Observations of isolated wave–turbulence interactions in the stable atmospheric boundary layer

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SUMMARY

Turbulence measurements of the stable atmospheric boundary layer made at Halley station in Antarctica are presented. The interaction between small-scale turbulence and larger solitary wave disturbances is investigated. Spectral methods are used to separate the wave and turbulence parts of the flow, and the appropriateness of this approach is discussed. Both individual wave events and an average of 18 wave events are studied. The transport of wave-induced velocity by wave-induced turbulence is found to be mostly upwards, whilst a strong downward turbulent heat flux accompanies each wave. Waves are often found to be associated with a burst of low-level turbulence which occurs later than the perturbation in the mean shear. The effect of the turbulence on the wave is shown to be small, whilst the instantaneous local effect of the wave on the turbulence appears to be approximately linear in the wave amplitude. Comparisons are made between the observations and numerical predictions. These tend to confirm the need for at least second-order closure schemes for the turbulence modelling.

KEYWORDS: Antarctic boundary layer Second-order closure Solitary waves Turbulence measurements

1. INTRODUCTION

During the southern winter of 1986 an extensive programme of atmospheric boundary-layer measurements was undertaken at Halley station, Antarctica by the British Antarctic Survey. The experiment was referred to as the Stable Antarctic Boundary-Layer Experiment (STABLE). This article concentrates on a type of disturbance which was observed repeatedly throughout the experiment, namely a large-amplitude solitary wave in the surface layer. In particular we address the interaction of such motions with boundary-layer turbulence. Edwards and Mobbs (1997), hereafter referred to as EM, describe the results of numerical modelling of these events and give a brief account of previously reported work on similar atmospheric waves and on wave–turbulence interaction. In this paper we make a detailed analysis of the events observed at Halley, and compare the results with the numerical simulations.

Most previous work on wave–turbulence interaction in the stable atmospheric boundary layer has concentrated on almost monochromatic waves (e.g. Einaudi and Finnigan 1981; Finnigan et al. 1984; Finnigan 1988; Einaudi and Finnigan 1993). In that case phasic averaging can be used to partition the wave and turbulent parts of the disturbance. This paper considers isolated disturbances for which phasic averaging cannot be used. Ideally, ensemble averaging would be used for this situation. However, insufficient similar events made this impractical as a sole method of decomposition of the disturbances and a Fourier filtering technique has been combined with ensemble averaging.

2. THE STABLE

The need for reliable data for strongly stable atmospheric boundary layers led to the running of the STABLE by the British Antarctic Survey. The experiments were carried out during the southern winter of 1986 at Halley station, which is situated near the seaward edge of the Brunt ice shelf (75.6°S, 26.7°W). Full details of the experiments can be found

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in King and Anderson (1988); the main features are given below. Two features of the site make it particularly suitable for stable boundary-layer studies:

1. Halley experiences three months of winter darkness and, in the absence of a diurnal heating cycle, the boundary layer often becomes strongly stable.
2. The ice shelf extends for 40 km inland in the direction of the prevailing wind, with no significant topography and small slope.

At the measurement site a triangular array consisting of one 32 m mast and two smaller masts, situated 200 to 300 m apart, carried most of the instrumentation. Three ultrasonic anemometers capable of measuring wind velocity and temperature at 20 Hz were mounted at heights of about 5, 16 and 32 m on the main mast. Cup anemometers were also mounted at three heights on the main mast and at about 8 m on each of the remote masts. There was also one wind vane on each mast. Platinum resistance thermometers were placed at five heights on the main mast and at various depths in the snow. A net radiometer was deployed at the snow surface. Above 32 m observations were made using an acoustic sounder and regular radiosonde ascents.

Three types of data streams were recorded: in one of these, ten minute means of all variables and their second-order correlations (such as \( \langle u'w' \rangle \), \( \langle v'w' \rangle \) and \( \langle w'T \rangle \) where \( u \), \( v \) and \( w \) are velocity components and \( T \) is temperature) were recorded. Over 2000 hours of this type of data were collected. About 100 files of essentially raw unprocessed data were also recorded. These files therefore contain data from the sonic anemometers recorded at 20 Hz and from the other instruments at 1 Hz. These runs are mostly of about one hour or half an hour in duration and it is these data which are used below. A similar number of files also exist in which these essentially unprocessed data were averaged over 10 s periods. These runs were usually of 12-hours duration.

Despite the lack of nearby topography, wave-like signals can be detected in a large proportion of the data. In fact the study of these waves was one of the principal aims of the experiment.

3. Data Processing

From amongst the unprocessed raw data files, 15 have been chosen during which large-amplitude isolated wave events were recorded. The measured time series of the basic-flow variables have been decomposed into mean and turbulent parts, defined as the low and high frequencies in a Fourier decomposition of the data.

