with the 36 hour $w$ pattern from the standard calculation (Fig. 1b) reveals striking similarities in the stratosphere. Both calculations exhibit prominent fan-like distributions of waves, the horizontal and vertical
wavelengths agreeing to within the limits of graphical analysis. The principal discrepancy in the stratosphere occurs in the region $1000 \text{ km} \leq x \leq 2000 \text{ km}$, $25 \text{ km} \leq z \leq 30 \text{ km}$. Here, the amplitude in the forced-wave calculation is too large. Outside this area, the stratospheric wave amplitudes show reasonably good agreement. Moreover, the positions of the amplitude maxima for each wave crest are very close. Indeed, the vertical-velocity fields in the stratosphere are in closer agreement than a direct comparison of Fig. 1(b) with Fig. 3 suggests. The reason is that the frontal circulation extends, albeit weakly, into the stratosphere. When this is taken into account, the patterns of vertical motion in the stratosphere show even greater similarity. In the troposphere, close to the region of the forcing, the vertical-velocity fields are highly dissimilar. This is a consequence of the different initial conditions used in each calculation.\footnote{Of course, repeating the forced-wave calculation with the initial conditions used in the standard calculation gives precisely the same solution as the standard calculation.} Despite the pronounced differences in the troposphere between the standard calculation and the forced-wave calculation, the fields of waves in the stratosphere have a great deal in common, suggesting that $\mathcal{F}$ can correctly be regarded as the wave forcing. Moreover, these similarities suggest that the front itself does not significantly refract the inertia–gravity waves as they propagate away from the source region.

Before concluding this section, we consider briefly the possibility that the waves are a result of dynamical instability. In particular, two possibilities for wave generation are considered, viz. shear instability and symmetric instability. A necessary condition for the waves to be generated by shear is that the Richardson number, defined by $Ri = \left(\partial b/\partial z\right)/\left[\partial(\mathbf{u} + \mathbf{U})/\partial z\right]^2$, be less than 0.25 and that a critical level (i.e. $(U + u) - c = 0$) be present in the same region. Contours of total cross-front velocity $U + u$ after 36 hours for standard calculation are shown in Fig. 1(c); those areas where $Ri < 0.25$ are shaded. Note that the horizontal phase speed of the waves varies from 6 to 28 m s$^{-1}$ (Table 4). Therefore, a critical level is possible only to the left of the frontal area away from the region of small $Ri$, making it unlikely that shear instability is the generation mechanism. Moreover, theories of shear instability generally predict that the maximum growth rates occur at much smaller wavelengths (e.g. Lalas and Einaudi 1976; Mastrantonio et al. 1976).

A necessary condition for the onset of (dry) symmetric instability is that the potential vorticity be less than zero. The most unstable modes are manifest as rolls oriented parallel to the vertical shear of the geostrophic wind within the unstable layer, and with the transverse motion approximately parallel to the isentropes. Because of diffusion and numerical truncation-error, potential vorticity is negative in a small region near the surface front and in a thin layer along the tropopause to the right of $x = 1000 \text{ km}$. In order to assess whether or not the waves are a result of symmetric instability, the 36-hour fields of along-front wind and potential temperature were used as an initial condition for an integration with no frontogenesis ($\alpha = \Lambda = 0$) and with no ageostrophic circulation ($u = v = 0$). If symmetric instability were a factor, the waves would quickly redevelop; however, they do not.

5. Balanced model.

The aim of the present section is to estimate the (nonlinear) back-reaction of the waves on the forcing. This is done by introducing a balanced model, the solutions from which are used to approximate $\mathcal{F}$. Since gravity waves are precluded from the balanced model, this estimate of $\mathcal{F}$ will be unaffected by the back-reaction.

Scale analysis, such as that used by Hoskins and Bretherton (1972), suggests that the along-front wind is almost geostrophic, and that its vertical shear balances the gradient of
cross-front temperature. Such a simplification, when applied to the governing equations, is commonly called the Geostrophic Momentum (GM) approximation (Hoskins 1975). An analysis by Reeder and Keyser (1988) showed that, provided $\alpha/f \ll O(1)$ and that $\Lambda \exp(-\alpha l)/N_a^2 \ll O(1)$, the GM approximation could be applied to the anelastic perturbation equations ((1) to (5)), yielding

