Introduction of a local mapping factor in the spectral part of the Météo-France global variable mesh numerical forecast model

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SUMMARY

For variable horizontal resolution simulations, a new formulation of spectral computations has been implemented in the ARPEGE/IFS spectral global numerical weather forecast model. The formulation makes use of a property of the conformal transformation which defines the stretched geometry: the local mapping factor is a combination of zero and first-order Legendre polynomials only. Consequently, the mapping factor dependency in the spectral part of the computations can be introduced by simple extra multi-diagonal operator multiplication. This idea is applied to both the semi-implicit and the horizontal diffusion schemes to eliminate problems occurring when variable resolution is used together with either long time-steps or steep orographies. The positive impact of these modifications is presented for semi-Lagrangian and high-resolution integrations. The relatively small extra computation and memory cost of the modified schemes is well suited to operational applications.

KEYWORDS: Numerical weather prediction Semi-Lagrangian methods Spectral models Variable resolution

1. INTRODUCTION

The ARPEGE/IFS system (Courtier et al. 1991), has been developed since 1987 as a cooperative project between Météo-France and the European Centre for Medium-range Weather Forecasts (ECMWF), and is now used operationally for Numerical Weather Prediction (NWP) in these two centres. Its main characteristics are a spectral horizontal representation of the fields with a spherical harmonic function basis, a hybrid sigma-pressure based coordinate, and vorticity-divergence as prognostic horizontal dynamical variables in spectral space. Apart from these main aspects, the complete ARPEGE/IFS system includes a large variety of geometrical and numerical options inside a unique code kernel, which is used for research (including climate), development, and operational purposes.

The forecast model contains options for Eulerian and different types of semi-Lagrangian transport schemes, leap-frog and two-time-levels time-discretization, 3-D primitive equation and 2-D shallow water system, etc. Similarly, the model can be applied over a global or limited-area domain (this latter option using bi-Fourier decomposition).

ALADIN, the limited area model (LAM) version of ARPEGE/IFS has been developed at Météo-France in collaboration with nine countries of Central and Eastern Europe and Morocco. It provides routine forecasts for these countries with the hydrostatic option, but a fully elastic option based on a hydrostatic-pressure hybrid vertical coordinate has also been implemented (Bubnová et al. 1995).

The global version is run operationally with a Gaussian mesh at ECMWF, but at Météo-France, due to the specific interest in short-range high-resolution forecasts, we use a variable horizontal mesh, performed by choosing a high-resolution pole, then applying a conformal transformation (Schmidt 1977) to increase the resolution in the area of interest centered on the high resolution pole, after defining a stretching coefficient. The application of Schmidt’s transformation to the problem of spectral global modelling is extensively described in Courtier and Geleyn (1988).

The operational version of ARPEGE/IFS used at Météo-France at the beginning of 1995 was a three-dimensional primitive equation Eulerian model T119 L24 c3.5 (triangular

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truncation 119, 24 vertical levels and a 3.5 stretching coefficient), with the high-resolution pole of the geometrical transformation located near the geographical centre of France (46.47°N, 2.58°E). This means that the horizontal resolution of the associated Gaussian grid smoothly varies from 32 km over the centre of France to 389 km at the low resolution pole, situated at the opposite point on the sphere, near New Zealand. With this method, a high resolution over Europe comparable to those of LAMs can be achieved, whilst avoiding the shortcomings of forcing by lateral boundary conditions. In October 1995, the resolution was increased to T149 L27 c3.5 with the use of a three-time-level vertical interpolating semi-Lagrangian scheme, the other characteristics of which are the same as in Ritchie et al. (1995).

A particularity arises due to the stretching transformation, in the same manner as in spectral LAMs working in a projected geometry: the spectral part of the equations is written in the transformed coordinate system while the grid-point part is treated in the geographical coordinate system. In the spectral part of the model computations, the horizontal derivatives are thus expressed with respect to the transformed coordinates, a form which is termed as ‘rescaled’ as opposed to the ‘geographical’ form in which horizontal derivatives are expressed with respect to the natural geographical coordinates. The choice of a conformal geometrical transformation leads to a very simple relationship between the rescaled first-order differential operator $\nabla'$ and its geographical counterpart $\nabla$:

$$\tilde{\nabla} = M \nabla'$$  \hspace{1cm} (1)

where $M$ denotes the local mapping factor (distance on the transformed sphere/distance on the geographical sphere) at the considered location.

As the vorticity and divergence fields are obtained as second-order derivatives of scalar fields (i.e. stream function $\psi$ and velocity potential $\chi$), the kinematic state variables of the model will be the rescaled vorticity and divergence:

$$\zeta' = \tilde{\nabla}^2 \psi$$
$$D' = \tilde{\nabla}^2 \chi$$  \hspace{1cm} (2)

It should be noted that due to spherical harmonic properties, the rescaled horizontal Laplacian operator is diagonal in the spectral space. As classically done in spectral models, the spectral part of the time-step integration includes horizontal derivatives, horizontal diffusion and semi-implicit corrections computations, while physical and dynamical explicit sources are computed in grid-point space, thus requiring spectral transforms between these two spaces (Orszag 1970; Machenhauer and Haugen 1987). A time-step of the model goes as follows: starting in spectral space, all necessary horizontal derivatives are computed from the $t$ time level, then transformed to grid-point space via ‘inverse’ spectral transforms. The computation of dynamical and physical tendencies in the grid-point space is then performed. After a ‘direct’ spectral transform to return to spectral space, the semi-implicit scheme (including solution of the Helmholtz semi-implicit equation) is solved in spectral space, and $t + \Delta t$ time level becomes $t$. The spectral part of the computations ends by an application of the horizontal diffusion scheme.

