The dynamics of error growth and predictability in a coupled model of ENSO

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\textbf{SUMMARY}

Using new and emerging ideas about the growth of singular vectors ('optimal perturbations') in dynamical systems, the dynamics of error growth and predictability in an intermediate coupled model of the El Niño Southern Oscillation (ENSO) is investigated. A mechanism is identified for error growth associated with penetrative-convection anomalies in the atmosphere. Conditions for error growth via this mechanism are most favourable in the central Pacific where sea surface temperatures (SSTs) are relatively warm, and where changes in SST are moderately sensitive to vertical movements of the main oceanic thermocline.

The singular vectors of the coupled system were computed using the tangent-linear coupled model and its adjoint. The singular-value spectrum was found to be dominated by one singular vector at all times of the year. The potential for error growth in the coupled model, measured in terms of the growth of energy of the dominant singular vector, is found to vary seasonally, being greatest during the boreal spring. These seasonal variations are associated with the seasonal cycle in SST. During boreal spring and early summer, the SST in the central Pacific is at its maximum, at which time conditions are most favourable for error growth. Springtime is also the time of the 'predictability barrier' for ENSO. The potential for error growth is also influenced by the ENSO cycle itself. The results suggest that error growth will be enhanced during the onset of El Niño and suppressed during the onset of La Niña, which indicates that El Niño may be less predictable than La Niña.

\textbf{KEYWORDS:} El Niño Southern Oscillation Error growth Ocean–atmosphere model Predictability Sea surface temperatures Singular vectors

I. INTRODUCTION

During the last 10 years of so, a hierarchy of coupled ocean–atmosphere models has been developed to study the El Niño Southern Oscillation (ENSO). These models range from so called intermediate coupled models, to coupled general-circulation models (GCMs). Intermediate coupled models are generally anomaly models in that they model anomalies of the coupled system about the observed mean climate which is prescribed. The atmospheric and oceanic components usually consist of simple models (e.g. Anderson and McCreary 1985; Zebiak and Cane 1987; Schopf and Suarez 1988; Kleeman 1993). Coupled GCMs on the other hand model both the mean climate of the system and associated anomalies. They usually consist of state-of-the-art atmospheric and oceanic GCMs and, as a result, require at least an order of magnitude more computer time to run than intermediate models (e.g. Gordon 1989; Philander et al. 1992; Latif et al. 1993). An excellent review of coupled models is given by McCreary and Anderson (1991) and the results from a variety of coupled models are presented by Neelin et al. (1992).

Both intermediate coupled models (Zebiak and Cane 1987; Kleeman 1993; Kleeman et al. 1995) and coupled GCMs (Latif et al. 1993) have been used in attempts to forecast ENSO, and show useful skill up to 11 months ahead. Statistical models have also been used to predict ENSO but will not be discussed here. A review of ENSO prediction studies is given by Latif et al. (1994).

Attempts to forecast ENSO have led to the discovery of an apparent predictability barrier around April/May (Latif et al. 1994). At this time of the year the anomaly correlation falls rapidly, regardless of when a forecast is started. Observations show that around this time of the year the zonal sea-surface-temperature (SST) gradient in the tropical Pacific is at a minimum, which suggests that the predictability barrier for ENSO is linked with

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the annual cycle (see Webster and Yang 1992). There is also evidence that at the same time of year the ocean and atmosphere are weakly coupled in the tropics owing to the development of a warm stable layer at the ocean surface in the eastern Pacific (Webster 1995). In general, the predictability barrier for ENSO is not well understood so there is an urgent need to ascertain the physical aspects of the coupled ocean–atmosphere system which limit its predictability in the tropics.

To understand the factors which limit the predictability of a dynamical system, it is instructive to appeal to the experiences of numerical weather prediction (NWP). NWP model forecasts are made in the following way:

(i) An NWP model is initialized by combining the most recent model forecast and the most recent observations to form a best estimate of the current state of the atmosphere. This step is achieved using data assimilation techniques.
(ii) The atmospheric analyses from (i) are used as initial conditions for the NWP model which is run forward in time to forecast the future state of the atmosphere.
(iii) The new model forecast is verified against new observational data at the time of the next analysis, usually 24 hours later.

The quality of the resulting forecast will depend upon a number of factors, including: (a) the accuracy of the observations used to initialize the model and accuracy of the boundary conditions used to drive the model; (b) the accuracy of the numerical methods and physical parametrizations used in the forecast model; and (c) the stability of the flow fields under consideration with respect to perturbations arising from data errors and model errors. The factors described by (a) and (b) will become less limiting in future years as forecast models are improved and as resources are increased. Therefore, (c) is the underlying factor which will ultimately limit the predictability of the atmosphere, ocean or coupled system.

The dominant variability in the tropical ocean–atmosphere system occurs on seasonal to interannual time-scales. Perturbations to these dominant low-frequency signals can result from the presence of higher-frequency (synoptic) weather events which are superimposed on the low-frequency variability. During a forecast made with a coupled model, these high-frequency perturbations can occur in addition to perturbations arising from the data errors and model errors described above. Some understanding of the predictability of the tropical ocean–atmosphere system can therefore be gained by studying the growth of errors and uncertainties during coupled model forecasts. With this in mind, we will now briefly review some of the current ideas regarding the growth of errors in dynamical systems.

Let $\psi$ represent a state vector that describes a dynamical system such as the atmosphere. Furthermore, suppose that the evolution of $\psi$ is governed by the equation:

$$\frac{\partial \psi}{\partial t} = L(\psi)$$

where $L(\psi)$ is a nonlinear operator. The evolution of $\psi$ in time between $t = 0$ and $t = \ell$ describes a trajectory in phase space. If $\psi(0)$ is perturbed by a small amount $\delta \psi$ then $\psi$ evolves along a new trajectory, and the time development of $\delta \psi$ will be described to first-order by the so called ‘tangent-linear equation’ of (1), which is given by:

$$\frac{\partial \delta \psi}{\partial t} = \left( \frac{\partial L}{\partial \psi} \right) \delta \psi.$$  

The tangent-linear equation (2) will be valid during the initial linear stages of development of $\delta \psi$ when $\delta \psi$ is small, and when nonlinear effects are unimportant.
In classical instability theory, the eigenvectors of \((\partial L/\partial \psi)\) are the so called ‘normal modes’ of the system. If \(\psi\) in (1) is steady, or rendered stationary in time, then the normal modes of (2) are exponentially growing or decaying disturbances (Pedlosky 1987, chapter 7). There is, however, a more general class of disturbances which are characterized not by asymptotic exponential growth, but instead by rapid transient growth over a finite time. To examine error growth characterized by these disturbances, consider as a measure of forecast-error growth the norm \(E = \langle \delta \psi, \delta \psi \rangle\), where \(\langle \ldots, \ldots \rangle\) represents a general inner product. By virtue of the linearity of (2), the evolution of \(\delta \psi\) during the time interval \(t = 0 \rightarrow \ell\) can be written as:

\[
\delta \psi(\ell) = R(0, \ell)\delta \psi(0) \tag{3}
\]

where \(R(0, \ell)\) is the ‘propagator’ of (2) over the time interval \(t = 0 \rightarrow \ell\). The forecast-error norm \(E\) at time \(\ell\) can therefore be expressed in the form:

\[
E(\ell) = \langle \delta \psi(\ell), \delta \psi(\ell) \rangle = \langle R(0, \ell)\delta \psi(0), R(0, \ell)\delta \psi(0) \rangle = \langle R^*(\ell, 0)R(0, \ell)\delta \psi(0), \delta \psi(0) \rangle > \tag{4}
\]

where use has been made of the definition of the adjoint of a linear operator (Courant and Hilbert 1989), and \(R^*\) denotes the adjoint of \(R\). \(R^*(\ell, 0)\) is in fact the propagator of the adjoint of Eq. (2), and the order of the arguments indicates integration backwards in time from \(t = \ell\) to \(t = 0\).

Consider the factor \(\lambda\) by which the forecast error \(E\) grows over the time interval \(\ell\) so that:

\[
\lambda = \frac{E(\ell)}{E(0)} = \frac{\langle R^*(\ell, 0)R(0, \ell)\delta \psi(0), \delta \psi(0) \rangle}{\langle \delta \psi(0), \delta \psi(0) \rangle}. \tag{5}
\]

The behaviour of \(\lambda\) will be governed by the operator \(R^*(\ell, 0)R(0, \ell)\) (hereafter denoted \(R^*R\)). In particular, the perturbation \(\phi(0)\) for which \(\lambda\) is a maximum is the eigenvector of the operator \(R^*R\) which has the largest eigenvalue.

The eigenvectors of \(R^*R\) are, by definition, the singular vectors of \(R\). They are also often referred to as ‘optimal perturbations’ since they are optimal in the sense that they yield the fastest growth of the error norm \(E\) over the time interval \(\ell\). In this regard, the singular vectors are characterized by rapid transient growth. Since \(R^*R\) is a symmetric operator, the singular vectors are orthogonal and form a complete set. The singular vectors are therefore an ideal set of physically based functions for describing the evolution of \(E\). This was the view adopted by Lorenz (1965) who argued that the growth of forecast errors in NWP models could be explained in terms of the growth of the singular vectors of the forecast-error norm. Since then, these ideas have been expanded upon by others in studies of atmospheric predictability (e.g. Lacarra and Talagrand 1988; Farrell 1990; Molteni and Palmer 1993; Mureau et al. 1993). Using a linear approximation to the Zebiak and Cane (1987) coupled model, Blumenthal (1991) and Xue et al. (1994) have drawn on similar ideas in attempts to explain the predictability of ENSO. These ideas have also been used to explore the dynamics of mid-latitude cyclogenesis and blocking in the atmosphere (Farrell 1985, 1987, 1988, 1989), and the growth of instabilities in the ocean (Farrell and Moore 1992; Moore and Farrell 1993, 1994).
The traditional approach to the study of instability growth has been to seek disturbances in the form of normal modes. For example, the normal modes of the large-scale atmospheric circulation are generally composed of large-scale wave trains extending far around the globe (e.g. Frederiksen 1982). The singular vectors of the atmosphere on the other hand have smaller scale structures and are more usually confined to regions where local energy sources and sinks are favourable for rapid disturbance growth (Molteni and Palmer 1993). The localized nature and structure of singular vectors, as well as their rapid growth rates, are consistent with the structure of error growth in some NWP models (Toth and Kalnay 1993). This is one reason why the success of normal-mode theory for explaining error growth in numerical models has been limited. In addition, it has been known for some time that certain configurations of the atmospheric circulation are less predictable than others. For example, certain phases of the Pacific/North American (PNA) teleconnection pattern (Wallace and Gutzler 1981) appear to be highly unpredictable. In particular, flows characterized by a negative PNA index are found to be more unpredictable than those with a positive PNA index. However, as shown by Palmer (1988), the growth rate of the most unstable mode is relatively insensitive to the sign of the PNA index. The growth factors of the singular vectors, however, are strongly influenced by the PNA index (Molteni and Palmer 1993), with the singular vectors of a negative PNA state growing more rapidly than those associated with a positive PNA state. Molteni and Palmer's work strongly suggests that error growth during a PNA forecast occurs via the growth of non-modal disturbances, such as singular vectors, and not via normal-mode growth.