$$ v_{gm} = \frac{1}{f} \frac{\partial \phi_{gm}}{\partial x}, \quad (34) $$

$$ \frac{\partial v_{gm}}{\partial t} + (U + u_{gm}) \frac{\partial v_{gm}}{\partial x} + w_{gm} \frac{\partial v_{gm}}{\partial z} + V_s v_{gm} + V_s w_{gm} + f u_{gm} = \phi_{gm}, \quad (35) $$

$$ \frac{\partial \phi_{gm}}{\partial z} = b_{gm}, \quad (36) $$

$$ \rho_n \frac{\partial u_{gm}}{\partial x} + \frac{\partial (\rho_n w_{gm})}{\partial z} = 0, \quad (37) $$

and

$$ \frac{\partial b_{gm}}{\partial t} + (U + u_{gm}) \frac{\partial b_{gm}}{\partial x} + w_{gm} \frac{\partial b_{gm}}{\partial z} + B_s v_{gm} + N_a^2 w_{gm} = \phi_{gm}, \quad (38) $$

In the present paper, (34) to (38) are referred to as the GM equations. The subscript 'gm' has been added to each of the independent variables in order to distinguish solutions of the GM model from those of the anelastic model. A key feature of these equations is that while the along-front wind is assumed to be strictly geostrophic (Eq. (34)), advection by the ageostrophic wind in the $x, z$ plane ($u_{gm}, w_{gm}$) is retained. An important consequence of the GM approximation is that, unlike the anelastic model, (34) to (38) filter gravity-wave solutions. Of course, the two models will exhibit differences in solution which have nothing to do with gravity-wave radiation, but instead reflect the approximations underpinning the GM model. Some of these differences have been discussed by Reeder and Keyser (1988) and Garner (1989).

The results from an integration of the GM model analogous to the standard calculation is shown in Fig. 4. An outline of numerical methods used to solve the GM equations can be found in Reeder and Keyser (1988). Comparing Fig. 4(a) with Fig. 1(a), it can be seen that the standard calculation and its GM equivalent predict along-front wind and total potential-temperature fields showing close agreement. The most prominent differences between the solutions arise in the vertical-velocity fields (cf. Figs. 4(b) and 1(b)); in particular the GM solution fails to reproduce the pattern of stratospheric waves (as expected) and the secondary updraught on the warm side of the front in the middle troposphere (the subject of continuing research).

Figure 5 shows the difference between the standard and GM calculations at 36 hours. One of the most striking features is how closely the fields $u - u_{gm}$ and $w - w_{gm}$ (shown in Fig. 5(a, b)) match the fields of $u - u_{SE}$ and $w - w_{SE}$ (shown in Part I Fig. 9). Here, $(u - u_{SE}, w - w_{SE})$ is the velocity field in the $x, z$ plane derived by subtracting the velocity calculated from the Sawyer–Eliassen equation from the velocity field predicted by the full model. Thus, it appears that diagnosing the balanced part of the circulation at a particular time using the Sawyer–Eliassen equation (Eq. (13) in Part I) delivers a very good approximation to the circulation predicted by explicitly integrating the GM equations. Also shown in Fig. 5 is $\theta - \theta_{gm}$, the unbalanced part of the potential temperature.

From (15) to (19), it can be shown that, inter alia,

$$ \hat{u} = - \left( \frac{m}{k} \right) \hat{w} \quad \text{and} \quad \hat{w} = i \left( \frac{\omega}{N^2} \right) \hat{b}, \quad (39) $$
where the density scale-height has been neglected for simplicity. These two equations describe the phase relationship between the velocity components and buoyancy for linear gravity-waves. In particular, the buoyancy is in quadrature with the vertical velocity, and the horizontal (cross-front) and vertical velocities are either in phase or out of phase by 180° depending on the sign of \( m/k \).

Comparison of each of the fields displayed in Fig. 5 shows that, in the stratosphere, \( u - u_{gm} \) and \( w - w_{gm} \) are in phase, and that \( \theta - \theta_{gm} \) is in quadrature with \( w - w_{gm} \). These observations agree with (39) since \( k > 0 \) and \( m < 0 \) for a gravity wave which is propagating upward and to the right (see (29) and (30)). Although the patterns of \( u - u_{gm}, w - w_{gm} \) and \( \theta - \theta_{gm} \) are more complex in the troposphere, the phase relations defined by (39) still hold if we take \( sgn(m/k) > 0 \).