In order to include all the effects of the waves, which had periods in the range from 25 s to 100 s, all the analysis has been carried out using data segments of length at least 150 s (longer segments containing 4096 points, i.e. 204.8 s, were used to calculate the Fourier transforms for reasons of efficiency). If \( n \) frequencies are retained in the mean part and the length of the time series is \( T \), the shortest period represented in the low-frequency part equals \( T/n \). This is referred to later as the 'cut-off period' \( p_c \). This method of decomposition works well when there exists a wide spectral gap between wave and turbulent parts of the flow. Clearly \( p_c \) should lie in this gap. However, one of the most critical problems with this type of analysis is that in real wave–turbulence situations, such a gap is rarely wide or clearly defined. On the other hand, it is evident in the plots of the raw data presented in section 4 that there is some scale separation between the processes which we wish to define as wave and turbulence. Thus the choice of cut-off period is an important issue which is investigated in a later section by comparing results for different values for \( p_c \). The potential lack of a wide spectral gap also affects the modelling of this type of flow.
This is because in order to derive the Reynolds averaged Navier–Stokes equations and the equations for the turbulence correlations, an important property of the decomposition (averaging) operator is required. Denoting the operator by $\langle \cdot \rangle$, the condition is

$$\langle a \langle b \rangle \rangle = \langle a \rangle \langle b \rangle,$$  \hspace{1em} (1)

for any quantities $a$ and $b$. When $\langle \cdot \rangle$ denotes a separation of harmonic components, Eq. (1) will only be a good approximation if the spectral gap is large. In a later section we consider the errors incurred by making the assumption that Eq. (1) holds.

An advantage of ensemble averaging, as a form of decomposition operator, is that it does not rely on the existence of a spectral gap between wave and turbulence. In the Halley data there are not enough similar events to use such an average to define the wave–turbulence decomposition, but the averaging procedure can be used in conjunction with the Fourier technique. For this purpose an ensemble average for a time-series $q(t)$ is defined by:

$$\{q\}(t) = \frac{1}{n} \sum_{r=1}^{n} q_r(t - t_{0r}), \quad t_1 < t < t_2,$$  \hspace{1em} (2)

where the subscript $r$ indexes $n$ different data sets, and the centre of the $r$th wave, defined as the zero of the wave-induced vertical velocity, is at time $t = t_{0r}$. This average is useful if the wave events are similar, in which case the average may provide a better estimate of the underlying wave-form. That is, variance due to random fluctuations of arbitrary scales (turbulent or otherwise) will be reduced. For this study 18 fairly similar waves have been averaged together, the resulting data set is referred to later as run s18 (or wave s18).

The expected effect of forming the ensemble average of $n$ events and then calculating turbulence correlations using the Fourier decomposition is shown in the appendix to be a reduction of the $r$th order moments by a factor of $1/n^{r-1}$.

The contributing waves were not normalized before the averaging procedure of Eq. (2). Several possible normalizations might be suggested, such as with respect to wavelength (i.e. period), amplitude, wave propagation velocity or background flow velocity. Normalizing with respect to period would present particular problems for a time series of a turbulent quantity: it would be necessary to interpolate values between observations, which would affect the spectrum of the turbulence and the correlations between variables. The wave velocity also presents problems, since this cannot be found accurately for some waves owing to severe instrument riming on the small, remote masts. These points are merely symptoms of the underlying problem, which is that the dynamics of the waves are not known in advance. Thus it is not possible to normalize correctly with respect to variation in a given parameter because the dependence on that parameter is unknown. For instance some properties are likely to depend nonlinearly on amplitude.

Data from instruments other than the sonic anemometers have been used for two purposes: firstly to estimate mean gradients and secondly to estimate the wave propagation velocity. Velocity gradients in the mean flow have been found from the cup anemometer data from the main mast. Mean temperature gradients have been found using the platinum resistance thermometers on the mast. All these data were logged at a frequency of 1 Hz, so the frequencies present in the Fourier transforms of these time series are not precisely the same as those in the 20 Hz data. We therefore choose the cut-off period closest to that used for the 20 Hz data in each case. Vertical gradients were calculated by fitting quadratic splines to the mean values of variables at three neighbouring heights. Horizontal gradients were calculated from the time derivative by assuming that the wave pattern moves without
change of form at the mean flow speed \( s \), so that:

\[
\frac{\partial}{\partial x} = -\frac{1}{s} \frac{\partial}{\partial t}.
\]

The wave speed was not used in this context because it was not always known. The result of this is that horizontal gradients are likely to be overestimated. The ratio of wave speed to mean flow speed at 16 m was usually between one and two, although in the case referred to later as wave 2 it was 3.3. From the mean gradients, mean values of the Richardson number, \( R_i \), have also been calculated.