With these ideas in mind, we have conducted an investigation into the dynamics of singular-vector growth in a coupled model of ENSO in an attempt to understand the dynamics underlying the predictability of ENSO. The coupled model is described in section 2, and the derivation of the tangent-linear coupled model and its adjoint is described in section 3. The procedure for computing singular vectors is described in section 4. In section 5 we examine the growth of singular vectors on the observed seasonal cycle, and investigate the dynamical and physical processes which account for seasonal variations in the growth rate of the dominant singular vectors. The influence of the ENSO cycle on the singular vectors is investigated in sections 6 and 7. Error growth in the coupled model due to singular vectors is explored in section 8, and in section 9 we present numerous sensitivity studies to reveal the robust nature of our results. We conclude with a summary and discussion of our results in section 10.

2. The coupled model

The model we have used is the intermediate coupled model of Kleeman (1993) (hereafter referred to as K93) which is currently used operationally at the Australian Bureau of Meteorology Research Centre as part of the seasonal outlook system. The forecast skill of the model is similar to that exhibited by the models of Zebiak and Cane (1987) and Latif et al. (1993), and useful forecasts can be made up to 11 months ahead. The individual components of the coupled model will now be described briefly.

(a) The atmospheric model

The atmospheric component of the coupled model is a linear, two-pressure-level model on a β-plane, described in detail by Kleeman (1991). It is global in extent and is a steady-state version of the model used by Kleeman (1989). The model pressure levels are centred at 250 mb and 750 mb, and the model equations of motion are linearized about a
state of rest. Following Gill (1980), Rayleigh-friction and Newtonian-damping terms are also included. The equations for the 750 mb flow field are:

\[
\begin{align*}
\epsilon_a U - f V &= -\frac{\partial \Phi}{\partial x} \\
\epsilon_a V + f U &= -\frac{\partial \Phi}{\partial y} \\
\epsilon_a (\Phi + RT'/2) + c_s^2 \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) &= -RQ/2
\end{align*}
\]

where \( U \) and \( V \) are the zonal and meridional wind anomalies respectively, \( f \) is the Coriolis parameter, \( \Phi \) is the geopotential height anomaly, \( Q \) is the 500 mb heating anomaly, and \( R \) is the universal gas constant. The direct thermal forcing term, \(-RT'/2\) relaxes the 750 mb geopotential height anomaly \( \Phi \) to the surface temperature anomaly \( T' \). This formulation has been widely used in the literature (e.g. Gill 1985; Davey and Gill 1987) and is included to mimic the effects of surface processes associated with radiation, sensible-heat fluxes and shallow convection which are not explicitly modelled. The physical basis for this term is discussed in some detail by Kleeman (1991). \( Q \) does not force the barotropic mode and so the 250 mb circulation is equal and opposite to that at 750 mb. The Rayleigh friction and Newtonian cooling coefficients, \( \epsilon_a, c_s \), correspond to a damping time-scale \( \sim 3 \) days, and \( c_s \), the phase speed of the baroclinic mode, is 60 m s\(^{-1}\). Equations (6) were solved numerically on a horizontal grid with 2.8125° resolution.

It has been known for some time that heating anomalies associated with deep penetrative convection in the atmosphere are important during the onset of ENSO events in the western tropical Pacific. With this in mind, \( Q \) in (6) represents latent-heating anomalies arising from anomalous deep penetrative convection. The observed seasonal cycle of the coupled ocean–atmosphere system is prescribed in the model through \( Q \), which is given by:

\[
Q = \begin{cases} 
\max(Q_c, -\overline{Q}) & \text{if } m(\overline{T} + T') > m_{cr} \\
-\overline{Q} & \text{otherwise}
\end{cases}
\]

(7)

where \( m(\overline{T}) \) represents the moist static energy of an air parcel at the observed basic-state temperature \( \overline{T} \), and \( m_{cr} \) is the critical value at the observed threshold for penetrative convection. Equation (7) is subject to the condition that the mean basic-state latent heating \( \overline{Q} = 0 \) if \( m(\overline{T}) < m_{cr} \). In (7), \( Q_c \) is the anomalous heating due to changes in precipitation associated with changes in the atmospheric circulation, and \( \overline{Q} \) is computed from observations. Penetrative convection in the atmosphere occurs predominantly in areas where SST > 28°C (Hirst 1986). Equation (7) therefore reflects the fact that latent-heating anomalies due to changes in \( \overline{T} + T' \) can switch off penetrative convection, which yields heating anomalies \( Q = -\overline{Q} \).

The heating term \( Q_c \) is computed from:

\[
Q_c = \frac{L_v \rho_s c_v}{c_p \rho_a l_1} \left[ W[q_{\text{diff}}(\overline{T} + T')'] + \overline{W}[q_{\text{diff}}(\overline{T} + T') - q_{\text{diff}}(\overline{T})] \right] + \frac{L_v}{c_p \rho_a l_1} \left[ (q + q) \nabla \cdot U + q \nabla \cdot \overline{U} + (\overline{U} + U) \cdot \nabla q + U \cdot \nabla \overline{q} \right].
\]

(8)

where \( L_v \) is the latent heat of vaporization, \( c_p \) the specific heat at constant pressure, \( \rho \) is the air density where subscripts 4 and 2 refer to the lowest and mid atmospheric
levels respectively and \( c \) the transfer coefficient for heat and moisture. This expression results from integrating the atmospheric-moisture equation vertically, where \( I_1 \) and \( I_2 \) are integration constants, and an overbar denotes the observed mean value. In (8), \( q_{\text{diff}}(T) = q_{\text{sat}}(T) - q(T) \) is the air-sea specific humidity difference, where \( q_{\text{sat}}(T) \) and \( q(T) = 0.8q_{\text{sat}}(T) \) are the saturation specific humidity and specific humidity respectively at the surface temperature \( T = \text{SST} - 1.5 \degree \text{C} \). \( q \) is computed from a linearized form of the Clausius-Clapyron relation. In (8), \( \overline{q} \) represents the specific humidity of air at the observed mean basic-state temperature \( \overline{T} \), and the wind speed is denoted \( W = |\text{U}| \). In (8), the term in \([\ldots]\) is due to evaporation anomalies while the term in \([\ldots]\) represents the effects of anomalous moisture convergence. A full discussion of the form of \( Q_c \) and the condition for penetrative convection is given by Kleeman (1991).

The performance of the atmospheric model has been subjected to rigorous examination (Kleeman 1991; Kleeman et al. 1992), and in general the model gives a good depiction of the wind anomalies associated with ENSO SST anomalies.

(b) The ocean model

The ocean component of the coupled model solves the linear shallow-water equations for a single baroclinic mode on an equatorial \( \beta \)-plane. The model domain is limited to the tropical Pacific between 124\degree\text{E} and 80\degree\text{W}. The long-wave approximation is made, so the equations of motion become:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \epsilon_o \right) u - \beta y v + g \frac{\partial h}{\partial x} &= \frac{\tau_x}{\rho_w H_o} \\
\beta y u + g \frac{\partial h}{\partial y} &= \frac{\tau_y}{\rho_w H_o} \\
\left( \frac{\partial}{\partial t} + \epsilon_o \right) g' h + c_o^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0
\end{align*}
\]

(9)

where \( u \) and \( v \) are the zonal and meridional current anomalies, and \( h \) is the thermocline-displacement anomaly which results from the wind-stress anomaly \( \tau \); \( \rho_w \) is the density of sea water, \( H_o \) is the equivalent depth commensurate with the 1st baroclinic mode with phase speed \( c_o = \sqrt{g' H_o} = 2.8 \text{ m s}^{-1} \) and \( g' \) is the reduced gravity. The Rayleigh friction and Newtonian cooling coefficient in the ocean is \( \epsilon_o \), which corresponds to a damping time-scale of \( \sim 2.5 \text{ years} \).

The shallow-water equations are solved by rewriting (9) in terms of new variables \( q = g' h/c_o + u \) and \( r = g' h/c_o - u \) then expanding \( q \) and \( r \) in terms of parabolic cylinder functions (see Gill 1982, chapter 11). Only the first six cylinder-function amplitudes are retained which correspond to the equatorial Kelvin wave, and the first five equatorial Rossby-wave modes. The boundary conditions for (9) are:

\[
u = 0 \text{ at the eastern boundary, and } \int_{-\infty}^{\infty} u \, dy = 0 \text{ at the western boundary,}
\]

(10)

where the western boundary condition results from the long-wave approximation (Cane and Sarachik 1981). From (10), appropriate boundary conditions for the cylinder-function amplitudes can be derived.
The changes in ocean circulation $u$, $v$ and $h$ give rise to changes in the ocean SST. The SST anomalies, $T'$, along the equator are modelled by the equation:

$$\frac{\partial T'}{\partial t} = \eta h - \epsilon T' + \kappa \frac{\partial^2 T'}{\partial x^2}. \quad (11)$$

The parameter $\eta$, hereafter referred to as the ‘thermocline coefficient’, is the constant of proportionality (or regression coefficient) that relates $T'$ to $h$. Different values are used for $\eta$ in the west and east Pacific to reflect the fact that the main thermocline is deeper in the west than in the east. As a result, a given heat-content anomaly is associated with larger SST anomalies in the east than in the west. In the east Pacific from the coast of Central America to $140^\circ W$, $\eta = \eta_E = 3.4 \times 10^{-8} \text{degC m}^{-1} \text{s}^{-1}$ which corresponds to a steady-state solution of (11) with $T' = 2 \text{degC}$ when $h = 15 \text{ m}$. In the west Pacific, $\eta = \eta_W = \eta_E/5 = 6.8 \times 10^{-9} \text{degC m}^{-1} \text{s}^{-1}$, while in the central Pacific between $140^\circ W$ and the date-line, $\eta$ varies linearly between $\eta_E$ and $\eta_W$.

In (11), $|h| \leq h_{\text{max}}$ where $h_{\text{max}} = 22.5 \text{ m}$ is a cut-off value of $h$ which crudely mimics nonlinearities within the real ocean and prevents a runaway instability from developing in the coupled system. In the real ocean when the thermocline is shallow, the mixed-layer temperature approaches the sub-thermocline temperature, and so upwelling produces no further change in SST. When the thermocline is deep, it lies beneath the region of strong upwelling so that thermocline-displacement entrainment becomes negligible, and no further reduction in entrainment cooling is possible.

A fixed Gaussian-structure function is used in the meridional direction to compute off-equatorial SST anomalies. The e-folding length-scale of the meridional structure function is $10^\circ$, which is close to the atmospheric equatorial radius of deformation. Such a simplification was also made by Neelin (1991), and reflects the assumption that effects such as meridional advection and the differing meridional structure of the horizontal modes are unimportant to the primary ENSO mechanism. In (11), the relaxation term, $\epsilon T'$, is included to model negative feedback effects (Battisti and Hirst 1989; Neelin 1990) where $\epsilon = 2.72 \times 10^{-7} \text{s}^{-1}$, and $\kappa \frac{\partial^2 T'}{\partial x^2}$ is a weak diffusion term with $\kappa = 7 \times 10^4 \text{m}^2 \text{s}^{-1}$.

K93 has demonstrated that the hindcast skill of the coupled model is greatest when $T'$ is influenced only by $h$. The skill of the model was also found to be fairly insensitive to the details of the thermocline parametrizations used in (11), and useful forecasts can be made up to 11 months ahead. Including the effects of anomalous horizontal advection and anomalous upwelling on SST anomalies was found to degrade the skill of the model. Furthermore, Kleeman et al. (1995) have found that if the coupled model is initialized by assimilating subsurface thermal data into the ocean model before making a forecast, the skill of the model is improved, and useful forecasts can be made up to 15 months ahead.