The forced-wave calculation described in section 4 is repeated with one important modification, viz. the forcing terms \( F_u, F_v \) and \( F_h \) are approximated by their GM equivalents \( F_{u_{gm}}, F_{v_{gm}} \) and \( F_{h_{gm}} \) respectively. In other words, (1) to (5) are numerically integrated
Figure 5. Isotachs of the differences between the standard and GM calculations at $t = 36$ hours. (a) Cross-front velocity $u - u_{gm}$, contour interval 1 m s$^{-1}$; (b) vertical velocity $w - w_{gm}$, contour interval 0.6 cm s$^{-1}$; and (c) potential temperature $\theta - \theta_{gm}$, contour interval 0.5 K. Dashed lines denote negative values.
from an unperturbed initial state, with the right-hand side replaced by

\[ F_{w_{gm}} = -\frac{1}{\rho_a} \left\{ \frac{\partial}{\partial x} \left( \rho_a u_{gm}^2 \right) + \frac{\partial}{\partial z} \left( \rho_a w_{gm} u_{gm} \right) \right\} \]

\[ F_{v_{gm}} = -\frac{1}{\rho_a} \left\{ \frac{\partial}{\partial x} \left( \rho_a u_{gm} v_{gm} \right) + \frac{\partial}{\partial z} \left( \rho_a w_{gm} v_{gm} \right) \right\} \]

and

\[ F_{b_{gm}} = -\frac{1}{\rho_a} \left\{ \frac{\partial}{\partial x} \left( \rho_a u_{gm} b_{gm} \right) + \frac{\partial}{\partial z} \left( \rho_a w_{gm} b_{gm} \right) \right\} \]

Figure 6 shows the vertical-velocity field after 36 hours of integration. Comparison of Figs. 3 and 6 shows qualitative agreement, both in the stratosphere and in the troposphere. This suggests that the cross-frontal circulation is the dominant contribution to \( F_x \), \( F_y \) and \( F_b \), and that, qualitatively at least, the back-reaction of the waves on the forcing is small. However, the precise interpretation is uncertain because, as noted earlier, differences exist between the standard calculation and the GM solution which are related to the GM approximation itself and have little to do with gravity waves. Interestingly, the results of this section suggest that it may be possible to infer the associated far-field gravity-wave solution from balanced models.

6. MOMENTUM FLUX CALCULATIONS

In general, radiating gravity-waves induce a vertical flux of horizontal momentum. This aspect of their dynamics has important implications for the global circulation, particularly in the stratosphere and mesosphere. The momentum of such tropospheric features as jets and fronts may be redistributed by gravity waves, and may be carried far from its source before being transferred to the mean flow. The flux of momentum associated with gravity waves is particularly important in the summer hemisphere since the vertical propagation of planetary waves is greatly affected at this time by the reversal of the zonal-mean winds.

Problems concerning wave/mean-flow interactions are often handled by separating the flow into mean and departure quantities. Commonly, the mean flow is defined as the
horizontal average of the flow and the ‘waves’ are defined as departures from the mean. In the present section this paradigm is followed, and estimates of the generalized momentum-flux, or Eliassen–Palm flux (Andrews and McIntyre 1976), for the waves generated in the standard calculation are presented.

In this study, mean quantities are defined as,

$$\langle r \rangle = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} r \, dx,$$  \hspace{1cm} (40)

where $r$ is any given quantity, and $[x_1, x_2]$ defines the averaging interval. Consequently, departures from the mean, or ‘waves’, are defined as

$$r' = r - \langle r \rangle.$$  \hspace{1cm} (41)

We note that the decomposition of the flow defined by (40) and (41) is arbitrary and depends on the choice of $x_1$ and $x_2$.

Rewriting (1) in flux form and applying the averaging operator, (40) yields

$$\frac{\partial \langle u \rangle}{\partial t} + \langle w \rangle \frac{\partial \langle u + U \rangle}{\partial z} + V_s \langle u \rangle - f \langle v_s \rangle = -\frac{1}{\rho_s} \frac{\partial E}{\partial z} + A + R + \langle D \rangle.$$  \hspace{1cm} (42)