Wave propagation speeds and directions were found by correlating observations at the main mast and the two remote masts. The data used were unfiltered time series from cup anemometers and wind vanes. The travel times between the three masts were determined from the time-lag for maximum cross-correlation of the signals. Poor data in some cases resulted in a low value of the maximum cross-correlation, and for these the propagation speed and direction were discarded. For the good cases the width of the cross-correlation peak could be used to estimate the uncertainty in the estimates. Generally, speeds were accurate to \( \pm 10\% \).

All the results shown in the next section have been transformed, using the wave propagation directions found, to a frame of reference where the wave velocity is along the positive \( x \) axis. For run s18, and also for the event described as wave 3, the direction was obtained from the average of \( c\hat{k} \), where \( c \) is the phase speed and \( \hat{k} \) is the unit wave vector. The observed wave directions were strongly clustered around 120°.

4. Results

All of the waves which were used to form the averaged run s18 occurred in conditions of low wind speed (less than 10 m s\(^{-1}\)) and strong stability (mean temperature gradients at 16 m were typically around 1 K m\(^{-1}\)). Six out of the seven waves for which a wave propagation direction could be clearly determined came from between 100° and 130°, all of these six were travelling faster than the background flow, with wave speeds between 2.2 and 8.3 m s\(^{-1}\). Where the wave was clearly visible in the data it was characterized by a large drop in temperature associated with a vertical-velocity perturbation which was first positive, then negative, and a unimodal variation in mean horizontal wind. The waves do vary to some extent in amplitude and period and there are also differences in the background flow velocities. To investigate the effects of these differences on the turbulence, three individual waves are analysed in this section, in addition to the averaged wave s18. The chosen cases are the second, third and sixth, chronologically, of the eighteen. Waves 2 and 3 of the set were recorded on 6 June 1986 while wave 6 was observed on 22 June 1986. Wave 2 is a typical average-amplitude wave event while wave 3 is amongst the smallest in amplitude and period. Wave 6 on the other hand is one of the largest waves in amplitude and wavelength (larger amplitudes tend to be associated with longer periods). Wave 6 also has a much greater background flow speed than waves 2 and 3.

The intermediate height of 16 m is chosen for most of the analysis in this paper. Of the three heights at which 20 Hz data are available, this is the height at which the vertical-velocity perturbation is usually greatest. Some plots of turbulence quantities measured at 5 m are also shown. A cut-off period \( p_c \) of 24 s is used in most cases. This choice is motivated by the scales which are present in the majority of cases but the effect of using other values for \( p_c \) is investigated. Taking \( p_c \) equal to 24 s means that the mean flow contains the lowest eight harmonics present in the original time series.
All the results are shown in a frame of reference where the wave velocity is along the positive x axis, with the centre of the wave at 50 s. The centre is defined by the zero of wave-induced vertical velocity. This also coincides with the minimum of temperature and usually also the maximum horizontal-velocity perturbation. Temperature values are given relative to an arbitrary zero, which varies between heights and between data runs.

(a) The averaged wave s18

The speed of the averaged flow for s18 was about 1 m s\(^{-1}\). This low value is caused by the cancelling of flows from different directions. The average of the individual mean flow speeds for the 18 cases is 3.8 m s\(^{-1}\). It was not possible in all cases to estimate the wave propagation speed, owing to poor data from the remote masts (caused by severe riming) but from the cases where this was possible an average speed of 3.5 m s\(^{-1}\) and a direction of 117° was obtained.

![Figure 1](image_url)

Figure 1. Mean flow variables for run s18 at a height of 16 m: (a) mean horizontal-velocity components (\(\langle u \rangle\) and \(\langle v \rangle\)) and temperature (\(\langle T \rangle\)), and (b) mean vertical-velocity component (\(\langle w \rangle\)).

Figures 1 to 5 relate to run s18 at 16 m. The results were obtained using a value for \(p_e\) of 24 s. The mean flow shown in Fig. 1 shows clearly the typical features of the waves observed: there is a large drop in temperature of 2 K, whilst the perturbation in the horizontal flow speed is almost 1 m s\(^{-1}\), which is comparable with the speed of the background flow. The wave-induced vertical velocity \(\langle w \rangle\) has a maximum of about 0.15 m s\(^{-1}\). Apart from a slight deviation in \(\langle w \rangle\) at 20 s, all four variables have an overall symmetry (or antisymmetry in the case of \(\langle w \rangle\)) about the 50 s point on the time axis. The most unexpected aspect of the plots is that there is no significant wave component in the normal velocity \(\langle v \rangle\). This is later seen not to be the case for the individual waves.