The point which is examined in this paper is the extension of formulations of the spectral horizontal diffusion and the semi-implicit scheme to the case of stretched geometry. Courtier and Geleyn (1988) made the assumption that formulations valid for constant mesh could be extended for variable mesh, at least for the shallow-water version of ARPEGE. A preliminary version of ARPEGE (2-D and 3-D models) was used until late 1992 with
these hypotheses, but a significant level of noise over topography and dramatic instabilities especially in semi-Lagrangian integrations have led us to test different ways of taking into account the variations of the mapping factor in spectral computations. The cheapest way, in terms of computation time and memory, to reformulate spectral computations has been found after several trials and consists in using an interesting property of Schmidt's (1977) conformal transformation: the mapping factor can be written as a combination of Legendre polynomials of orders zero and one only, implying that multiplications or divisions by low-degree polynomial expressions of the mapping factor in spectral space can be performed by low-multidiagonal matrix products in spectral space, thus resulting in a minor increase in computational cost and memory.

The aim of this paper is to describe how this property has been used to reformulate spectral computations for operational semi-implicit and horizontal diffusion schemes. The spectral expression of a scalar multiplication by a low-order polynomial of the mapping factor is described in section 2. In section 3 the assumptions of Courtier and Geleyn (1988) concerning the formulation of the semi-implicit scheme are presented, and it is shown why they are not satisfactory, especially with the large time-steps used within semi-Lagrangian schemes. Then there is a description of how the semi-implicit scheme can be modified to correctly take into account the variations of the mapping factor. In section 4 it is shown that a unique horizontal diffusion scheme on the transformed sphere cannot accurately describe homogeneous horizontal mixing, especially in the stretched zone. A new horizontal diffusion scheme, which is a combination of two operations, is then described. Finally, a short conclusion is given in section 5.

2. LOCAL MAPPING FACTOR IN SPECTRAL SPACE

In the transformed space, the expression for the local mapping factor $M$ is given by:

$$M(\lambda, \xi) = a + b\xi$$  \hspace{1cm} (3)

where $\lambda$, $\xi$ are respectively the longitude and the sine of latitude on the transformed sphere, $a = (c^2 + 1)/2c$, and $b = (c^2 - 1)/2c$ ($c$ is the stretching coefficient, presently set to 3.5 in the operational version as mentioned above). From (3) it can be seen that the mapping factor varies from $c$ at the high resolution pole of the transformed sphere, to $1/c$ at the opposite pole, and that the resolution ratio between the two poles is $c^2$. In spectral space, a useful basic property of Legendre polynomials is:

$$\xi P_{(m,n)} = e_{(m,n)}P_{(m,n-1)} + e_{(m,n+1)}P_{(m,n+1)}$$  \hspace{1cm} (4)

where $n$ and $m$ are respectively the total and zonal wave numbers ($|m| \leq n$), and $e_{(m,n)} = \sqrt{(n^2 - m^2)/(4n^2 - 1)}$. The Legendre polynomials considered here are the normalized Legendre functions of the first kind, satisfying the following normalization relationship:

$$\frac{1}{2} \int_{-1}^{1} (P_{(m,n)})^2 \, d\xi = 1$$  \hspace{1cm} (5)

Due to the completeness and orthogonality of the $P_{(m,n)}$ basis, Eq. (4) remains valid replacing Legendre polynomials by the spectral coefficients of any horizontal field $X$:

$$\xi X_{(m,n)} = e_{(m,n)}X_{(m,n-1)} + e_{(m,n+1)}X_{(m,n+1)}$$  \hspace{1cm} (6)

The combination of Eqs. (3) and (6) gives the expression for the mapping factor multiplicative spectral operator:

$$[MX]_{(m,n)} = be_{(m,n)}X_{(m,n-1)} + aX_{(m,n)} + be_{(m,n+1)}X_{(m,n+1)}$$  \hspace{1cm} (7)
and thus:

\[
[M^2X]_{mn} = b^2 e_{mn} e_{m,n-1} X_{m,n-2} + 2abe_{mn} X_{m,n-1} + (a^2 + b^2(e^2_{mn} + e^2_{m,n+1})) X_{mn} + 2abe_{m,n+1} X_{m,n+1} + b^2 e_{m,n+1} e_{m,n+2} X_{m,n+2}
\]

(8)

These expressions show that multiplications by \( M \) or \( M^2 \) of any horizontal spectral field can be performed by mean of the multiplication by a tridiagonal or pentadiagonal matrix for each of its zonal wave number column vector \( X_{m,m} \). Since the matrix coefficients are 'mathematical constants', this process can be very highly optimized and thus lead to a very small computational overhead. This simple idea is developed and applied in the rest of this paper.