(c) Coupling procedure

Before coupling the ocean and atmosphere models, the ocean model was spun up using the wind-stress anomalies computed from the Florida State University wind products (Legler and O’Brien 1984). In coupled mode, the SST anomalies computed from (11) were passed to the atmospheric model every month, where they were superimposed on the monthly Climate Analysis Center SST climatology. The resulting surface-wind anomalies computed by the atmospheric model were converted to surface wind-stress anomalies using the linear stress law:

$$\tau = \rho_s c_D |W| U \quad (12)$$

where $\rho_s$ is the density of air and $c_D$ is a drag coefficient. $|W|$ is a representative value
of the mean wind speed, and also acts as a coupling coefficient between the ocean and atmosphere. The effect of varying $|W|$ on the coupled-model behaviour is discussed by K93. Unless otherwise stated, $|W| = 6.5 \text{ m s}^{-1}$. The surface wind stress computed from (12) was used to force the ocean model.

3. The tangent-linear model and adjoint model

We wish to study the growth of forecast errors and uncertainties in the model using the framework of singular vector/optimal perturbation ideas discussed in section 1. The singular vectors we seek are the eigenvectors of $R^TR$ in (5) where now we view $R$ as the propagator of the tangent-linear coupled model, and $R^*$ as the propagator of the adjoint tangent-linear coupled model.

The tangent-linear coupled model is given by the first-order linearization of (6)–(11), and a small perturbation to a variable in these equations will be denoted by a prefix $\delta$. The form of the tangent-linear atmospheric-model equations remains unchanged with $\delta U, \delta V, \delta \Phi, \delta T'$ and $\delta Q$ replacing $U, V, \Phi, T'$ and $Q$ respectively in (6).

The nonlinearity described by the 28 °C SST threshold for penetrative convection in (7) was linearized so that:

$$\delta Q = \begin{cases} \delta Q_c & \text{if } m(\overline{T} + T') > m_{cr} \\ 0 & \text{otherwise} \end{cases}$$

(13)

where $\delta Q_c$ is given by the linearized form of (8), namely:

$$\delta Q_c = \frac{L_v \rho_0 c_v}{c_p \rho_1 I_1} \left[ \delta W \left[q_{\text{diff}}(\overline{T} + T') + (\overline{W} + W)\left[q_{\text{diff}}(\overline{T} + T' + \delta T') - q_{\text{diff}}(\overline{T} + T')\right]\right]ight.$$

$$+ \frac{L_v I_2}{c_p \rho_1 I_1} \left[ \nabla \cdot ((\overline{q} + q) \delta U) + \nabla \cdot (q(\overline{U} + U)) \right].$$

(14)

The form of the tangent-linear ocean model is given by (9) with $\delta u, \delta v$ and $\delta h$ replacing $u, v$ and $h$. The SST equation (11) is linearized as follows:

$$\frac{\partial \delta T'}{\partial t} = \gamma \delta h - \epsilon \delta T' + \frac{\partial^2 \delta T'}{\partial x^2},$$

(15)

$$\gamma = \begin{cases} \eta & \text{if } |h| \leq h_{\text{max}} \\ 0 & \text{if } |h| > h_{\text{max}} \end{cases}.$$

The condition for deep penetrative convection (7) and the thermocline-depth anomaly cut-off, $h_{\text{max}}$, applied in the nonlinear coupled model are discontinuous and so strictly speaking they cannot be linearized. However, the conditions $m(\overline{T} + T') > m_{cr}$ and $|h| \leq h_{\text{max}}$ are satisfied over large areas in the tropics which persist for long periods of time, and rapid spatial and temporal variations in $m(\overline{T} + T')$ and $|h|$ do not generally occur. In addition, these areas change their shape and location relatively slowly. As a result, the approximate linearizations (13) and (15) were found to perform well in tests of the tangent-linear coupled model which we performed, and we are satisfied that (13) and (15) represent reasonable linearizations of the processes at hand in the model. However, it should be stressed that linearizations of highly nonlinear or discontinuous processes should be approached with caution, and their validity checked and tested against the nonlinear model before they are used.
Figure 1. Time series of the r.m.s. difference between perturbations in the tangent-linear coupled model and the same perturbations as they evolve in the nonlinear coupled model, where $\Delta T'$ represents the difference in sea-surface-temperature (SST) perturbations and $\Delta h$ the difference in thermocline-depth perturbations. Time series of the correlation between the perturbation SST fields, $\delta T'$, and the perturbation thermocline-depth fields, $\delta h$, in the two models are also shown. Identical perturbations are used in both models and correspond to (a) the fastest-growing system singular vector with initial r.m.s. $\Delta T' \sim 0.1$ degC and r.m.s. $\Delta h \sim 0.1$ m, (b) as (a) but with initial r.m.s. $\Delta T' \sim 1$ degC and r.m.s. $\Delta h \sim 1$ m, (c) random noise with initial r.m.s. $\Delta T' \sim 1$ degC and r.m.s. $\Delta h \sim 1$ m, and (d) as (c) but with initial r.m.s. $\Delta T' \sim 1$ degC and r.m.s. $\Delta h \sim 5$ m. In each case a nonlinear coupled-model hindcast for the period January 1972–January 1973 was used as the background state to which perturbations were added.

For numerical consistency, the linearization of the coupled model was performed on the discretized model equations, and the tangent-linear coupled model was tested under a variety of conditions. These tests were performed by integrating perturbation fields $\delta T'$ and $\delta h$ forward in time, (a) with the tangent-linear coupled model, and (b) by adding the same perturbations to the nonlinear coupled model and differentiating the perturbed and unperturbed nonlinear model fields. The difference between the perturbation thermocline-depth fields of (a) and (b) will be referred to as $\Delta h$, and the corresponding perturbation SST differences as $\Delta T'$. Figure 1 shows time series of the normalized root mean square (r.m.s.) $\Delta h$ and normalized r.m.s. $\Delta T'$ which resulted from four different cases. The r.m.s. differences were normalized in each case by the initial r.m.s. difference at time $t = 0$. Also
shown in Fig. 1 are time series of the correlations between the perturbation fields \( \delta h \) and \( \delta T' \) from (a) and (b). In each case a nonlinear coupled-model hindcast for the period January 1972–January 1973 was used as the background state to which perturbations were added. Figures 1(a) and 1(b) show time series for the case when a spatially coherent and rapidly growing perturbation in the form of one of the system singular vectors (which are described in detail later) was used in (a) and (b). In Fig. 1(a) the perturbation has an initial r.m.s. \( \Delta T' \sim 0.1 \) degC and r.m.s. \( \Delta h \sim 0.1 \) m, while in Fig. 1(b) the initial r.m.s. \( \Delta T' \sim 1 \) degC and r.m.s. \( \Delta h \sim 1 \) m. The r.m.s. differences increase slowly in time over a period of a year, and show a similar trend regardless of the amplitude of the initial perturbation, while the correlations become smoother with increasing perturbation amplitude as the effect of computer-rounding errors is reduced. However, even though \( \Delta h \) and \( \Delta T' \) grow in time, the shape of the perturbation fields in the two models remains very similar, as evidenced by the high pattern correlations. Figure 1(c) shows the case when the perturbation takes the form of random noise, and the initial perturbation r.m.s. amplitudes of \( \Delta T' \) and \( \Delta h \) are 1 degC and 1 m respectively. The r.m.s. differences increase more slowly in this case compared with the case when a coherent perturbation with the same r.m.s. amplitude was used (cf. Fig. 1(b)), although the temporal variation of the correlations between the perturbation fields are similar in the two cases. Figure 1(d) shows the case when a random perturbation was used with initial r.m.s. amplitudes for \( \Delta T' \) and \( \Delta h \) of 1 degC and 5 m respectively. This case was chosen because SST and thermocline-depth errors of this magnitude are commonly associated with currently used ocean data analysis schemes. In this example the r.m.s. differences increase very little over time, and the perturbation fields remain highly correlated. Based on the results of the tests presented in Fig. 1, and the results of many other similar tests, we are satisfied that the tangent-linear coupled model represents a good approximation of perturbations in the nonlinear model for periods of many months duration and for both random perturbations and spatially coherent perturbations which grow rapidly in time. Indeed, Fig. 1(d) shows that in some cases the tangent-linear coupled model agrees very well with perturbations in the nonlinear model for periods of up to 12 months.

The adjoint \( R^* \) of a linear operator \( R \) is defined by the 2nd and 3rd lines of (4). The form of \( R^* \) will depend on the definition of the error norm, \( E \), and inner product \( \langle \ldots , \ldots \rangle \). However, once the adjoint for one inner product has been determined, it is a simple matter to apply a linear transformation to find the adjoint associated with any other inner product. We have used the total perturbation energy of the coupled model as the error norm for the system, and the adjoint of the tangent-linear coupled model has been constructed using the appropriate inner product. To this end we have defined a non-dimensional perturbation state vector \( \delta \psi \) in terms of the model prognostic variables, namely:

\[
\delta \psi = \begin{pmatrix}
\frac{R\delta T'c_o^2}{g\delta h/c_o^2} \\
\frac{\delta u/c_o}{g\delta h/c_o^2}
\end{pmatrix}.
\]  \hspace{1cm} (16)

The atmospheric model is always in steady state with the perturbation atmospheric heating, \( \delta Q \), and hence the perturbation SST, \( \delta T' \), provides sufficient information to describe the perturbation energy of the atmosphere. Because the long-wave approximation has been made in the ocean model, there is no contribution to \( \delta \psi \) from the perturbation meridional ocean velocity, \( \delta v \). A diagnostic equation for \( \delta v \) can be derived in terms of \( \delta u \) and \( \delta h \).

The total non-dimensional perturbation-energy norm for the tangent-linear coupled
model is given by,
\[
E(t) = \langle \delta \psi(t), \delta \psi(t) \rangle = \delta \psi(t)^T S_t \delta \psi(t)
\]
\[
= \sum_i \frac{1}{2} m_i (\delta U_i^2 + \delta V_i^2 + \delta \Phi_i^2/c^2) + \sum_j \frac{1}{2} m_j^0 (\delta u_j^2 + \delta h_j^2)
\]
(17)
where \( \delta U = \delta U/c_o, \delta V = \delta V/c_o, \delta \Phi = \delta \Phi/c_o, \delta u = \delta u/c_o, \delta h = g' \delta h/c_o \) and \( c = c_s/c_o \).
From here on, we will drop the hats and all variables will be assumed to be non-dimensional unless otherwise stated. The summations in (17) are over all atmospheric and oceanic grid boxes with masses \( m_i \) and \( m_j^0 \) respectively. The operator \( S_t \) is a weighting matrix which yields the total perturbation energy of the system described by \( \delta \psi(t) \). For numerical consistency, the adjoint model was derived directly from the discretized form of the tangent-linear equations.