Figure 2 shows the diagonal and off-diagonal components of the Reynolds stress tensor \((u'u')\), the diagonal stresses being the component energies, \((u'^2)\). Note that according to the theory in the appendix, all the turbulence correlations are expected to be 1/18th of the size that they would be for a single wave event with the same mean flow. Comparison with the results for individual waves does indeed show that the turbulence correlations for wave s18 are at least an order of magnitude smaller than for individual waves presented later. There is a clear wave-induced variation in each of the three diagonal stress components, with peaks in \((u'^2)\), \((w'^2)\) and \((v'^2)\) at about 40 s, 55 s and 70 s respectively. The wave in \((w'^2)\) is smaller than that in the other diagonal stresses and the background value of this quantity is also smaller, showing the influence of the strongly stable conditions. The off-diagonal stresses also show larger oscillations centred around 50 s but the influence of the wave is much less clear than in the diagonal components. There is some evidence from the \((u'w')\)
record of a net upward flux of wave velocity. The effect of the wave on the temperature variance \( \langle T'^2 \rangle \) is even less clear and consequently the heat fluxes in Fig. 3(b) also exhibit much apparently random variation. However, there is a fairly clear oscillation in \( \langle u'T' \rangle \) and an even clearer wave in \( \langle w'T' \rangle \) which has a minimum at about 55 s. This suggests that a wave-induced downward turbulent heat flux is likely to occur for a typical wave.

Estimates of mean horizontal and vertical shear are shown in Fig. 4. The vertical shear in the background flow is negative, suggesting that there is a maximum of mean flow speed below 16 m. There is a clear wave variation in \( \delta \langle u \rangle / \delta z \) which is very significant in relation to the background value. The normal velocity, by comparison, has little vertical shear. The horizontal-shear components also show wave variation, and the plots suggest that the variations in horizontal shear are actually larger than those in vertical shear.

In Fig. 5 an attempt has been made to estimate the components, \( P_{11} \) and \( P_{22} \), of the energy production tensor for \( \langle u'^2 \rangle \) and \( \langle v'^2 \rangle \). \( P_{11} \) is given by:

\[
P_{11} = -2 \langle u'^2 \rangle \frac{\partial}{\partial x} \langle u \rangle - 2 \langle u'w' \rangle \frac{\partial}{\partial z} \langle u \rangle
\]

(4)
while \( P_{22} \) is given by:

\[
P_{22} = -2 \langle u'w' \rangle \frac{\partial}{\partial x} \langle v \rangle - 2 \langle u'w' \rangle \frac{\partial}{\partial z} \langle v \rangle.
\]  \hfill (5)

The result is made clearer by plotting the two terms on the right-hand side of Eq. (4) separately. The first is labelled \( p1x \) and the second \( p1z \). There is strong negative production of \( \langle u'^2 \rangle \) at the centre of the wave where this variable is decreasing most rapidly. This follows a maximum of positive production. Thus to some extent the early maximum in \( \langle u'^2 \rangle \) is related to the production tensor. On the other hand the production estimate offers no explanation for the maximum which occurs later in \( \langle v'^2 \rangle \), although subsequent analysis will suggest that the lack of shear in \( \langle v \rangle \) is due to the averaging process.

Graphs of variables at the other two heights (5 m and 32 m) generally show less clearly defined wave signals but the values of the variables are similar. At 5 m the background values of the diagonal stress components appear to be larger than at the other two heights. A significant signal occurs in the vertical momentum flux at around 80 s, but the graphs of the diagonal stress components show no wave signal near this point at all. This is later seen to contrast with some of the individual cases in which a low-level burst of turbulence energy occurs after the mean wave perturbation.
Figure 6. The structure of wave 2 at 16 m: (a) mean flow variables \( \langle u \rangle \), \( \langle v \rangle \), and \( \langle T \rangle \); (b) mean flow variable \( \langle w \rangle \); (c) diagonal stress components; and (d) off-diagonal stress components. See text for further explanation.

(b) Wave 2

Plots comparable with those shown in the case of run s18 are now examined for the single event referred to as wave 2. The same cut-off value of 24 s for \( \rho_c \) was used and variables are again described first at a height of 16 m. Mean values of the four basic variables are shown in Figs. 6(a) and (b) along with the diagonal stress components in Fig. 6(c) and the components of the Reynolds stress tensor in Fig. 6(d). The mean variables show a very similar pattern to s18 with one significant difference; in this case the wave in the horizontal velocity is almost entirely in \( \langle u \rangle \) rather than \( \langle u \rangle \), that is the main horizontal wave perturbation is normal to the direction of propagation. The background flow is similar but the magnitude of the wave in \( \langle T \rangle \) and \( \langle w \rangle \) is almost twice as large as in s18.