3. Application to the geographical divergence formulation for the semi-implicit scheme

The original semi-implicit scheme will be presented in this section and it will be shown why it cannot be conveniently used with large time-steps. The new formulation will then be discussed together with some results.

The linear terms responsible for gravity-wave generation are treated in an implicit way, in order to allow a time-step length compatible with operational utilization (Robert et al. 1972, Robert 1981). The semi-implicit temporal discretization is designed as an explicit scheme with implicit correction. This means that the leap-frog time discretized form of the scheme can be written symbolically as:

\[
X_{t+\Delta t} = X_{t-\Delta t} + 2\Delta t M X_t + 2\beta \Delta t \Lambda \left( \frac{X_{t+\Delta t} + X_{t-\Delta t}}{2} - X_t \right)
\]

(9)

where \( X \) represents the state of the model, \( M \) the explicit model evolution, \( \Lambda \) the linear model containing the physics of the gravity waves. The model time-step is denoted by \( \Delta t \) and \( \beta \) is the semi-implicit correction coefficient. In this paper, the discussion will be illustrated on the leap-frog version of the equations, but the same principles and conclusion are strictly valid for any other temporal scheme (e.g. two-time-level scheme). When \( \beta = 0 \), the scheme becomes explicit, while \( \beta = 1 \) gives the complete semi-implicit scheme. Since horizontal advection is entirely explicit, the above symbolic formulation applies to both Eulerian or the semi-Lagrangian version of the model. Grouping all the \( t + \Delta t \) terms on the left-hand side, (9) can be rewritten as a linear equation for \( X_{t+\Delta t} \):

\[
(X_{t+\Delta t} - \beta \Delta t \Lambda X_{t+\Delta t}) = \mathcal{K}^*.
\]

(10)

This equation is solved in spectral space, where it takes a simple form. The right-hand side term \( \mathcal{K}^* \) represents the whole explicit part and the \( t - \Delta t \) and \( t \) parts of the temporal Laplacian corrections terms and is calculated in the grid-point space before being transferred to spectral space via spectral transforms.

The basic state used to perform linearization is isothermal (with temperature \( T \)), dry and at rest, without orography, thus resulting in a uniform surface pressure \( \Pi \). No implicit contributions are assumed for moisture and vorticity, thus the linear system \( \Lambda \) only contains three equations, for divergence, temperature, and surface pressure. This minimal set has been extensively shown to be sufficient for fast-gravity-wave control and
for conservation of basic invariants (Simmons and Burridge 1981). As mentioned in the introduction, all horizontal derivatives quantities (such as divergence, vorticity, and all other scalar derivatives) are in rescaled form in spectral space.

(a) Original semi-implicit scheme

The original ARPEGE semi-implicit scheme follows the assumptions of Courtier and Geleyn (1988) by using a linearized value \( \hat{M} \) for the mapping factor in spectral space calculations instead of the local value \( M \). The chosen value \( \hat{M} = \text{Max}(M) = c \) insures a fully semi-implicit treatment of the waves at the high resolution pole.

The starting point for this discussion is the spectral linear system \( \Lambda \) in temporally, horizontally and vertically discretized form (see Courtier et al. 1991 for further details).

The vertical domain is divided into \( L \) layers indexed by \( \ell \) (\( \ell \) ranging from 1 at the top to \( L \) at the bottom of the domain). These layers are vertically separated by \( L + 1 \) interfaces indexed by \( \bar{l} \) (\( \bar{l} = 0 \) at the top to \( \bar{l} = L \) at the bottom).

For a given spectral location \( (m, n) \) and time level \( t \), the 3-D prognostic perturbation variables are treated as vertical column vectors \( X_{(m,n,\ell)}^t = (X_{m,n,1}^t, \ldots, X_{m,n,L}^t) \) while the 2-D perturbation of surface pressure is treated as a scalar value. The spectral index \( (m, n) \) is dropped in the following equations for clarity.

If variables \( T, D' \) and \( \Pi \) respectively denote the perturbation of temperature, rescaled divergence and surface pressure, the linear system \( \Lambda \) can be written as:

\[
\Pi_{t+\Delta t} + \beta \Delta t \bar{M}^2 v D'_{t+\Delta t} = \mathcal{F}^* \tag{11}
\]

\[
D'_{t+\Delta t} + \beta \Delta t \bar{v}^2 (\gamma T_{t+\Delta t} + \mu \Pi_{t+\Delta t}) = \mathcal{D}^* \tag{12}
\]

\[
T_{t+\Delta t} + \beta \Delta t \bar{M}^2 \tau D'_{t+\Delta t} = \mathcal{F}^* \tag{13}
\]

where \( \mathcal{F}^*, \mathcal{D}^* \) and \( \mathcal{F}^* \) represent the total ‘grid-point’ part of the scheme (see Eq. (10)) and \( \tau, v, \gamma, \mu \) are linear vertical operators defined as in Courtier et al. (1991).