4. THE PROCEDURE FOR COMPUTING SINGULAR VECTORS

The desired singular vectors of the tangent-linear coupled model are given by the solutions of the eigenvalue equation,
\[
E(\ell) = \lambda E(0)
\]
or equivalently by the generalized eigenvalue equation,
\[
R^*(\ell, 0) S_t R(0, \ell) \delta \psi(0) = \lambda S_0 \delta \psi(0).
\]
(19)
A single application of the operator \( R^*(\ell, 0) S_t R(0, \ell) \) on a vector \( \delta \psi(0) \) can be achieved as follows: (i) Choose \( \delta \psi(0) \) as the initial condition for the tangent-linear coupled model and integrate forward in time from \( t = 0 \rightarrow \ell \). This procedure is equivalent to operating on \( \delta \psi(0) \) with \( R(0, \ell) \). (ii) Apply the operator \( S_t \) to \( \delta \psi(\ell) = R(0, \ell) \delta \psi(0) \). (iii) Choose \( \delta \psi^*(\ell) = S_t R(0, \ell) \delta \psi(0) \) as the initial conditions for the adjoint model and integrate backwards in time from \( t = \ell \rightarrow 0 \). This step corresponds to acting on \( \delta \psi^*(\ell) \) with the operator \( R^*(\ell, 0) \) to yield the vector \( \delta \psi^*(0) = R^*(\ell, 0) S_t R(0, \ell) \). In practice we work with the transformed vector \( \delta \phi = G \delta \psi \), where \( G \) is the Cholesky factor of the initial perturbation-energy weight operator \( S_0 \), which yields the generalized eigenvalue equation (19).

To examine the dynamics of small error growth in the coupled model, we are interested in the spectrum of growing singular vectors of \( E \) (i.e. those for which \( \lambda > 1 \)). The total number of singular vectors of \( E \) is 1820, the number of grid points in the model. Iterative Lanczos and Arnoldi techniques (Golub and van Loan 1990) were used to compute the eigenvectors and eigenvalues of (19) using the algorithms of Sorensen (1992), and without explicitly computing the elements of the matrix on the left-hand side.

5. SINGULAR VECTORS OF THE OBSERVED SEASONAL CYCLE

First we will consider error growth on a basic state consisting of only the observed seasonal cycle. In this case, \( T' = q = U = V = \Phi = u = h = 0 \) in the tangent-linear and adjoint coupled models. The year was divided into quarters starting with January to March (JFM), followed by April to June (AMJ) and so forth. The singular vectors for each quarter were computed which corresponds to \( \ell = 3 \) months in (5) and (19). This is the seasonal
time-scale over which forecast errors for ENSO are found to grow (Latif et al. 1994; Palmer and Anderson 1994). The perturbation-energy growth factors, $\lambda$ (i.e. the squares of the singular values), of the first six members of the singular-vector spectrum of the total perturbation-energy norm for the tangent-linear coupled model are shown in Table 1. The perturbation-energy growth factors for each quarter are shown in descending order, with $\lambda_1$ being the largest, $\lambda_2$ the next largest, and so on. The value of $\lambda_3$ in the column labelled JFM represents the factor by which the total perturbation energy of the fastest-growing singular vector grows during that quarter.

Table 1 shows three interesting properties of the singular vectors. Firstly, the singular-vector spectrum of the seasonal cycle for each quarter is dominated by the fastest-growing singular vector, with energy-growth factor $\lambda_1$. Subsequent members of the singular-vector spectrum generally have somewhat smaller growth factors. Secondly, the largest growth factor occurs during the AMJ quarter, which corresponds to the time of the ENSO predictability barrier. Thirdly, we note that out of a possible 1820 singular vectors, only the first five or six grow with time (i.e. have growth factors > 1). This is less than 1/3 of 1% of the entire spectrum of singular vectors. The remaining singular vectors are either neutral or have perturbation energy which decays in time.

To examine the dynamical and physical factors responsible for the seasonal dependence in the value of $\lambda_1$, we must consider the energetics of developing perturbations in the tangent-linear coupled model. In the ensuing discussions, perturbation-energy growth is taken to be synonymous with error growth.

(a) The energetics and dynamics of singular-vector growth in the atmosphere
The non-dimensional perturbation-energy equation for the atmosphere can be written as:

$$\frac{1}{2} (\delta U^2 + \delta V^2 + \delta \Phi^2/c^2) = -\frac{1}{2e_a} \nabla \cdot (\delta \Phi \delta U) - \frac{\delta T^* \delta \Phi}{4c^2} - \frac{\delta Q_e \delta \Phi}{4e^2}. \quad (20)$$

Atmospheric energy ($E_A$) Flux divergence Direct Latent

The first term on the right-hand-side of (20) is the contribution to the atmospheric perturbation energy due to the perturbation-energy flux divergence. The second term results from the Newtonian cooling/damping term in (6) and represents the contribution to the energy from direct thermal effects. The third term is the contribution of latent-heat release from the surface and from moisture convergence to the energy budget of the tangent-linear
atmospheric model. Integrating (20) over the entire atmosphere yields,

$$\int \int E_A \, dx \, dy = - \int \int \frac{\delta T' \delta \Phi}{4 \epsilon c^2} \, dx \, dy - \int \int \frac{\delta Q_c \delta \Phi}{4 \epsilon c^2} \, dx \, dy$$  \hspace{1cm} (21)$$

where the energy flux divergence term vanishes.

Using (8) for $Q_c$ and (14) for $\delta Q_c$ we can write (21) as,

$$\int \int E_A \, dx \, dy = - \int \int \frac{\delta \Phi \delta T'}{4 \epsilon c^2} \, dx \, dy$$

Direct thermal

$$- \frac{A}{4 \epsilon c^2} \int \int \delta \Phi \{ \delta W q_{diff} + (\bar{W} + W) \delta q_{diff} \} \, dx \, dy$$

Surface latent heat

$$- \frac{B}{4 \epsilon c^2} \int \int \delta \Phi \nabla \cdot ((\bar{q} + q) \delta U) \, dx \, dy$$

Moisture-flux convergence

$$- \frac{B}{4 \epsilon c^2} \int \int \delta \Phi \nabla \cdot (\delta q (\bar{U} + U)) \, dx \, dy$$  \hspace{1cm} (22)$$

Moisture-flux convergence

where $A = L_v \rho_4 c_E / c_p \rho_2 I_1$ and $B = L_v I_2 / c_p \rho_2 I_1$. The globally integrated atmospheric perturbation-energy budget consists of contributions from direct thermal forcing, surface latent heating and latent-heat release associated with moisture-flux convergence. The moisture-flux convergence term has two components in (22); one term is associated with the basic-state specific humidity $(\bar{q} + q)$ and the other with the basic-state atmospheric circulation $(\bar{U} + U)$.

For the case considered above where the basic state is given by the observed seasonal cycle, $T' = q = U = V = \Phi = W = 0$ in (20), (21) and (22). Figure 2 shows time series of the perturbation-energy budget of the tangent-linear atmospheric model for the fastest-growing singular vector for each quarter of the year. The step-like appearance of the time series is due to the fact that the coupling frequency between the ocean and atmosphere is only one month. The total perturbation energy of the atmosphere is shown in Fig. 2(a) which indicates that atmospheric energy growth is most rapid for the fastest-growing AMJ singular vector. The contributions of the various terms in (22) to the atmospheric perturbation-energy budget are shown in Figs. 2(b)–(e). The energy budget during each quarter is dominated by only two terms: the direct thermal-forcing terms denoted $-\delta T' \delta \Phi / 4 \epsilon c^2$ and the moisture-flux convergence term denoted $\delta \Phi \nabla \cdot (\bar{q} \delta U) / \epsilon$. In addition, the growth of energy associated with these two terms is most rapid during the AMJ quarter.

The moisture-flux convergence and surface latent-heating terms in (22) can be broken down further. Figure 3 shows time series of the individual contributions to these terms for the fastest-growing AMJ singular vector. It would appear from Fig. 3 that terms involving the gradient of $\bar{q}$ and the divergence of $U$ are relatively unimportant to the energetics of singular-vector growth. In fact, Fig. 2 and Fig. 3 indicate that the major contributors
Figure 2. Time series of the various contributions (see text) to the globally integrated atmospheric perturbation-energy budget for the fastest-growing singular vector during each quarter of the year. All terms have been non-dimensionalized as described in the main text.
to the growth of the atmospheric component of the fastest-growing AMJ singular vector are the direct thermal term and the term $\delta \Phi \nabla \cdot \delta U / \epsilon$. This is in fact the case for the fastest-growing singular vector of each quarter. Given the importance of $\delta \Phi \nabla \cdot \delta U / \epsilon$ to the energetics of singular-vector growth in the coupled model, any references hereafter to perturbation-energy growth or error growth associated with moisture-flux convergence will refer to this quantity.

Contour maps of the dominant terms in the atmospheric energy budget of the fastest-growing AMJ singular vector are shown in Fig. 4 during May and June. Only the fields over the tropical Pacific Ocean are shown since they are negligible elsewhere. Figures 4(a) and 4(b) show the time development of the total atmospheric perturbation energy $E_A$. The energy is mainly concentrated in a region centred just south of the equator and close to the date-line, and most of the total energy growth of the singular vector occurs in this region. Figures 4(c) and 4(d) show the contribution to the energy budget of the direct thermal-forcing, $-\delta T' \delta \Phi / 4c^2$. The globally integrated energy budget of Fig. 2 suggests that this term makes the largest contribution to the atmospheric energy budget. However, $-\delta T' \delta \Phi / 4c^2$ occupies a relatively large region and so, while its global integral is relatively
Figure 4. Contour maps showing the development of the dominant terms in the atmospheric perturbation-energy budget (see text) over the tropical Pacific Ocean for the fastest-growing singular vector of the April–May–June quarter. Maps are shown for May and June of the following non-dimensional fields: Atmospheric perturbation energy ([a] and [b]), the direct thermal-forcing term ([e] and [f]), the dominant moisture-flux convergence term ([c] and [d]) and the energy-flux divergence term ([g] and [h]). Light shading indicates positive values and dark shading indicates negative values. The contour interval is $2 \times 10^{-3}$ in (a) and (b) and $5 \times 10^{-3}$ in (c)–(h).
large, locally its contribution to the energy budget is quite small. Its development in time essentially mirrors the development of the SST perturbation, $\delta T'$, which will be discussed later.

The growth of energy due to the dominant moisture-flux convergence term $\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U}/\epsilon$ is shown in Figs. 4(e) and 4(f). The contribution of this term to the energy budget is localized mainly around a region centered just south of the equator and east of the date-line. A much smaller region of energy growth associated with moisture-flux convergence can be found north of the equator off the coast of Central America. As we will show, energy growth due to moisture-flux convergence is the most important term in the atmospheric energy budget, and in the central Pacific it is an order of magnitude larger than the direct thermal-forcing term.

Figures 4(g) and 4(h) show the contribution to the perturbation-energy budget of the energy-flux divergence term $-[(\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U})_x + (\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U})_y]/\epsilon$. The global integral of this term is zero since it cannot create or destroy perturbation energy. The energy-flux divergence term merely acts to redistribute the perturbation energy. In Figs. 4(g) and 4(h) regions of darkest shading indicate energy-flux divergence, while regions of light shading are regions of energy convergence. It is apparent that energy generated by moisture-flux convergence and direct thermal forcing is transferred out of the regions of generation and deposited in a region to the west where the perturbation energy of Figs. 4(a) and 4(b) is a maximum. This pattern of energy-flux divergence is consistent with Rossby-wave packets radiating energy westward from the regions of atmospheric heating.