The magnitudes of the stresses are up to ten times larger than in s18 but this is less than the factor expected due to the averaging process, and some quantities, such as \( \langle u^2 \rangle \), are of a similar size. Despite the reversal in the forms of \( \langle u \rangle \) and \( \langle v \rangle \) compared with s18, there is still a large peak in \( \langle u^2 \rangle \) at around 40 s, followed by smaller peaks in the other diagonal stress components. The off-diagonal stresses all have different forms to s18. The most marked wave variation is in \( \langle u'w' \rangle \) which is very small outside the wave but shows a clear downward flux of momentum during the wave, peaking at about 65 s. A similar negative flux occurs in s18 but there it is preceded by a positive peak of larger magnitude. The temperature variance \( \langle T^2 \rangle \) is plotted in Fig. 7. The graph shows a clear wave signal but this may be somewhat artificial. The reason for this is that the wave-form in the raw temperature data has an almost triangular shape which is not well fitted by a small number of sinusoidal functions. The sharply pointed part of the wave will therefore be defined to
be part of the turbulence. Whether this definition is correct, however, is a complex question which underlies the whole method of analysis. The turbulence heat fluxes also show clear wave signals, due, at least in part, to the sharp peak in the temperature record. The wave in the vertical heat flux takes a very similar form to that in the averaged run s18.

The mean vertical gradients are plotted in Fig. 8(a). Despite the lack of a wave in \( \langle u \rangle \) at 16 m, \( \partial \langle u \rangle / \partial z \) at this height has a wave component larger than any of the other estimated mean shear components. This may be related to the large coincident wave in \( \langle u'^2 \rangle \), although the sign of \( \langle u'w' \rangle \) at 40 s would suggest negative rather than positive production of \( \langle u'^2 \rangle \). Some evidence for the significance of the production tensor is given by the fact that the only portion of the wave in which Eq. (5) would suggest a positive value for production of \( \langle v'^2 \rangle \) is around 70 s, close to the maximum of the wave in \( \langle v'^2 \rangle \) at 60 s. However, the production tensor clearly cannot alone explain the observed distributions of turbulence variables.

One of the features of this wave, which is visible in the unprocessed data, is a burst of high-frequency turbulence. This occurs in the 5 m record of all four basic variables, near to the end of the wave event. The burst shows up in the plot of the three diagonal stress components at 5 m in Fig. 8(b). Large wave signals also occur in the horizontal turbulent fluxes of momentum and heat but not in the vertical fluxes. The wave in \( \langle u'^2 \rangle \) is of a similar size to that at 16 m but the disturbances in the other diagonal stress components are both significantly larger. The larger values of these components are associated with
larger values of background shear, but in this case the wave variation in the diagonal stress components occurs after the wave in the mean shear.

(c) Wave 3

Wave 3 has a period only about half as large as the other two individual cases studied. For this reason a smaller cut-off period is more appropriate to separate wave-like and turbulent parts of the basic variables $u_1$ and $T$. The wave is also considerably less clearly defined at 16 m and 32 m than in the other cases described. The analysis below, therefore, concentrates on the values of variables at 5 m, where the appearance of the mean flow resembles that in the larger amplitude waves at 16 m. The results shown are for a cut-off period of 12 s. Unfortunately, no clear signal was measured at the remote masts for this wave owing to the lack of a wave perturbation in the horizontal wind higher up. Thus it is not possible to determine whether the shorter period is due in part to a greater wave speed.

![Graphs](image)

**Figure 9.** The structure of wave 3 at 5 m: (a) mean vertical velocity, (b) mean temperature, (c) off-diagonal stress components, and (d) heat flux components.

Despite the fact that wave 3 resembles wave 2 in many respects and actually occurred only 20 minutes later, the relationship between the two is not a simple linear one. For example, changes in the background flow have occurred such that, although the temperature perturbation has only one half the amplitude of that in wave 2 at 5 m, the amplitude of the vertical-velocity perturbation is actually twice as great. The mean vertical velocity and temperature are shown in Fig. 9, along with the components of the stress tensor and the heat flux. In general, wave-like variations in the mean variables are smaller than those in wave 2 by up to a factor of four. The values of the mean shear components (Fig. 10), on
the other hand, display even more variation between the two waves; \( \partial \bar{u} / \partial z \), the wave-like component of \( \partial (u) / \partial z \), is about ten times smaller in wave 3 than wave 2, while \( \partial \bar{u} / \partial x \) is larger. Our notation here follows EM and is defined later in section 5 in Eqs. (10) to (14).