Here

$$\langle v_s \rangle = \langle v \rangle - \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \frac{\rho_s \langle u' \theta' \rangle}{\langle \theta \rangle} \right)$$

is the mean along-front residual circulation, and

$$E = \rho_s \langle (u' + U')w' \rangle - f \rho_s \frac{\langle u' \theta' \rangle}{\langle \theta \rangle}$$

is the vertical component of the generalized momentum flux (i.e. the negative of the Eliassen–Palm flux). Moreover,

$$A = -\frac{1}{x_2 - x_1} \left[ \phi \right]_{x_1}^{x_2}$$

is the average pressure gradient, proportional to the difference in pressure at the left and right end-points of the averaging interval, and

$$R = -\frac{1}{x_2 - x_1} \left[ \langle u^2 + 2uU - \langle u + U \rangle u \rangle \right]_{x_1}^{x_2}$$

represents the terms in (42) describing fluxes at the end points of the averaging interval. In the calculations that follow, the averaging interval is taken to be $[-2000 \text{ km}, 3000 \text{ km}]$, i.e. the entire width of the model domain excluding the lateral sponge-regions.

The mean values of the generalized momentum-flux for the standard run and for the GM run, $E_{z}^G$ and $E_{gm}^G$, are $1.14 \times 10^{-3}$ and $9.6 \times 10^{-4} \text{ kg m}^{-1} \text{s}^{-2}$, respectively. The overbar ($\overline{\cdot}$) represents an average taken over the time interval 24 to 48 hours and over the height range 15 to 29 km. This height range was chosen to lie below the sponge, but well above the tropopause. The most significant point to note here is that $E_{z}^G$ and $E_{gm}^G$ are nearly equal even though the GM solution explicitly filters gravity waves. The ‘waves’ that contribute to $E_{gm}^G$ are not inertia–gravity waves but are simply departures from the mean, attributable to the horizontal structure in the basic state and the penetration of the frontal
TABLE 5. AVERAGE VALUES OF THE TERMS IN EQUATION (42)

<table>
<thead>
<tr>
<th></th>
<th>( \frac{\partial u}{\partial t} )</th>
<th>( \frac{\partial (u^2 + E^{cl})}{\partial z} )</th>
<th>( V_{1} (\bar{a}^{cl}) )</th>
<th>( f (\bar{v}_{a}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>( 9.5 \times 10^{-7} )</td>
<td>( 1.4 \times 10^{-6} )</td>
<td>( -1.3 \times 10^{-6} )</td>
<td>( -1.3 \times 10^{-5} )</td>
</tr>
<tr>
<td>GM</td>
<td>( -9.6 \times 10^{-7} )</td>
<td>( 1.8 \times 10^{-6} )</td>
<td>( -1.3 \times 10^{-6} )</td>
<td>( -1.3 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

Average values of the terms in Eq. (42) for the standard run and GM runs. Each of the terms is averaged over the height range \( 15 \leq z \leq 30 \) km and the time interval \( 24 \leq t \leq 48 \) hours.

circulation, albeit very weakly, into the stratosphere. For this reason, the magnitude of \( \bar{E}^{cl} \) (and \( \bar{E}_{gm} \)) is sensitive to the choice of averaging interval \([x_1, x_2] \). These results highlight the danger in identifying inertia–gravity waves as departures from an arbitrary mean state, and sound a note of caution when interpreting observations.

An estimate of the generalized momentum-flux associated with the inertia–gravity waves identified in the standard calculation is \( E_{wave}^{cl} \equiv \bar{E}^{cl} - \bar{E}_{gm} = 1.8 \times 10^{-4} \) kg m\(^{-1}\) s\(^{-2}\). This estimate compares favourably with the 36-hour value of \( \rho_s (u - u_{SE})(w - w_{SE}) \) = \( 1.3 \times 10^{-4} \) kg m\(^{-1}\) s\(^{-2}\) found in Part I. The overbar \((\bar{ }) \) represents an average over the domain \(-2000 \leq x \leq 3000 \) km and \( 15 \leq z \leq 29 \) km. Note that the latter calculation isolated the inertia–gravity wave signal in the stratosphere by first defining the frontal circulation by means of the Sawyer–Eliassen equation.

Table 5 lists the mean values of each term contributing to the momentum budget, given by (42), for both the standard and GM runs. The table shows that in each case the dominant balance is between the divergence of the generalized momentum-flux and the difference in pressure at the lateral boundaries. Figure 7 depicts the generalized momentum-flux attributable to inertia–gravity waves \( \rho_s^{-1} \partial (\bar{E}^{cl} - \bar{E}_{gm}) / \partial z \) and the unbalanced pressure difference at the lateral boundaries \( \bar{A}^{cl} - \bar{A}_{gm} \), showing that these two terms are essentially mirror images of one another. The overbar \((\bar{ }) \) represents an average in time from 24 to 48 hours. Note that the acceleration of the mean flow is two orders of magnitude smaller than either of these two terms. This result has implications for the parametrization of inertia–gravity waves.