Figure 4 begs the question: Why is the central Pacific preferred over other regions for the growth of the dominant moisture-flux convergence term $\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U}/\epsilon$? To answer this question, we need to consider the two most important factors for the growth of this term. Firstly, in the coupled model, atmospheric heating anomalies, $Q$, associated with penetrative convection are favored only where the SST $\geq 28^\circ C$ (cf. Eq. (7)). As a result, heating perturbations, $\delta Q$, can only develop over regions of warm water where $\bar{T} \geq 28^\circ C$. This condition will mainly be satisfied over the western and central Pacific. Secondly, for $\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U}/\epsilon$ to be non-zero, there must be a divergent perturbation wind field, $\delta \mathbf{U}$. To spin-up such a wind field, we need atmospheric heating through the growth of SST perturbations, $\delta T'$. Growth of $\delta T'$ will be favored where the sensitivity of SST to thermocline perturbations is greatest (i.e. where $\eta$ and $\gamma$ are largest) which is in the east and central Pacific. Clearly then, the two conditions favorable for the growth of $\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U}/\epsilon$ will be simultaneously satisfied only in the central Pacific. To illustrate this, Fig. 5 shows contour maps of the monthly mean SST climatology imposed in the coupled model. Superimposed on each map are contours of the energy release due to the dominant moisture-flux convergence term $\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U}/\epsilon$ for the fastest-growing singular vector of each quarter of the year. Also shown in each panel of Fig. 5 are the regions which delineate the different values assumed for the thermocline coefficient, $\eta$. In the central Pacific where $\eta = \eta_C$ and $\eta_W \leq \eta < \eta_E$, $\delta T'$ is moderately sensitive to thermocline perturbations, $\delta h$. Figure 5 shows that perturbation-energy growth due to moisture-flux convergence is confined mainly to the region of moderate SST sensitivity bounded by the $28^\circ C$ isotherm of $\bar{T}$. It is in this region that the two conditions, discussed above, favorable for the growth of $\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U}/\epsilon$ are simultaneously satisfied. We have further demonstrated this condition by repeating the calculation of the AMJ singular vector in a model where $\eta$ assumes a constant value of $(\eta_W + \eta_E)/2 = 2.04 \times 10^{-5} \text{degC m}^{-2} \text{s}^{-1}$ everywhere. Figure 6 is similar to Fig. 5 and shows for this new singular vector the regions of energy growth due to moisture-flux convergence superimposed on the monthly mean SST. In this case, the sensitivity of $\delta T'$ to $\delta h$ is the same everywhere, and $\delta \Phi \bar{q} \nabla \cdot \delta \mathbf{U}/\epsilon$ grows to occupy most of the region bounded by the $28^\circ C$ isotherm.
Figure 5. Contour maps of the observed monthly mean sea surface temperature (SST) in the tropical Pacific. Superimposed on each map are shaded contours of $\delta \Phi \cdot \vec{V} \cdot \delta \vec{U} / \epsilon$ (see text) for the fastest-growing singular vector during each quarter. Also shown are the values assumed for the thermocline coefficient, $\eta$, in each region. The contour interval for SST is 1 degC, and a change in grey scale for $\delta \Phi \cdot \vec{V} \cdot \delta \vec{U} / \epsilon$ corresponds to an interval of $1.5 \times 10^{-2}$. The 28°C isotherm is highlighted.
The seasonal dependence of the size of the perturbation-energy growth factors of the fastest-growing singular vectors given in Table 1 is due to the seasonal cycle of SST, $\bar{T}$. Figures 5(a)–(f) show that, during the first half of the year, the warm pool in the western tropical Pacific begins to warm and spread eastwards south of the equator. This can be seen by following the development of the 28°C isotherm. During April, May and June, the warm pool pushes eastward along the equator as well, creating the most favourable conditions for energy growth through the action of $\delta\Phi/\bar{\nabla} \cdot \delta \mathbf{U}/\epsilon$. Not only is there a larger body of warm water in the central Pacific for penetrative convection to draw upon at this time of the year, but also the region of perturbation-wind divergence, $\bar{\nabla} \cdot \delta \mathbf{U}$, which is accessible for energy growth is largest. During the latter half of the year, the west Pacific warm pool shrinks westwards as the surface waters begin to cool, creating conditions less favourable for energy growth via penetrative convection.

To quantify these ideas further, we have computed the singular vectors of the seasonal cycle for two other experiments. In the first experiment only the climatological basic-state SST, $\bar{T}$, was allowed to vary seasonally. All other basic-state fields assumed their annual mean values. Table 2 shows the perturbation-energy growth factors of the first six singular vectors for each quarter for this case, and shows that the largest values of $\lambda_j$ still occur during the AMJ quarter. In the second experiment, $\bar{T}$ assumed its annual mean value and
all other climatological basic-state fields were allowed to vary seasonally. Table 3 shows the singular-vector growth factors for each quarter in this case, and demonstrates that \( \lambda_1 \) changes little throughout the year. These experiments therefore clearly demonstrate that the seasonal dependence of the growth of the dominant singular vector during the year is controlled by the seasonal cycle in the basic-state SST, \( \bar{T} \).

The dynamics of the second fastest-growing singular vector (with singular value \( \lambda_2 \)) during each quarter is similar to that of the fastest-growing singular vector. Energy growth is dominated by \( \delta \Phi q \nabla \cdot \delta U / \epsilon \), but now this term grows mainly over the regions of warm water off the coast of central America north of the equator in the east Pacific.

\[ \frac{1}{2} \frac{\partial}{\partial t} (\delta u^2 + \delta h^2) = \frac{1}{2} (\delta u \delta \tau_e + \delta v \delta \tau_e) - \frac{\epsilon_o}{2} (\delta u^2 + \delta h^2) - \frac{1}{2} \nabla \cdot (\delta h \delta u). \]  

(23)

The first term on the right-hand side of (23) is the contribution to the oceanic perturbation-energy budget due to surface perturbation-wind forcing, which we refer to as the source term. The second and third terms are the contributions from dissipation and ocean perturbation-energy flux divergence respectively. Integrating (23) over tropical Pacific Ocean
yields,

\[ \frac{\partial}{\partial t} \int \int E \, dx \, dy = -\frac{\epsilon_0}{2} \int \int E_0 \, dx \, dy + \frac{1}{2} \int \int (\delta u \delta \tau_x + \delta v \delta \tau_y) \, dx \, dy \]

Dissipation

Source

\[ + \frac{1}{2} \int \delta h \delta u \mid_{x=0} \, dy. \]

Energy loss

(24)

The third term on the right-hand side of (24) results from integrating the energy flux divergence over the tropical Pacific. This integral does not vanish because we have made the long-wave approximation in the ocean. The boundary conditions on \( \delta u \) are given by (10) with \( \delta u \) replacing \( u \). By making the long-wave approximation we are essentially ignoring the generation of short Rossby waves at the western boundary. As a result, energy will be lost through the western boundary whenever a westward-propagating Rossby wave is reflected there. This is given by the ‘Energy loss’ term of (24).

Figure 7 shows time series of the perturbation energy of the ocean and its components for the fastest-growing singular vector for each quarter. The total perturbation energy (Fig. 7(a)) grows at a similar rate regardless of the time of year. Figure 7(b) shows that the dominant contribution to the total energy is from the perturbation-wind stress. Energy dissipation and energy loss through the ‘leaky’ western boundary are negligible as shown in Fig. 7(c) and 7(d) respectively.

To understand fully the development of the atmospheric component of the fastest-growing singular vectors, it is necessary to consider the oceanic component, since atmospheric energy growth is initiated by the ocean. To this end, Fig. 8 shows contour plots of \( \delta h \) and \( \delta T' \), as well as vector plots of \( \delta \tau \) for the fastest-growing singular vector of the AMJ quarter. These fields are very similar to those of the singular vectors at other times of the year, and so are representative of the dynamics of oceanic singular-vector growth throughout the year. We will proceed with a discussion of Fig. 8 by considering perturbation-energy growth in the atmosphere due to the dominant moisture-flux convergence term \( \delta \Phi \eta \nabla \cdot \delta U / \epsilon \). For energy growth via this term, we require \( \bar{T} > 28 \, ^\circ C \) as discussed above, and a divergent perturbation-wind field (i.e. \( \nabla \cdot \delta U \neq 0 \)). For the best result, the scale of \( \delta U \) should be \( \sim \) the scale of the region bounded by 28 \, ^\circ C \) water and the region of moderate \( \eta \) (cf. Fig. 5), so as to maximize \( \nabla \cdot \delta U \) there. This divergent circulation is set up in the tangent-linear coupled model by short atmospheric Rossby waves in the vicinity of the date-line. Figures 8(h) and 8(i) show the perturbation-wind field associated with these Rossby waves in the form of a band of westerly wind perturbations centred on the equator and flanked to the north and south by vortex circulations. These Rossby waves are generated and maintained by a growing SST perturbation, \( \delta T' \), which develops to the east of the Rossby-wave wind field as shown in Figs. 8(e) and 8(f). (Notice also the development of a divergent circulation off the central American coast in response to the secondary maximum in \( \delta T' \) in the east Pacific.) The spatial phase difference between \( \delta T' \) and \( \delta \tau \) in the tropics is well understood and has been discussed extensively in the literature (e.g. Gill 1980; Neelin 1991; Clark 1994). According to Eq. (15), \( \delta T' \) is maintained by the passage of downwelling equatorial ocean waves which increase \( \delta h \). Figures 8(a)–(c) show the development of \( \delta h \) during the growth of the singular vector. The initial configuration of \( \delta h \) is shown in Fig. 8(a) with the thermocline perturbed downwards along the
Figure 7. Time series of the various contributions to the perturbation-energy budget (see text) for the ocean averaged over the entire tropical Pacific for the fastest-growing singular vector of each quarter. All terms have been non-dimensionalized as described in the main text.
Figure 8. Contour maps of $\delta h$ ((a)–(c)) and $\delta T'$ ((d)–(f)), and vector plots of $\delta T$ ((g)–(i)) for the fastest-growing singular vector of the April–May–June quarter. The contour interval is 1.5 m for $\delta h$ and 0.1 degC for $\delta T'$. The maximum vector length for $\delta T$ is shown in m s$^{-1}$. The darkest shading corresponds to negative values of $\delta h$ and $\delta T'$. See text for explanation of symbols.
entire length of the equator, the perturbation increasing westward. As the ocean evolves, an equatorially trapped Kelvin wave moves out from the western boundary along the equator, deepening the thermocline as it propagates eastwards. By the beginning of May (Fig. 8(b)) the maximum thermocline perturbation occurs in the central Pacific, giving rise to a $\delta T'$ which drives the divergent perturbation-wind field of Fig. 8(h) and causes energy growth in the atmosphere via $\delta \Phi \hat{q} \nabla \cdot \delta U / \epsilon$. As the singular vector evolves, thermocline deepening is maintained in the central and eastern Pacific by the action of equatorially trapped waves (Fig. 8(c)), and $\delta T'$ continues to grow (Fig. 8(f)), further fueling perturbation-energy growth in the atmosphere.