Figure 10. Mean shear components for wave 3 at 5 m: (a) horizontal shear components, and (b) vertical shear components.

Given the variations in the forms of the mean waves discussed above, it is difficult to use these results to assess whether the wave-like part of the turbulence fluxes, \( r_{ij} \) and \( r_i \), in the notation of EM, depend linearly or non-linearly on the mean wave amplitude. However, the values obtained for the Reynolds stress components and turbulent heat fluxes show variations in amplitude between the two cases, which are within the scale of the variation in mean quantities described above. In other words, they do not provide any evidence of a very strongly nonlinear dependence of \( r_{ij} \) and \( r_i \) on the mean wave components \( \bar{u}_i \) and \( \bar{T} \). As in run s18 and in wave 2 at 16 m, there is a clearly defined wave-induced increase in the downward heat flux around the centre of the wave. A similar feature occurs for \( \langle u'u' \rangle \), accompanied by an opposite change in \( \langle v'w' \rangle \). This represents upward turbulent transport of wave velocity at this height.

(d) Wave 6

The mean flow variables and stress components for wave 6 at 16 m are shown in Fig. 11. Here there is significant wave perturbation in both \( \langle u \rangle \) and \( \langle v \rangle \) of a much larger amplitude than in wave 2. The background flow is at a larger angle to the wave direction and has a much higher velocity. The wave-induced vertical velocity is similar in magnitude but the temperature wave is larger. In fact the actual temperature wave must have been much larger still as the measurements went very rapidly off the scale of the instrument, which has a range of 5 K, remaining off scale for 32 s. There is therefore no way of calculating valid turbulence correlations based on the fluctuating temperature for this wave. The period of the wave is not much longer than wave 2 but the wave propagation speed is 8.3 m s\(^{-1}\) as compared with 4.6 m s\(^{-1}\) for wave 2, so the spatial wavelengths of the two waves may have differed by a larger factor. The diagonal stress components are around ten times greater than in wave 2, or around five times the values in s18 (after rescaling). However, the pattern is very similar, a large wave in \( \langle u^2 \rangle \) at about 40 s is followed by smaller waves in the other two components with peaks at around 70 s. The off-diagonal stresses are different from both s18 and wave 2 although comparison of the mean wave components and the off-diagonal stresses indicates that the transport of wave velocity is mostly upward, as in wave 3. The wave variations in the mean shear components (not shown) are of a similar size to wave 2 but the background values are much larger. This explains the increased
values of turbulence energies, but note that the size of the temperature wave shows that the background temperature gradient is also much greater than in wave 2.

This wave also exhibits a low-level burst of turbulence similar to that observed in wave 2, visible in the turbulence stresses shown in Fig. 12. The burst of turbulence energy occurs at around 90 s, which is after the perturbation in the main components of the mean shear. The burst is not as isolated as in wave 2 because here there is a similar increase of turbulent energy at about 50 s. However, in this wave the burst of energy at 90 s is
associated with an enhanced downward heat flux. There is also a clear wave-induced variation in \( \langle u'w' \rangle \) which, as in wave 3, is accompanied by an opposite variation in \( \langle v'w' \rangle \), showing upward transport of wave velocity. The existence of a similar but larger flux of wave velocity at 16 m suggests that the wave-induced turbulence is acting to reduce the mean wave amplitude, on average, between these heights. The magnitude of this effect can be roughly estimated at between \( 2 \times 10^{-3} \) and \( 4 \times 10^{-3} \) m s\(^{-2}\). This is about one hundred times smaller than an estimated value of \( \partial \langle u \rangle / \partial t \).

\( \text{\textit{(e) The effect of varying the cut-off period}} \)

![Figure 13. Components of the decomposition of vertical velocity for wave 6 at 16 m: (a) unfiltered time series, and (b) mean and turbulent parts (\( \langle w \rangle \) and \( \langle w' \rangle \)) using a cut-off value of 24 s.]

![Figure 14. Components of the decomposition of vertical velocity for s18 at 16 m: (a) unfiltered time series, and (b) mean and turbulent parts (\( \langle w \rangle \) and \( \langle w' \rangle \)) using a cut-off value of 24 s.]