Elimination of Eqs. (11)–(13) for divergence yield the following Helmholtz equation:

\[
(1 - \beta^2 \Delta t^2 \bar{M}^2 B \bar{v}^2) D'_{t+\Delta t} = \mathcal{D}^* - \beta \Delta t \bar{v}^2 (\gamma \mathcal{F}^* + \mu \mathcal{F}^*). \tag{14}
\]

The matrix \( B = \gamma \tau + \mu v \) is therefore a \( L \)-order vertical operator, the diagonalization of which can be performed once at the beginning of the integration:

\[
B = QAQ^{-1}. \tag{15}
\]

Since \( A \) is an \( L \)-order diagonal operator containing the eigenvalues \( a_l \) of gravity modes given by the columns \( Q_l \) of \( Q \), the ‘horizontal’ and ‘vertical’ parts of the Helmholtz equation (14) can now be separated in the eigenvector space. For each gravity mode \( l \), an uncoupled equation has to be solved:

\[
(1 - \beta^2 \Delta t^2 a_l \bar{M}^2 \bar{v}^2) [Q D'_{t+\Delta t}]_l = [Q (\mathcal{D}^* - \beta \Delta t \bar{v}^2 (\gamma \mathcal{F}^* + \mu \mathcal{F}^*))]_l. \tag{16}
\]

Due to the diagonality of the rescaled Laplacian in spectral space, the solution of such an equation is equivalent to a set of divisions (or to the inversion of a diagonal horizontal operator if spectral coefficients are grouped according to their zonal wave number) \( m \). The inversion of (16), and a multiplication by \( Q^{-1} \) gives \( D'_{t+\Delta t} \). Finally, a direct elimination in Eqs. (11) and (13) ends the semi-implicit calculations by yielding \( \Pi_{t+\Delta t} \) and \( T_{t+\Delta t} \).
(b) Shortcomings of the original semi-implicit scheme

In this algorithm $D$ is replaced by $\tilde{M}^2 D$ in all parts of the linear system (i.e. also in the linear part of the right-hand sides of Eqs. (11) and (13)). This results in an overestimation of semi-implicit corrections by a factor $\tilde{M}^2 / M^2$ at each location. For the usual value $\tilde{M} = c = 3.5$, the overestimation reaches a factor of 152 at the low-resolution pole. As a consequence, the local displacement of each gravity mode is bounded by semi-implicit tendencies to one minimal grid-mesh per time-step, in the whole domain. This condition is far too restrictive in the low-resolution area and results in a dramatic distortion of gravity-wave propagation.

The semi-implicit corrections are proportional to $(\Delta t)^2$. The overestimation of the corrections does not therefore cause problems where the time-step is short, as it will typically be if Eulerian advection is used.

On the other hand, when using the large time-steps allowed by a semi-Lagrangian scheme, the semi-implicit correction terms become important and reach an unrealistic magnitude near the low-resolution pole, especially over mountainous areas such as Antarctica. Figure 1 shows 72 h semi-Lagrangian T119 c3.5 forecasts of 850 hPa temperature over Antarctica (near the low resolution pole in our case), from 15 July 1994 00 UTC with a 3 minute (Fig. 1(a)) and a 15 minute (Fig. 1(b)) time-step. The comparison of the two simulations shows that with a time-step of 15 minutes, the unrealistic semi-implicit corrections result in a spurious trend toward cold temperatures over Antarctica, leading to erroneous low temperatures in this region (the maximum deviation from the reference analysis reaches $-69$ K). This phenomenon is not present in the simulation with the 3-minute time-step simulation.

(c) The new method for the semi-implicit scheme

Contrary to the assumption of Courtier and Geleyn (1988), the semi-implicit correction is now written with the geographical divergence in the temperature and continuity equations. The new semi-implicit system is therefore obtained by replacing $\tilde{M}$ by $M$ in (11)–(13), which yields the new Helmholtz equation in gravity eigenmode space, for each mode $l$:

\[
(1 - \beta^2 \Delta t^2 \alpha / \tilde{\nabla}^2 M^2) [Q D_{r+\Delta t}^*]_l = [Q(\tilde{\nabla}^2 - \beta \Delta t \tilde{\nabla}^2 (\gamma T^* + \mu P^*))],
\]

as a replacement for Eq. (16) (note the position of $M$ with respect to $\tilde{\nabla}^2$ due to the non-commutativity of these two horizontal operators).