The small momentum-flux accompanying the modelled inertia–gravity waves is in accord with observations. While such waves contribute significantly to the variance of atmospheric wind and temperature fields, and may have important roles in the transport of chemical tracers, they appear to play only a minor part in the vertical transport of horizontal momentum; higher-frequency short-wavelength waves are believed to be the principal contributors to the vertical flux of horizontal momentum.

7. SUMMARY AND CONCLUSION

A dry, two-dimensional, anelastic numerical model has been used to investigate the origin of stratospheric inertia–gravity waves associated with upper-level jets and fronts. The analysis has focused on a particular model run described in Part I (run 3 in that paper). For convenience, we have referred to this model run as our ‘standard calculation’.
Equations for the propagation of a wave packet through the model basic-state stratosphere have been derived, i.e. ray equations, and have been used to identify the source region of the waves and to investigate their propagation characteristics. In essence, the derivation of the ray equations assumed that the phase of the wave varies more rapidly in time and space than the background flow. In the ray calculations, the horizontal and vertical wavelengths at $t = 36$ hours were estimated graphically from the standard calculation, and were used as initial conditions when integrating the ray equations backwards in time to locate the source region of the waves. Only the model basic-state wind and temperature fields were used in these calculations. Despite the limitations of the analysis, the ray-tracing technique provided useful insights into the source of the waves, suggesting that the inertia–gravity waves had their origins in the frontal region.

As first shown by Jones (1969), the basic-state deformation acts to decrease the horizontal wavelength along a ray exponentially. For the rays presented here, horizontal wavelengths decreased from around 1000 km to around 500 km in the time interval from 15 to 36 hours. On the other hand, the vertical shear of the basic-state cross-front wind (which itself decreases exponentially with time) increases the vertical wavelength along a ray linearly. For the rays presented here, the vertical wavelengths increased from about 3 km to about 10 km during the interval 15 to 36 hours. As the basic state is time dependent, the frequency, as measured along a ray, changes also. During the time interval from 15 to 36 hours, the intrinsic frequency increased from around $1.3 \times 10^{-4}$ s$^{-1}$ to around 6 or $7 \times 10^{-4}$ s$^{-1}$.

The governing equations were rewritten with the non-linear advection terms involving, inter alia, the frontal circulation transposed to the right-hand side. This resulted in a set of linear equations with nonlinear forcing terms (cf. Lighthill’s (1952) theory of jet noise). The frontal circulation was viewed as a coherent broadscale time-dependent eddy which was shown to dominate the forcing. Moreover, it was shown that inertia–gravity waves were generated by the frontal circulation.
The forcing terms were determined from the standard calculation and were used in an integration of the linear wave-equations. In contrast to the standard calculation, the initial conditions comprised no perturbation velocity or perturbation buoyancy (i.e. no front). While the evolution of the flow in the troposphere (the near field) was very different to that in the standard calculation, the stratosphere (the far field) developed a field of inertia–gravity waves which was very similar to that in the standard calculation.

A numerical model incorporating the Geostrophic Momentum (GM) approximation has been integrated for the same basic-state and initial conditions as the standard calculation. An important difference between the two integrations is that the inertia–gravity waves present in the standard calculation are precluded from the GM calculation. This difference has been used to provide an estimate of the (nonlinear) back-reaction of the waves on the forcing. This was achieved by re-running the forced-wave calculation with the forcing terms approximated by their GM counterparts. The resulting field of stratospheric waves showed qualitative agreement with that in the standard calculation, although quantitative differences existed.

The mean flux of horizontal momentum attributable to the inertia–gravity waves has been shown to be approximately $1.8 \times 10^{-4} \text{ kg m}^{-1}\text{s}^{-2}$, where the flux was calculated as the difference between that for the standard run and that for the GM model. The divergence of the generalized momentum-flux associated with the wave field was shown to be balanced by the difference in pressure between the lateral boundaries of the model domain.

ACKNOWLEDGEMENTS

We should like to thank Steve Eckermann, David Karoly, Dan Keyser, Michael McIntyre and Joe Zehrder for their helpful criticisms and suggestions. This work was supported by grants from the Australian Research Council and the National Greenhouse Advisory Committee.

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