An obvious feature of the oceanic singular-vector structure is the very-short-wavelength structure of the initial SST perturbation $\delta T'$ shown in Fig. 8(d). This structure is a property of the weighting function $S$ used to compute the total perturbation-energy norm in Eq. (17). The solution of the generalized eigenvalue problem represented in (19) requires the inversion of $S_0$. The eigenvalues $\nu$ of $S_0$ decrease very rapidly, and there are 5 orders of magnitude difference between the largest and the smallest. The eigenvectors of $S_0$ and $S_0^{-1}$ are identical while the eigenvalues of $S_0^{-1}$ are $1/\nu$. Therefore projections onto the eigenvectors with smallest $\nu$ will be amplified by a factor $\sim 10^5$ by $S_0^{-1}$. While this may
appear to be a highly undesirable property of the matrix $S_0$, we demonstrate in section 9(d) that the singular-vector spectrum is relatively insensitive to the treatment of $S_0$.

In fact, the development of the fastest-growing singular vectors of the tangent-linear coupled model is insensitive to the initial $\delta T'$ since all of the information for the future development of the disturbance is contained within the ocean-thermocline field. To demonstrate this, we initialized the tangent-linear coupled model with the $\delta h$ and $\delta u$ fields of the AMJ singular vector described above, and with $\delta T' = 0$ everywhere. Figure 9 (solid curves) shows time series of the development of the perturbation energy of the atmosphere and ocean, as well as the dominant terms in the atmospheric energy budget. The perturbation energy of the atmosphere is zero in Fig. 9(a) during the first month because $\delta T' = 0$. However, as the singular vector evolves, the energy grows rapidly as before. The dominant contributions to the atmospheric energy budget are as before $-\delta T' \delta \Phi / 4c^2$ (Fig. 9(c)) and $\delta \Phi \nabla \cdot \delta U / \epsilon$ (Fig. 9(d)). The energy of the ocean begins to increase only after the end of the second month when $\delta \tau$ is non-zero. Also shown in Fig. 9 (dashed curves) are the results of an experiment in which the tangent-linear coupled model was initialized with the $\delta T'$ field for the AMJ singular vector shown in Fig. 8(d) and with $\delta h = \delta u = 0$. In this case the perturbation energy of the coupled model does not grow.
The results displayed in Fig. 9 show that the initial SST perturbation $\delta T'$ is unimportant for the development of the singular vectors of the tangent-linear model. More importantly, they show that the precursors for energy growth in the coupled system reside in the ocean. Moreover, this suggests that all we need to achieve error growth in the coupled model via the mechanisms described above is the passage of upwelling or downwelling equatorial waves through regions of the ocean where the conditions for the development of penetrative-convection anomalies in the atmosphere are favourable. This scenario is highly likely when a coupled model such as that described here is used for forecasting ENSO, since internally generated noise will generate equatorial ocean waves. In addition, transient equatorial ocean waves associated with the spin-up and initialization of the ocean model will be present for quite some time after the ocean is coupled to the atmosphere.

(c) The dynamics of the set-up of the atmospheric perturbation circulation

The energy-budget analyses of section 5(a) show that perturbation-energy growth in the atmosphere is dominated by the moisture-flux convergence term $\delta \Phi \nabla \cdot \delta U/\varepsilon$ and the direct thermal forcing $-\delta T' \delta \Phi/4c^2$. Figure 2 shows that the contributions to the atmospheric energy budget from surface latent heating and the moisture flux convergence term $\delta \Phi \nabla \cdot (\delta q \vec{U})/\varepsilon$ are relatively unimportant. However, as we will now demonstrate, these terms are essential to the dynamics of the development of the singular vector.

In order to investigate the dynamics of the set-up of the perturbation circulations in the atmosphere that are favourable for energy growth, we initialized the tangent-linear coupled model with the structure of the AMJ singular vector described above, but retained only a subset of the terms in the atmospheric heating function, $\delta Q$, given by (14). Table 4 shows the factor $\mu$ by which the total perturbation energy grows when only the terms indicated in the left-hand column are retained in (14). Our choice of which terms to retain was influenced by Fig. 3 which shows that terms involving $\nabla q$ and $\nabla \cdot U$ make a negligible contribution to the energy of the singular vector.

Table 4 shows that the direct-thermal term $-R\delta T'/c^2$, the moisture-flux convergence terms $\bar{q} \nabla \cdot \delta U$ and $\bar{U} \cdot \nabla \delta q$, and the surface latent-heating term $\bar{W} \delta q_{\text{diff}}$ account for most of the energy growth of the AMJ singular vector, where $\mu = 5.5$. Table 1 shows that with all terms retained in (14), $\mu = 6.5$. If only $\bar{q} \nabla \cdot \delta U$ and/or $-R\delta T'/c^2$ are retained (i.e.

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<tr>
<td>Source terms included</td>
<td>$\mu$</td>
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<tr>
<td>$-R\delta T'/c^2$, $\bar{q} \nabla \cdot \delta U$, $\bar{U} \cdot \nabla \delta q$, $\bar{W} \delta q_{\text{diff}}$</td>
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</tr>
<tr>
<td>$-R\delta T'/2c^2$</td>
<td>2.0</td>
</tr>
<tr>
<td>$-R\delta T'/2c^2$, $\bar{q} \nabla \cdot \delta U$</td>
<td>2.5</td>
</tr>
<tr>
<td>$-R\delta T'/2c^2$, $\bar{q} \nabla \cdot \delta U$, $\bar{U} \cdot \nabla \delta q$</td>
<td>4.2</td>
</tr>
<tr>
<td>$-R\delta T'/2c^2$, $\bar{q} \nabla \cdot \delta U$, $\bar{W} \delta q_{\text{diff}}$</td>
<td>3.1</td>
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See text for explanation of symbols.
the dominant contributors to the atmospheric energy budget of Fig. 2), the energy of the singular vector can grow by only a factor of 2 or so, which is less than one third the growth factor achieved when all heating terms are retained. If in addition to $\bar{q} \nabla \cdot \delta U$ and $-R \delta T'/c^2$ we retain either the surface latent-heating term $\bar{U} \delta q_{\text{diff}}$ or the other moisture-flux convergence term $\bar{U} \cdot \nabla \delta q$, Table 4 shows that $\mu$ can attain values as high as 4.2.

Therefore, even though $\bar{q} \nabla \cdot \delta U$ and $-R \delta T'/c^2$ are the heating terms which contribute most to the atmospheric energy budget, these terms develop only in response to $\delta U$ and $\delta T'$. A divergent perturbation-wind field $\delta U$ is necessary for the growth of $-\delta T'/\delta \Phi/4c^2$ and $\delta \Phi \bar{q} \nabla \cdot \delta U/e$. This $\delta U$ develops in response to $\bar{W} \delta q_{\text{diff}}$ and $\bar{U} \cdot \nabla \delta q$ which contribute relatively little to the total energy budget of (22).

6. THE INFLUENCE OF THE ENSO CYCLE ON SINGULAR-VECTOR GROWTH

In the preceding sections, we have demonstrated a possible mechanism by which errors can grow rapidly in coupled models of the ENSO. In addition, we have shown how the growth of errors in this way can be more rapid during the April–June period which coincides with the predictability barrier identified experimentally for ENSO. We will now demonstrate how the above mechanism for error growth can be significantly enhanced or suppressed by the presence of the ENSO cycle itself.

When the coupling strength $|W|$ of (12) is increased from 6.5 m s$^{-1}$ to 9.0 m s$^{-1}$, a self-sustaining oscillation can develop in the coupled model with a regular period of approximately 3 years. Figure 10 shows a Hofmøller diagram of $T'$ along the equator during one cycle of the coupled oscillation which develops in the coupled model. The amplitude of $T'$ in the east Pacific is close to 3 degC and the oscillation is similar to the observed ENSO cycle in that the SST anomalies in the east Pacific have a standing-wave character, while there is a pronounced eastward-propagating component to the oscillation in the central Pacific. In addition, the warm ('El Niño') phase and cold ('La Niña') phase of the oscillation are asymmetric.

We have computed singular vectors for the oscillation, and the perturbation-energy growth factors, $\lambda_1$, of the fastest-growing singular vector during each quarter of each year are also shown in Fig. 10. For comparison, we show in Table 5 the growth factors of the singular vectors for the seasonal cycle alone for each quarter when $|W| = 9.0$ m s$^{-1}$. As before, Table 5 shows that the size of $\lambda_1$ varies seasonally and is largest during AMJ. During the first 12 months of the oscillation in Fig. 10, a cold event develops and matures in the east Pacific. The onset of this cold event is preceded by a 6-month period during which the SST in the central Pacific is up to 2 degC colder than normal. During this time, the west Pacific warm pool shrinks westwards, and conditions for perturbation-energy growth

| TABLE 5. PERTURBATION-ENERGY GROWTH FACTORS FOR THE FIRST SIX SINGULAR VECTORS WHICH MAXIMIZE THE GROWTH OF TOTAL PERTURBATION ENERGY OF THE SEASONAL CYCLE DURING EACH QUARTER WHEN $|W| = 9.0$ m s$^{-1}$ (SEE TEXT) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\lambda_1$    | 5.2             | 8.9             | 6.3             | 5.5             |
| $\lambda_2$    | 2.0             | 3.8             | 2.4             | 2.1             |
| $\lambda_3$    | 1.4             | 1.5             | 1.6             | 1.7             |
| $\lambda_4$    | 1.3             | 1.4             | 1.4             | 1.4             |
| $\lambda_5$    | 1.1             | 1.3             | 1.2             | 1.3             |
| $\lambda_6$    | 1.0             | 1.1             | 1.1             | 1.0             |
Figure 10. A Hofmoller diagram of sea-surface-temperature anomalies, $T'$, which develop during a complete cycle of the self-sustaining oscillation of the coupled model when the coupling strength $|W| = 9.0 \text{ m s}^{-1}$. Also shown are the perturbation-energy growth factors $\lambda_1$ of the fastest-growing singular vectors which grow on this oscillation during each quarter (shown by initial letter of the month) of each year. Shaded regions correspond to areas where $|h| > h_{\text{max}}$ so that the thermocline nonlinearity is invoked and $\gamma$ in (15) is zero (see text). The contour interval for $T'$ is 0.5 degC. The thermocline coefficient $\eta$ in the west, centre and east are shown by subscripts, $W$, $C$ and $E$ respectively.

are unfavourable. The growth factors for JFM and AMJ during the first year confirm this. Around JAS of the first year, SSTs in the central Pacific return to normal, and subsequently begin to increase during the onset of the warm event. At this time, the west Pacific warm pool begins to advance eastward, and by JFM of year 2 the SST is up to 2 degC warmer than normal in the central Pacific. During the onset of the warm event, conditions for energy growth are extremely favourable in the central Pacific because of the large body of warm water there. This is reflected in the growth factors $\lambda_1$ which reach values of almost 20, which is some four times larger than the value attained by the seasonal cycle alone at the same time of the year (see Table 5).

As the warm SST anomalies in the central Pacific decline, the values of $\lambda_1$ fall rapidly. During this time, another important effect comes into play which strongly constrains the growth of the singular vectors. The shaded regions in Fig. 10 show the areas in which the nonlinearity in the thermocline term given by (15) is invoked. In these regions, $\gamma = 0$ in the tangent-linear SST Eq. (15). In other words, $\delta T'$ is unable to grow in the areas which are shaded, since subsequent $\delta h$ has no influence on SST because the nonlinear limit of $h$ has been reached. When this occurs, Fig. 10 shows that the growth factors $\lambda_1$ are relatively small compared with those of the seasonal cycle (Table 5).
The results of this section highlight three important points. Firstly, the growth of errors can be strongly influenced by the phase of the ENSO cycle. During the onset of a La Niña event, energy growth is suppressed, while during the onset of El Niño, energy growth is enhanced. This is consistent with the results of Kleeman and Power (1994) who examined the growth of atmospheric noise in the same coupled model, and found that the tendency for error growth was greatest just before El Niño events. Secondly, during extreme ENSO events, nonlinearities in the influence of thermocline movements on SST act to stabilize the coupled system, producing conditions which are unfavourable for rapid energy growth. Finally, we note that these two effects can modify the seasonal variations that occur in the size of the dominant growth factor $\lambda_1$ which were discussed earlier. The ensuing seasonal dependence in $\lambda_1$ for a given ENSO event will depend upon how that particular event is phase locked with the seasonal cycle.