Because of the property of the mapping factor previously described in section 2 by Eq. (8), $(1 - \beta^2 \Delta t^2 \alpha / \tilde{\nabla}^2 M^2)$ can be written as a non-symmetrical matrical pentadiagonal operator for each zonal wave number $m$. The solution of the Helmholtz equation is therefore equivalent to the inversion of a set of $(N + 1)$ pentadiagonal matrices for each mode $l$, where $N$ is the maximum total wave number. These inverse matrices could be pre-computed before the integration, but this would require large amounts of memory since the inverse matrices are no longer pentadiagonal but full matrices. The solution which has been chosen in practice is thus to perform an ‘LU’ decomposition of each pentadiagonal matrix into products of a lower triangular tridiagonal matrix by an upper triangular tridiagonal matrix at the beginning of the integration. The obtained tridiagonal matrices can then be stored without high memory requirements. At each time-step and zonal wave number $m$ a lower triangular tridiagonal linear system and an upper triangular tridiagonal linear system have then to be solved successively.
Additional memory savings are obtained for non-zero zonal wave numbers \( m \). Since \( \tilde{\nabla}^2 \) becomes an invertible diagonal matrix, it can be factorized outside the Helmholtz equation, which then becomes:

\[
(\tilde{\nabla}^{r-2} - \beta^2 \Delta t^2 a_i M^2)\{Q D_{i+\Delta t}^r\} = [Q(\tilde{\nabla}^{r-2} \Omega^* - \beta \Delta t (\gamma \mathcal{J}^* + \mu \mathcal{B}^*))]_t
\]  (18)
The decomposition of the symmetrical pentadiagonal matrical operators \((\hat{V}_{r}^{-2} - \beta^2 \Delta t^2 a_i M^2)\) thus leads to identical upper and lower operators, allowing the storage of 3 diagonals instead of 5.

\(M^2 D'_{t+\Delta t}\) is computed using formula (8) after computing \(D'_{t+\Delta t}\) by a multiplication with \(Q^{-1}\). Once known \(D'_{t+\Delta t}\) and \(M^2 D'_{t+\Delta t}\), \(\Pi_{t+\Delta t}\) and \(T_{t+\Delta t}\) are computed with the new formulation of Eqs. (11)–(13).

\[\text{(d) Results with the new semi-implicit scheme}\]

The largest advantage is to be found in semi-Lagrangian integrations as discussed in section 3(b). Figure 2(a) shows the 850 hPa temperature field (T850) after 72 h for the same semi-Lagrangian experiment as in Fig. 1(b), except for the use of new semi-implicit scheme instead of the original one. Compared to the reference analysis (Fig. 2(b)), the maximum deviation of T850 is now bounded between \(-19.9\) K and \(15.5\) K (instead of \(-69.7\) K and \(19.5\) K) and the global root-mean-square (RMS) have now been reduced to reasonable values (2.8 K instead of 6.2 K). Moreover, no systematic trend could be detected through the forecast duration. The semi-implicit instability has thus been completely removed from semi-Lagrangian integrations with the new scheme. The differences between Figs. 1(a), 2(a) and 2(b) show that with the large time-step, temporal truncation errors become significant in the low-resolution area, as suggested by the improved results obtained with the new scheme with \(\Delta t = 3\) minutes (not shown) when compared to the original one. However, with the time-step chosen for operational configuration (e.g. 8 minutes for T149), this problem disappears.

When using small time-steps as in Eulerian mode, no stability problems are encountered with the original semi-implicit scheme, but an improvement in the forecast scores can be expected with the new scheme especially near to the low-resolution pole, resulting in an improvement of global scores. This idea has been tested on a set of six independent situations in 1994: 15/01; 15/03; 15/05; 15/07; 15/09; 15/11. Forecasts on this set of situations have been performed with both versions of the semi-implicit scheme, and the global scores have been computed with the operational reference analyses. All integrations have been done with a version of ARPEGE similar to the operational version used during spring 1995, from 00 UTC at 72 h range with a 200 s time-step. The 500 hPa and 200 hPa geopotential height (Z500 and Z200) global score averages are compared for the experimental set (new semi-implicit scheme) and the reference set (original semi-implicit scheme) forecasts. The results, presented in Fig. 3, show that the scoring advantage for the new semi-implicit scheme progressively increases with the forecast range. At 72 h range, RMS for Z500 and Z200 have decreased by 3% and 6.5% respectively (Figs. 3(a) and 3(b)), while the bias improvements are respectively 21% and 24% (Figs. 3(c) and 3(d)). Comparable results are obtained for other fields: global average scores for mean sea level pressure, 850 hPa and 500 hPa temperature and humidity. However, as for the semi-Lagrangian forecasts, the two semi-implicit formulations give similar scores over the domain \(\xi > \sqrt{2}/2\) covering western Europe (not shown).

The new semi-implicit scheme has been introduced in the operational Eulerian model (T119 L24 c3.5) in March 1995, resulting in a significant improvement to the scores, not only in the low-resolution hemisphere, but also in the high-resolution hemisphere (positive feedback due to the assimilation cycle).
4. Application to Homogeneous Horizontal Diffusion

(a) Original scheme for diffusion

The expression of the mapping factor in spectral space can also be used to improve the efficiency of the horizontal diffusion for the variable mesh version of ARPEGE. In
spectral models, the horizontal diffusion scheme has two aims:

- Simulate the sub-grid scale horizontal mixing which takes place in the real atmosphere.
- Avoid accumulation of energy at the small-scale end of the spectrum which contaminates lower wave numbers and represents noise in the forecast fields.
In the regular mesh version of ARPEGE as in other spectral models, horizontal diffusion is performed by applying in spectral space a $\hat{\nabla} r$ operator on prognostic variables (Sardeshmukh and Hoskins 1984), where $r$ is the order of horizontal diffusion. The order $r$ is generally chosen as an even integer number, but this constraint is not compulsory for spectral models because of the form of the Laplacian operator in spectral space. For any variable $X$, the effect of horizontal diffusion is then given by:

$$\frac{\partial X}{\partial t} = -K_X \hat{\nabla}^r X \quad (19)$$

where $K_X$ is a horizontally constant coefficient which can optionally have vertical variations. Due to the diagonality of the horizontal Laplacian operator in spectral space, the implicit discretisation of Eq. (19) is equivalent to a set of divisions for each zonal wave number $m$. Such a scheme is convenient in spectral global models because of its low cost, and is used in constant resolution versions of ARPEGE/IFS (for example in the T213 L31 operational version used at ECMWF).

Courtier and Geleyn (1988) have made the assumption that such a scheme could be still used in a variable-mesh spectral model, replacing the geographical derivative operator by the rescaled derivative operator in Eq. (19), which then becomes:

$$\frac{\partial X}{\partial t} = -K_X \hat{\nabla}^r X \quad (20)$$

The horizontal diffusion scheme defined by (20) is equivalent to:

$$\frac{\partial X}{\partial t} = - \frac{K_X}{M^r} \hat{\nabla}^r X \quad (21)$$

and thus cannot represent homogeneous horizontal mixing which occurs in the atmosphere, in particular if a high order ‘$r$’ is used. This formulation leads to insufficient damping in the high-resolution hemisphere. The asymmetry between the two hemispheres increases with the stretching coefficient, but also with the order of the horizontal diffusion derivative operator, and can become very large. For example, with $r = 5$ and $c = 3.5$, $M^r$ varies from 1/525 to 525.

Insufficient damping in the high-resolution hemisphere can lead to noisy forecasts, in particular in cases when a jet lies over a mountainous region. To illustrate this phenomenon, a semi-Lagrangian experiment is presented with increased horizontal resolution (T127 c3.5), in which the sharpness of the orography near the area of interest is significantly enhanced compared to the T119 c3.5 operational one. The time-step is set to 15 minutes.

Figure 4 shows a 12 h semi-Lagrangian forecast of Z500 (geopotential height at 500 hPa) from the analysis of 25 February 1989 00 UTC over Europe, with the original type of horizontal diffusion scheme defined by a fifth-order operator. In this experiment, a strong north-westerly jet lies over the Cantabrian Mountains which appear as a sharp 1000 m high cliff in the model orography. Without additional geographical diffusion, the flow is affected by strong oscillations centered near the coastline but extending toward the ocean and the north-easter part of the peninsula. These oscillations are also present in Eulerian mode with a time-step of 3 minutes (not shown) which indicates that they are not due to a spurious resonant response of the semi-Lagrangian discretization to orographic forcing (Rivest et al. 1994).
(b) Unified scheme for diffusion

A homogeneous horizontal diffusion scheme is added to represent correctly homogeneous mixing which exists in the atmosphere. So formula (20) becomes:

$$\left( \frac{\partial X}{\partial t} \right)_u = -\left( K_X \hat{V}^\nu + K_{uX} M^u \hat{V}^\nu \right) X$$

(22)

The horizontal diffusion scheme now has two components:

- The first component $-K_X \hat{V}^\nu X$ is kept as a numerical component to damp accumulation of energy at the end of spectrum.
- The second component $-K_{uX} M^u \hat{V}^\nu X$ called the 'geographical component' is used to represent homogeneous damping.

Orders $r$ and $u$ can be different. The vertical dependencies of $K_X$ and $K_{uX}$ are identical and chosen in order to have a correct smoothing in the stratosphere and to avoid some spurious reflection of waves at the top boundary of the model.

The implicit temporal discretization of this equation results in the inversion of a banded matrix with $2u + 1$ non-zero diagonals for each zonal wave number $m$. This problem could practically be solved in a similar way as presented in section 3 by mean of 'LU' factorizations. However, for the typical values of $u$ needed to obtain a sufficiently selective diffusion operator ($u \simeq 5$), this method would require excessive computational and storage costs (the problem would be 11-diagonal). The idea used to reduce the size of the problem is to replace the function $M^u$ in Eq. (22) by an approximated form $\tilde{M}^u(\xi)$ written as a
low degree polynomial of $\xi$ as proposed by M. Déqué (personal communication). For our purpose, a second-degree polynomial in $\xi$ approximation has been chosen, and appears to be satisfactory for approximating the $M^u$ function up to $u \simeq 5$, in a sense that the violation of homogeneity of horizontal mixing due to this approximation remains weak. The expression of $\tilde{M}^u(\xi)$ is:

$$\tilde{M}^u(\xi) = a_0 + a_1\xi + a_2\xi^2$$  \hspace{1cm} (23)

where the constants $a_0$, $a_1$ and $a_2$ are computed in order to minimize the squared difference between $M^u$ and $\tilde{M}^u$, with constraints on the value of $\tilde{M}^u(-1)$ and on the positive sign of $d\tilde{M}^u(\xi)/d\xi$, while $c_e = [M^u(1)]^{1/3}$ is kept free.