The unfiltered vertical velocity \( w \) at 16 m and the two components of its decomposition are shown for wave 6 in Fig. 13 and for run s18 in Fig. 14. In both cases the value of \( p_c \) was 24 s — the value used above. Fig. 15(a) shows the two components of the decomposition for wave 6 using a shorter cut-off value of 12 s and Fig. 15(b) shows the decomposition for s18 with a 6 s cut-off. In the case of the 24 s cut-off, the mean wave-form has fewer turning points and resembles more closely numerical and theoretical models of solitary waves. Even if the two components of the decomposition do not represent distinct physical processes however, the analysis may be viewed as a study of the interaction of different scales of motion.
To examine the effect on the results of using a different cut-off, turbulence quantities for run s18 at 16 m have been recalculated using a much smaller value of \( p_e \) of 6 s. There are a number of differences between these results and those obtained earlier. The diagonal stress components and Reynolds stresses are shown in Fig. 16. There is still a very clear wave-induced variation in the diagonal stress components although the values obtained are significantly smaller. In fact comparing Fig. 16 with Fig. 2 suggests that around half of the energy formerly defined as turbulent was due to the scales between 6 s and 24 s. Another very significant feature of the energy plot is that the maximum of wave-induced turbulence occurs later when the smaller value of \( p_e \) is used. This may be a result of the time taken for energy to be transferred from larger to smaller scales. This plot also shows clearly the very strong anisotropy of the background flow, in which the vertical component energy is about one sixth of the largest horizontal component. That is:

\[
\overline{u'^2} \approx \frac{1}{6} \overline{u'^2}.
\]

(6)

This is in spite of the fact that the smaller scales are expected to be more isotropic. The off-diagonal stresses do not show such a marked reduction in amplitude, the main peaks being of a similar size to those in Fig. 2. However, the forms of the graphs of these stresses are quite different for the two different values of \( p_e \). As with the diagonal stresses, the main signal occurs later for the smaller cut-off value, although there is a minimum in \( \langle u'w' \rangle \) at
about 50 s. The wave part of this quantity at 70 s again suggests upward transport of wave velocity.

Wave-induced values of the off-diagonal stresses might be expected to peak earlier than wave variations in turbulent kinetic energy. This is because they are limited mainly by the return to isotropy effect of the pressure–strain correlation, while the turbulent kinetic energy is limited by the dissipation. Because the pressure–strain term involves one spatial gradient of a turbulent quantity while the dissipation involves two, the spectrum of the dissipation would be expected to be more skewed towards higher wave numbers. Higher wave numbers can be expected to appear later in the wave as the energy provided by the mean flow is transferred down scale. Although the higher frequencies do appear to be represented later in the wave, any lag between the diagonal and off-diagonal stresses in this respect is not clear from the results. If the turbulent kinetic energy was more skewed towards higher wave numbers than the off-diagonal stresses, this might also manifest itself in a smaller reduction in amplitude of the wave signal in the results with \( p_e \) equal to 6 s. In fact the opposite is true, that is the diagonal components are reduced more than the other stresses. There is thus no evidence in the data of a lag of the type described.

(f) Summary of the data analysis

The direction of the mean horizontal velocity perturbation varied widely between different waves; thus the lack of a signal in \( \langle v \rangle \) in run s18 may not be representative of a typical wave. In spite of this the wave-induced variations in the components of turbulent kinetic energy showed similar forms in most cases: typically a maximum in \( \langle u'^2 \rangle \) early on in the wave was followed by smaller maxima in \( \langle w'^2 \rangle \) and \( \langle v'^2 \rangle \). The background values of turbulent kinetic-energy components varied significantly owing to the varying background shear but always showed strong anisotropy. Off-diagonal stresses did not show such consistent patterns: the transport of wave-induced velocity by wave-induced turbulence was mostly upward, although the reverse occurred in one case. There was some evidence that shear production is a dominant factor in the turbulence distributions. One consistent feature was a large downward turbulent heat flux near the centre of the wave period. This was usually accompanied by a large increase in temperature variance.

An interesting feature of some of the waves was a burst of turbulence at 5 m, after the mean shear perturbation. This was accompanied in one case by an increase of downward turbulent heat flux. In the averaged wave the only sign of the burst was in \( \langle u'w' \rangle \). Using a smaller value of \( p_e \) showed that much of the energy in the turbulence was due to periods between 6 s and 24 s. The higher frequencies still showed wave-induced variations, which tended to appear later in the wave. Analysis of a smaller event showed no evidence that the instantaneous response of the turbulence to the wave in the mean flow was strongly nonlinear. In one case it was possible to estimate the effect of the turbulence on the mean wave; it was found that the turbulence acts to reduce the wave amplitude but that the effect is very small.

5. Comparison with numerical models

The results of the data analysis, in particular the averaged run s18, are now compared with the numerical results described in detail in EM. Two numerical simulations are considered: a fully time-dependent simulation and a so called ‘steady-state’ solution. In both cases the appropriate equations were solved in a two-dimensional domain 100 m high by about 1.5 km long which translated at a constant speed in the horizontal. Solutions were sought corresponding to the flow which resulted when a wave-like perturbation was imposed on the mean flow. The initial state constituted a boundary layer which was
evolving only very slowly and was horizontally uniform. The 'steady solution' is not a steady solution of the full governing equations; rather it constitutes a steady solution of the turbulence equations with the imposed mean flow artificially held fixed.