7. SINGULAR-VECTOR GROWTH DURING ACTUAL ENSO HINDCASTS

As discussed in section 1, the coupled model has been used to hindcast all of the warm and cold events which have been observed between 1972 and 1991 as described in K93 and Kleeman et al. (1995). We have repeated a subset of the successful K93 hindcasts and have computed the spectrum of singular vectors at different times of the year to see how the ideas and results presented above carry over to actual observed ENSO events. In the experiments of K93, the ocean–atmosphere coupling strength $|\mathcal{W}| = 6.5$ m s$^{-1}$, and the model does not support self-sustaining oscillations. Table 6 summarizes the hindcast experiments which were repeated here. A total of seven successful hindcasts were chosen from K93. The hindcasts were initialized by spinning up the ocean model with the observed wind-stress anomalies for 2 years before coupling with the atmospheric model. The hindcast start date in Table 6 is the time at which the ocean and atmosphere models were coupled, hence the ocean model was spun-up before this with observed wind anomalies from the preceding 24-month period. The type of ENSO event and the month and year when it was observed to peak are indicated in the 2nd and 3rd columns, respectively, of Table 6. The duration of each hindcast was 24 months, and the spectrum of singular vectors was computed for each quarter during the first year of the hindcast. The growth factors of the fastest-growing singular vector in each quarter are given in Table 6.

Table 6 shows that, in general, the largest growth factors of the fastest-growing singular vectors occur during the AMJ quarter during the onset of El Niño events. Observed ENSO events are phase locked with the seasonal cycle, so that during the onset of an El Niño the regular seasonal warming of the central Pacific is enhanced, and conditions are

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<tr>
<td>Jan. 1973</td>
<td>La Niña</td>
<td>Nov. 1973</td>
<td>5.4</td>
<td>4.0</td>
<td>3.4</td>
<td>4.3</td>
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<tr>
<td>Jan. 1986</td>
<td>El Niño</td>
<td>Apr. 1987</td>
<td>10.3</td>
<td>21.3</td>
<td>8.9</td>
<td>2.4</td>
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<tr>
<td>Apr. 1986</td>
<td>El Niño</td>
<td>Apr. 1987</td>
<td>8.6</td>
<td>8.3</td>
<td>11.8</td>
<td>8.7</td>
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more favourable than normal for the growth of $\delta \Phi \bar{q} \nabla \cdot \delta U / \epsilon$. During La Niña events, the growth factors have a less pronounced seasonal dependence. During the onset of La Niña, the regular seasonal warming of the central Pacific is suppressed, and so conditions for the growth of $\delta \Phi \bar{q} \nabla \cdot \delta U / \epsilon$ are less favourable than normal. Table 6 suggests that the potential for error growth via the mechanism described in section 5 will be greater during an El Niño than during a La Niña. This would suggest that El Niño events could be less predictable than La Niña events, although as yet there is little practical evidence to support this idea.

8. SINGULAR VECTORS AS PRECURSORS FOR ENSO EVENTS

By examining the structure of the fastest-growing singular vectors of the tangent-linear coupled model, we identified in section 5 a mechanism by which error growth can occur in the nonlinear coupled model. If errors or internally generated noise are present which project heavily onto the fastest-growing singular vector, then rapid error growth is likely to occur in the nonlinear system. In this section we will investigate how these errors are likely to evolve by examining how the singular vectors of the tangent-linear coupled model evolve in the nonlinear coupled model.

We note first of all that the development of $\delta T'$ in Figs. 8(e) and 8(f) resembles the development of SST anomalies observed during ENSO events. To demonstrate this, Fig. 11 shows time series of contour maps of $h$ and $T'$ during May, June and July 1972 from a coupled-model hindcast started in January 1972. The hindcast procedure was identical to that described in section 7, and the event that we are attempting to capture is the 1972 El Niño. Figure 11(g) shows a time series of the NINO3 index (i.e. $T'$ averaged over the region 5°N to 5°S, 90°W to 150°W) for this hindcast. According to Fig. 11(g), the SST anomaly in the central Pacific peaks around October 1972, which is in agreement with observation.

The maps of $h$ and $T'$ in Figs. 11(a)–(f) show the onset and development of the 1972 El Niño. These fields bear a striking resemblance to the development of the singular vector depicted in Fig. 8. In fact, this singular vector can act as a precursor for ENSO events in the nonlinear coupled model. To demonstrate this, we initialized the nonlinear coupled model with the $\delta h$ and $\delta u$ fields of the singular vector shown in Fig. 8, with $T' = 0$ everywhere. The initial conditions were scaled so as to yield SST anomalies which after 3 months are similar to those observed during the 1972 El Niño (Figs. 11(d)–(f)). Two experiments were performed, one with positive initial conditions (i.e. $h(0) = \delta h(0)$ and $u(0) = \delta u(0)$), and the other with negative initial conditions (i.e. $h(0) = -\delta h(0)$ and $u(0) = -\delta u(0)$). Time series of the resulting NINO3 index for these two experiments are shown in Fig. 11(h), and indicate that positive (negative) initial conditions evolve into an El Niño (La Niña) event. (We note that the coupled model does not support growing normal modes when $|W| = 6.5$ m s$^{-1}$, and so the singular vector of Fig. 8 cannot be viewed as the development of a normal mode.)

It would appear that errors in the nonlinear coupled model which project onto the fastest-growing singular vector have the potential to evolve into ENSO events. Error growth arising from this mechanism is, therefore, likely to give rise to unsuccessful forecasts in which ENSO events are forecast in the model that are not observed in the real world. It is also conceivable that the sign of the error could be such that it can suppress the development of an ENSO in the model by cancelling out a legitimate ENSO precursor already present in the model.

Using a series of linear Markov models as an approximation to the Zebiak and Cane (1987) model, Xue et al. (1994) have computed the singular vectors arising from these models. They too find that their fastest-growing singular vector evolves into an ENSO
Figure 11. Contour maps of the thermocline-depth anomaly, $h$ ((a)-(c)) and sea-surface-temperature anomaly, $T'$ ((d)-(f)) during the onset of the 1972 El Niño as predicted by the coupled model during a hindcast started in January 1972. The contour interval is 3 m for $h$ and 0.1 degC for $T'$. The darkest shading corresponds to negative values of $h$ and $T'$. Also shown are time series of the Niño3 index during (g) the January 1972 hindcast of ENSO, and (h) integrations of the coupled model when initialized with the positive and negative April–May–June singular vectors.
event. Palmer et al. (1994) also present some preliminary results for the fastest-growing singular vector which maximizes the variance of SST in the coupled model of Battisti (1988). They also find that this singular vector evolves into a structure resembling ENSO with fastest growth rate during April. The structures of the fastest-growing singular vectors described in section 5 and shown in Fig. 8 are consistent with those studied by Xue et al. (1994) and Palmer et al. (1994), although these authors give no information about the dynamics or energetics responsible for the growth of their perturbations.

9. SOME SENSITIVITY STUDIES

In this section we examine the sensitivity of the singular-vector spectrum of the tangent-linear coupled model to variations in several model parameters.

(a) Variations in optimal growth time, $\epsilon$

Equation (5) suggests that the singular-vector spectrum of the propagator $R(0, \epsilon)$ is dependent upon the time interval $\epsilon$. We can think of $\epsilon$ as the "optimal growth time" since a singular vector of $R$ will yield the optimal growth of the error norm $E$ in (5) over this time interval.

We have examined the sensitivity of the singular-vector spectrum of the tangent-linear coupled model to variations in $\epsilon$. To do this, we computed the singular vectors of the annual mean observed fields for $\epsilon = 2, 3$ and 6 months, with the ocean–atmosphere coupling strength $|W| = 6.5 \text{ m s}^{-1}$. The annual mean fields were used so as to remove the effects
of the seasonal cycle. The perturbation-energy growth factors of the first six members of the singular-vector spectrum for each period are given in Table 7. For $\epsilon = 3$ months, the distribution of the growth factors is similar to that in Table 1 (i.e. the growth factors of the seasonal cycle) with one singular vector dominating the spectrum. If $\epsilon$ is decreased to 2 months, Table 7 shows that $\lambda_1$ decreases and no longer dominates the spectrum. Also, the number of growing singular vectors increases. Increasing $\epsilon$ from 3 to 6 months causes $\lambda_1$ to increase and become more dominant, and the number of growing singular vectors decreases. Further experiment has revealed that there are no growing singular vectors of the annual mean climatology for $\epsilon > 16$ months.

While the shape of the perturbation-energy growth-factor spectrum in Table 7 is altered somewhat by variations in $\epsilon$, the dynamical structure of the singular vectors is not (not shown). The growth mechanism for the fastest-growing singular vector in each case is the same as that described in section 5.

(b) Variations in ocean–atmosphere coupling strength $|W|$  

The effects of varying $|W|$ in (12) on the singular-vector spectrum can be seen by comparing Table 1 and Table 5 which show the perturbation-energy growth factors of the singular vectors of the seasonal cycle when $|W| = 6.5$ m s$^{-1}$ and $|W| = 9.0$ m s$^{-1}$ respectively. In general, $\lambda_1$ varies approximately in proportion to $|W|$, while the smaller growth factors change little. Also, the number of singular vectors that grow in time is similar in each case.

(c) Seasonal variations in the thermocline coefficient, $\eta$  

Sea-surface-temperature anomalies, $T'$, develop in the coupled model in response to thermocline-depth anomalies, $h$ (cf. (Eq. (11))). In the real ocean, this arises because of the presence of mean equatorial upwelling, $\bar{\omega}$, which in turn is controlled by the zonal component of the mean surface wind stress, $\bar{r}$, acting on the ocean. Although $\bar{\omega}$ is present throughout the year in the central and eastern Pacific, its strength will fluctuate due to seasonal variations in $\bar{r}$. We can parametrize this effect by varying $\eta$ in (11) and (15). Seasonal variations in $\eta$ will obviously be reflected in the singular-vector growth rates of the tangent-linear coupled model. The influence of the seasonal cycle in $\bar{\omega}$ was, therefore, investigated by replacing $\eta$ with $\alpha\eta$ in (11) and (15), where the coefficient $\alpha$ varies seasonally, and is computed from the monthly mean climatological wind stress. The negative-feedback term $\epsilon T'$ in (11) and (15) is also associated with $\bar{\omega}$ and so we replace $\epsilon$ with $\alpha \epsilon$. The annual cycle of $\alpha$ is shown in Fig. 12, and a value of $\alpha > 1$ indicates that $\bar{\omega}$ is greater than its
annual mean value assumed in the parametrization of section 2(b). Figure 12 shows that the coefficient $\alpha$ has its minimum value during April. Table 8 shows the growth factors of the first six singular vectors for the seasonal cycle when seasonal variations in $\eta$ and $\varepsilon$ are included in the model. Comparing the values of $\lambda_1$ in Table 8 with the corresponding values in Table 1 where $\eta$ was held constant in time, we see that the seasonal variations in $\alpha$ are reflected in the differences in these growth factors. During the AMJ quarter, $\lambda_1$ is reduced slightly by a reduction in $\bar{w}$ at this time of the year ($\alpha < 1$), while at other times of the year $\bar{w}$ is greater than the annual average ($\alpha > 1$) and $\lambda_1$ increases accordingly. However, the changes in $\lambda_1$ are small, and Table 8 shows that $\lambda_1$ still exhibits a pronounced maximum during AMJ. Therefore, we conclude that seasonal variations in the sensitivity of SST to thermocline-depth anomalies will have little impact on error growth in the coupled model.