The least square minimization takes into account the mapping factor, so the high-resolution hemisphere has an increased weight compared to the low-resolution hemisphere. The expression for the approximation of $[M^u X]_{(m,n)}$ is:

$$[\tilde{M}^u X]_{(m,n)} = a_2 e_{(m,n)} e_{(m,n-1)} X_{(m,n-2)} + a_1 e_{(m,n)} X_{(m,n-1)}$$
$$+ (a_0 + a_2 e_{(m,n)}^2 + e_{(m,n+1)}^2) X_{(m,n)}$$
$$+ a_1 e_{(m,n+1)} X_{(m,n+2)} + a_2 e_{(m,n+1)} e_{(m,n+2)} X_{(m,n+2)}$$  \hspace{1cm} (24)

Values of optimal $c_e$, $a_0$, $a_1$ and $a_2$ are given in Table 1 for integer values of $u$ from 3 to 6 and $e = 3.5$.

Figure 5(a) shows meridian variations of the exact and approximated mapping factor obtained by this method: in the approximate formulation, the exact mapping factor is generally over-estimated in the low resolution hemisphere and near the equator of the computational sphere, and under-estimated near the high resolution pole. Figures 5(b) to 5(e) show the variations of the ratio $1/[M(\xi)]^u$ and $M^u(\xi)/[M(\xi)]^u$, which represent the ratio of the original and of the new diffusion scheme intensities compared to the intensity of the true geographical diffusion. Near the high resolution pole, the new scheme is closer to homogeneous diffusion for low orders than for high orders, and remains better than the original scheme for the four values presented. However, it can be seen in Fig. 5(e) that the diffusion at the low-resolution pole becomes comparable for the original and the new scheme for $u = 6$. This second-degree approximation should thus not be used with orders larger than 5.

Changing $M^u$ into $\tilde{M}^u$ in Eq. (22) yields:

$$\left( \frac{\partial X}{\partial t} \right)_u = -(K_X \tilde{\nabla}^u + K_{ux} \tilde{M}^u \tilde{\nabla}^u) X$$  \hspace{1cm} (25)

Equation (25) is solved implicitly by inversion of a set of pentadiagonal matrices for each zonal wave number $m$, and the same method previously described for the new semi-implicit scheme is here still used.
Figure 5. Approximation of $M^r$ by a second-degree polynomial, for $r$ equal to 3, 4, 5 and 6. The abscissa is $\xi$, the sine of latitude on the computational sphere. (a) Comparison of the mapping factor and its approximations $(a_0 + a_1\xi + a_2\xi^2)^{1/r}$ for different values of $r$. Full line: $M$. Dotted line: approximation for $r = 3$. Dashed line: approximation for $r = 4$. Long dashed line: approximation for $r = 5$. Dashed-dotted line: approximation for $r = 6$. (b) Dotted line: ratio $\frac{M^3}{M^3}$. Full line: ratio $1/M^3$. (c) Dashed line: ratio $M^4/M^4$. Full line: ratio $1/M^4$. (d) Long dashed line: ratio $M^5/M^5$. Full line: ratio $1/M^5$. (e) Dashed-dotted line: ratio $M^6/M^6$. Full line: ratio $1/M^6$. 
Figure 6. Same as Fig. 4 but both components of unified horizontal diffusion scheme are used. Orders $r = u = 5$ for both components.

Figure 6 shows the same (Z500) forecast as Fig. 4, but performed with the unified horizontal diffusion scheme previously described, both components being defined by a fifth-order operator. The magnitude of the waves above the north-west of Spain has been significantly reduced with the new diffusion scheme, and some smaller oscillations noticeable in other mountainous regions are also damped (e.g. in the Apennine and Atlas regions).

(c) Spectral diagnoses

The first requirement of the diffusion, which is to prevent small-scale waves in the stretched area, is thus satisfactorily fulfilled with the unified scheme, but more globally, since the general aim of the horizontal diffusion is to control the energy spectrum of the flow, it is natural to diagnose this spectrum to validate the impact of the designed scheme. But this cannot be done in a straightforward manner in the case of a spectral variable resolution, because spectra obtained in the stretched geometry are physically meaningless since all scales are mixed through the conformal geometrical transformation. For this reason, the actual impact of the diffusion scheme in our case can only be evaluated via ‘geographical’ spectra (i.e. in the not-stretched geometry) and in domains sufficiently small to ensure a weak variation of the local mapping factor in their whole area. In order to allow an accurate comparison with an exact Laplacian based operator, such spectra are presented together with spectra obtained with ALADIN forecasts in a domain 3400 $\times$ 3400 km wide, centered on the high resolution pole of ARPEGE, where differences between schemes principally take place. The limited area model ALADIN uses the same equations as ARPEGE, but in Lambert projection, resulting in a quasi-constant mesh over this domain. To obtain ARPEGE spectra on this geometry, grid-point interpolation of the fields to the ALADIN grid, followed by a Fourier decomposition in each horizontal
Figure 7. Comparison of energy spectra of ARPEGE (T119 L24 c3.5) and ALADIN over a 3400 × 3400 km wide domain centered over the centre of France, after a 36 hour integration from 6 January 1995 00 UTC. ARPEGE uses a fifth-order horizontal diffusion. Full-line: reference spectra obtained from the quasi-operational version of ALADIN (mesh-size 14 km). Dashed line: Spectra for ARPEGE without geographical diffusion. Dotted line: Spectra for ARPEGE with both numerical and geographical diffusion. The characteristics of the ARPEGE model are the same as for the operational version used during spring 1995 for other features than horizontal diffusion scheme: (a) Layer 6; (b) Layer 11; (c) Layer 15; (d) Layer 18. Total wave number between 1 and 39 is marked on the abscissa.
direction is required. The double Fourier decomposition is limited to an elliptic truncation and spectra are computed with respect to total horizontal wave numbers. The wave number 1 thus represents wave lengths of 3400 km, and the smallest ARPEGE wave is represented by the wave number 39.