(d) Different treatments of the energy-weight matrix, $S$

In section 5(a) we found that the initial temperature structure, $\delta T'$, of a singular vector is dominated by short-wavelength features (cf. Fig. 8(d)). However, we also demonstrated that the evolution of a singular vector is insensitive to the initial structure of $\delta T'$. We will investigate this further in this section.
The singular vectors are solutions of the generalized eigenvalue Eq. (19). In section 5(a) we explained how the short-wavelength features in $\delta T'$ at initial time are associated with the eigenvectors with smallest eigenvalue of the energy-weight matrix $S_n$. This matrix has a blocked structure with separate blocks that are associated with the perturbation energy of the atmosphere and ocean. The short-wavelength eigenvectors which appear in $\delta T'$ are associated with the atmospheric block of $S_n$. By manipulating this block of the matrix, we can modify the eigenvectors of $S_n$ and so eliminate the short-wavelength features in the initial $\delta T'$ field.

We have considered two possible approaches to this problem. 
(i) Our first approach was to apply a 1-2-1 digital filter $F$ to $\delta T'$. This was done by solving the following positive-definite quadratic form for $\lambda$:

$$G^{-1} F^* R S_\tau R F G^{-1} \delta \phi(0) = \lambda \delta \phi(0)$$

(25)

where $G$ is the Cholesky factor of $S_n$, $\delta \phi = G \delta \psi$, and $F^*$ is the adjoint filter. Figures 13(a) and 13(b) show $\delta h$ and $\delta T'$ at time $t = 0$ for the fastest-growing singular vector computed from (25) during the AMJ quarter of the seasonal cycle. The SST $\delta T'$ is much smoother than that shown in Fig. 8(d) and the structure of $\delta h$ is essentially unchanged (compare Fig. 8(a) and Fig. 13(a)).

(ii) Our second approach was to compute $\delta T'$ on a coarser grid. This modifies the eigenvalue spectrum of $S_n$ as well as the structure of its shortest wavelength eigenvectors. The SST Eq. (15) in the calculations of section 5 was solved on a horizontal grid with 50 grid points along the equator in the Pacific Ocean. Figure 13 shows the singular vectors for the AMJ quarter of the seasonal cycle which result when we use 10 and 25 grid points to solve (15). Reducing the number of grid points yields a smoother $\delta T'$ as anticipated, and as before the structure of $\delta h$ is unchanged.

The perturbation-energy growth factors (not shown) of all three singular vectors displayed in Fig. 13 are very similar to one another and to those given in Table 1 for AMJ. The results of this section, therefore, confirm our earlier finding that the initial structure of $\delta T'$ is unimportant for the development of the dominant singular vector. The precursor for error growth exists within $\delta h$ and is essentially independent of the initial $\delta T'$.

10. DISCUSSION

It was Lorenz (1965) who first argued that the growth of errors in a numerical forecast model could be described by the singular vectors of the forecast-error norm. During the last decade or so, there has been a resurgence of these ideas which is beginning to change the way we think about the growth of errors and instabilities in dynamical systems. Using these ideas, we have examined the mechanisms which limit error growth and predictability in a coupled model of the ENSO.

We have found that errors can grow rapidly owing to the release of energy associated with penetrative-convection anomalies in the atmosphere. The most favourable conditions for error growth by this mechanism exist over regions where the SST $\geq 28^\circ$C, which occurs predominantly in the west and central tropical Pacific. Penetrative convection results from SST anomalies which develop in response to vertical movements of the main oceanic thermocline. The SST is most sensitive to thermocline movements in the east and central Pacific. As a result, conditions favourable for the growth of energy through perturbation penetrative convection only exist in the central Pacific where SST $\geq 28^\circ$C and where there is moderate sensitivity of SST to thermocline-depth anomalies.

We find that error growth via this mechanism is most pronounced in the observed seasonal cycle during the boreal spring. During this time of year, the zonal gradient of
Figure 13. Contour maps of the initial $\delta h$ and $\delta T'$ fields of the fastest-growing April–May–June singular vector when (i) a 1-2-1 filter is applied to $\delta T'$ ((a) and (b)), (ii) the number of SST grid points is 10 ((c) and (d)) and 25 ((e) and (f)). The contour interval is 1.5 m for $\delta h$ and 0.2 degC for $\delta T'$. The darkest shading corresponds to negative values of $\delta h$ and $\delta T'$. See text for explanation of symbols.
SST along the equator is a minimum, and the west Pacific warm pool has its maximum extension eastwards. The area of water in the central Pacific with SST \( \geq 28 \, ^\circ \text{C} \) is therefore greatest at this time of the year, which creates the most favourable conditions for error growth. The boreal spring also corresponds to the time of the ‘predictability barrier’ of the ENSO which has been identified from experiment. Around this time, the skill of model forecasts of ENSO is found to fall dramatically, regardless of when a forecast is started. Our results indicate that one possible cause of the predictability barrier is associated with the seasonal cycle of SST in the tropical Pacific which modulates the potential for error growth by atmospheric penetrative convection, thereby creating conditions which are more or less favourable for this process to occur at different times of the year.

The seasonal cycle of SST in the central Pacific will be influenced by a number of factors, which include equatorial upwelling in the ocean and surface heat fluxes (Köberle and Philander 1994) as well as the strength of the Asian monsoon which is responsible for maintaining the tropical Pacific trade winds. An understanding of how these environmental factors influence the annual cycle of SST will, therefore, further advance our understanding of the predictability of ENSO.

The ENSO cycle itself also exerts a strong influence on SST, and we have found that the onset and development of El Niño and La Niña events can significantly affect the potential for error growth in the coupled model depending upon how such an event is phase locked with the seasonal cycle. In general, we find that conditions become more favourable for error growth during the onset of El Niño because the regular warming of the central Pacific during the boreal spring is enhanced. During the onset of La Niña, however, conditions are less favourable for error growth, because the spring warming of the central Pacific is suppressed. These results suggest that El Niño events may be less predictable than La Niña events. The findings of Davey et al. (1994) lend support to our results. Using a GCM of the tropical Pacific Ocean coupled to an empirical atmospheric model, they found that the hindcast skill of ENSO events was greatest during the transition from warm to cold conditions in the central Pacific.

We have found that the fastest-growing disturbances in the coupled model can act as precursors for ENSO events. Therefore, when error growth occurs in the model, it is likely to manifest itself as an ENSO event whose development is different to events which are observed.

Our sensitivity studies have revealed that the mechanism for error growth that we have identified is a robust feature of the coupled system which is likely to manifest itself in other coupled models. In fact, the results of other experiments (not presented here) concerned with the effects of varying the ocean–atmosphere coupling frequency indicate that the potential for error growth in coupled GCMs could be far greater than that suggested here. These experiments highlight the importance of local coupling processes and ocean–atmosphere feedbacks in controlling the growth of errors. In coupled GCMs, where the component models are coupled at least once every day, the potential for error growth could be very large, not only because of the mechanism discussed in this paper, but also because of other complex atmospheric processes associated with radiation, moisture and clouds. There is some evidence for this in the coupled model studied by Gent and Tribbia (1993).

We are continuing our investigation of the effects of ocean–atmosphere coupling frequency on error growth and our findings will be presented at a later date.

Our results show that nonlinearities in the atmospheric heating can influence error growth in the coupled system and the predictability of ENSO. The model of Zebiak and Cane (1987) includes a nonlinearity which is different to that used in the coupled model employed here. Heating anomalies due to moisture convergence develop in the Zebiak and Cane model only when the total wind field (mean + anomaly) is convergent, namely when
\( \nabla \cdot (\overline{U} + U) \leq 0 \) (Zebiak 1986). The seasonal dependence of the hindcast skill of this model could be explained in terms of error growth resulting from a mechanism similar to that described above. All that would be required for error growth associated with moisture convergence is \( \nabla \cdot (\overline{U} + U) \leq 0 \). During spring, \( \nabla \cdot \overline{U} \leq 0 \) in the central Pacific where SST is increasing as part of the normal seasonal cycle. In addition, during the onset of El Niño, the normal warming of the central Pacific is enhanced and \( \nabla \cdot U \leq 0 \) (see Zebiak 1986, Figs. 4 and 5). Therefore, in the Zebiak and Cane model, wind errors \( \delta U \) resulting from SST perturbations will induce atmospheric heating perturbations in the tropical Pacific (which in their model are, to first order, proportional to \( \nabla \cdot (\overline{U} + U + \delta U) - \nabla \cdot (\overline{U} + U) \)), and so error growth can occur. The same arguments could be applied during La Niña events in the Zebiak and Cane model providing that \( \nabla \cdot (\overline{U} + U) \leq 0 \).

There are of course a number of important caveats which must be placed upon the work presented here, and which are listed below.

(i) It is yet to be determined how model dependent our results are, although our sensitivity studies strongly suggest that they are applicable to other models currently used to predict ENSO.

(ii) In this study, we have not investigated the possible mechanisms of error growth associated with horizontal advection and upwelling in the ocean. Based upon physical arguments, the influence of these processes on SST were not included in the model used here. Evidence strongly suggests that, for the most part, it is vertical movements of the main oceanic thermocline which control the development of SST anomalies during ENSO events. However, not all ENSO events develop in the same way so it is possible that horizontal advection and upwelling could play an important role during some events. Certainly tropical SST anomalies in coupled GCMs will always be influenced by these processes to some degree (e.g. Lau et al. 1992; Latif et al. 1994; Moore 1995). We are presently investigating the role that horizontal advection and upwelling in the ocean can play in controlling error growth in coupled models.

(iii) We have only considered the problem of the predictability of ENSO associated with the growth of errors or system noise. However, it is possible that the coupled ocean–atmosphere system possesses a number of preferred, yet different, circulation regimes, related to the regime centroids which have been identified in the atmosphere. Experience in NWP indicates that transitions between these different flow regimes can limit the predictability of the atmosphere (Palmer 1993; Palmer et al. 1994), and we believe that investigations along these lines for the coupled system are certainly warranted.

It is our belief that we can learn much more about the predictability of ENSO by using ensemble forecasting techniques. Recent experience at the ECMWF has shown that when singular vectors are used as perturbations to model initial conditions, ensemble forecasts yield valuable information about the predictability of the atmospheric circulation (Mureau et al. 1993; Buizza et al. 1994). In addition, adding singular-vector disturbances to the model initial conditions has been found to improve some forecasts where the singular vectors have acted as precursors for events that were otherwise miss-forecast. Work along these lines is in progress using the K93 model, and will be described elsewhere.

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