The spectra shown in Fig. 7 are obtained after 36 h forecast starting from 6 January 1995 00 UTC. The diffusion scheme in ALADIN uses a fifth-order operator, and reasonably reproduces the slope of the energy spectrum in the forecast flow (Banciu 1994, personal communication). For ARPEGE, the original diffusion scheme and the unified scheme with a fifth-order pseudo-geographical diffusion are shown on four model levels (close to 200, 500, 700, 850 hPa).

The departure of energy for last wave numbers in the lower levels is due to the flow adaptation to the finer orography of ALADIN. On the contrary, curves should be similar for the other wave numbers. This is not the case for the ARPEGE experiment with the original diffusion scheme (dashed curves), where amounts of energy are larger than in the reference ALADIN spectra, because of insufficient damping in the high-resolution domain. In the experiment with numerical and geographical components of the unified scheme (dotted curves), the energy is clearly reduced for wave numbers above 10.

5. CONCLUSION

The simple spectral expression of the mapping factor in the variable resolution version of ARPEGE/IFS allows a proper treatment of geographical derivative operators in the spectral part of the computations. This possibility has been used to improve both the semi-implicit and horizontal diffusion schemes for the specific purpose of variable resolution mode. Concerning the semi-implicit scheme, it has been shown that for operational values of the stretching coefficient, the original linearization of Courtier and Geleyn (1988) with a constant mapping factor was an over-simplification, and had to be replaced by an exact treatment to remove an instability problem clearly appearing when the model was used with long time-steps and in semi-Lagrangian mode. The new semi-implicit formulation also led to an improvement of forecast scores in Eulerian mode due to a more accurate evolution in the low-resolution hemisphere. The new horizontal diffusion scheme, which is able to simulate both geographical and rescaled Laplacian operators, improves the energy spectrum of the flow, and prevents the development of small-scale noise when fast winds occur above a steep orography near the high-resolution pole.

These modifications generate a minor increase in computation time (about 1.5% for Eulerian T119 L24 configuration with full physics). The increase of memory cost is roughly equivalent to one 3D spectral array for the semi-implicit scheme, and two 3D spectral arrays per 3D prognostic variable for the unified horizontal diffusion scheme. For typical implementations such as those in the operational version, the additional memory required by these two new schemes is thus about 10%.

From a practical point of view, the introduction of the local mapping factor into the spectral computations makes it possible to integrate the model in stretched semi-Lagrangian mode on an operational basis. Nevertheless, the application of the semi-Lagrangian technique to high-resolution models is a problem which is not yet completely addressed: though the semi-Lagrangian version of ARPEGE can be integrated accurately for CFL numbers larger than 6 in adiabatic mode, the introduction of physical contributions leads to more severe time-step limitations (CFL number around 3), principally due to the small depth of the model layers near the ground compared to the large value of the time-step. This class of residual problems, not specific to our GSVRM (i.e. global spectral variable resolution modelling) approach, mainly affects the quality of forecast
fields near the surface, and has been one of our recent research directions for applying the semi-Lagrangian technique to regional NWP.

The low-order decomposition of the mapping factor on the spectral basis is a crucial ingredient in the Schmidt (1977) transformation for the correct implementation of the GSVRM technique, since it allows an accurate treatment of the local mapping factor at a marginally increased cost. Moreover, there is no specific obstacle to extending the GSVRM approach to any semi-implicit semi-Lagrangian class of schemes provided that the mapping factor is fully taken into account in the spectral part of the computations. GSVRM can then be legitimately considered as a promising alternative to the limited area modelling (LAM) approach in the domain of high-resolution short-range forecasting. However, Caian and Geleyn (1995) have shown that indefinitely increasing the value of the stretching coefficient when keeping the total truncation as a constant does not bring increased accuracy in the forecast fields in the area of high resolution. The value \( c = 7 \) seems to be the limit after which spectral deformation and forecast skills degradation begin to appear, at current operational resolutions. To avoid this problem, a third approach is currently being explored at Météo-France which is to perform short-range forecasts with ALADIN driven by a moderately stretched version of ARPEGE. This strategy has the advantage of making possible high-resolution forecasts without the inconvenience of an exaggerated resolution gap between the two models.

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