The fully time-dependent simulation is described in detail in EM where it is referred to as the standard run or run 1. The steady-state simulation uses the same 11-equation turbulence closure with the same parameters and conditions. Results from the fully time-dependent model are derived from the solution at time \( T_f \) (chosen to be long enough for the wave to have travelled a distance greater than its horizontal wavelength). \( T_f \) is set to 100 s. The 11-equation closure is used because it was the most advanced scheme considered, with turbulence correlations analogous to those calculated from the data in the previous section being carried as dynamical variables. The advantage of comparisons with the steady-state simulation is that the mean flow more closely resembles that in run s18. The implications of holding the mean flow fixed are discussed in detail by Edwards (1992) where it is shown that, in this wave–turbulence situation, the structure of the turbulence reacts quickly to changes in the mean flow and is not sensitive to the fact that the mean flow is artificially imposed.

![Figure 17. \( \langle u' \langle u \rangle \rangle \) for wave 2. See text.](image)

It was noted earlier that the derivation of the governing equations of the numerical models relies on the assumption that Eq. (1) holds. The effect of failures of Eq. (1) in the present case can be assessed by calculating terms of the same type as those which are neglected when deriving the governing equations. Use of Eq. (1) leads to the neglect of the following terms:

\[
\left\langle u'_j \frac{\partial}{\partial x_j} \langle u_i \rangle \right\rangle \quad \text{and} \quad \left\langle \langle u_j \rangle \frac{\partial}{\partial x_j} u'_i \right\rangle ,
\]

from the mean-momentum equation and a number of terms including:

\[
\left\langle u'_i u'_j \frac{\partial}{\partial x_j} \langle u_i \rangle \right\rangle - \left\langle u'_i u'_j \right\rangle \frac{\partial}{\partial x_j} \langle u_i \rangle ,
\]

from the turbulent kinetic energy equation. A simple example of this type of term is shown in Fig. 17. This shows \( \langle u' \langle u \rangle \rangle \) for the case of an individual wave. This would be zero for a perfect decomposition. The fact that the sets of frequencies present in \( a' \) and \( \langle a \rangle \) are disjoint means that the zeroth harmonic of \( \langle a' \langle b \rangle \rangle \) must be zero, for any quantities \( a \) and \( b \). However, Fig. 17 shows that the instantaneous values of such quantities can be of a comparable magnitude to correlations which are normally retained. Thus the
standard equations will hold for the background flow, if this is defined to be the zeroth harmonic, but not for the full mean flow. The non-vanishing of these extra terms is one of the problems of using a spectral decomposition when the spectral energy gap between wave and turbulence is not very wide. This should be borne in mind when comparing the results of modelling and data analysis. The neglected terms represent correlations between wave and turbulent variables. Therefore the neglect of these terms may result in a greater independence between wave and turbulence than would occur in practice.

(a) The mean flow

The mean flow in the model runs differs significantly from that in the data run s18, in particular with respect to the background flow. The main reason for this is one of practical difficulty with the turbulence scheme; namely that the model was unable to cope with conditions as strongly stably stratified as those observed. Very large temperature gradients of around 1 K m⁻¹ occur in the data and are associated with strongly anisotropic turbulence, as described earlier. This situation, in which one turbulence energy component approaches zero, is referred to as a two-dimensional limit of turbulence. The problems of modelling such situations using second-moment closures are the subject of much current research. The main difference between the mean flow in data run s18 and in the models is thus that the models have a faster background flow with more shear and a much smaller temperature gradient. In both cases the mean flow is at an angle between about π/6 and π/4 to the wave propagation direction. Temperatures are referenced to the value at the ground.

The initial wave condition for the models was chosen to be similar to the observed waves in both form and amplitude. The major difference is therefore that the temperature perturbation is 50 times smaller owing to the reduced background stability. Graphs of the mean flow variables for the steady-state model are shown in Fig. 18. \( \langle u \rangle \), \( \langle v \rangle \) and \( \langle w \rangle \) are also shown for the time-dependent model in Fig. 19. In these and subsequent graphs the time axis has been calculated by assuming that the wave moves past a given point without change of form, at the wave speed defined in, or resulting from, the model. In the fully time-dependent model the speed of the disturbance at 16 m is 1.6 m s⁻¹, which is only just greater than the background flow velocity component \( \bar{u} \). Thus the apparent wave period is considerably longer than that in the data run s18 where the mean flow speed is approximately 1 m s⁻¹ and the speed of propagation of the wave is of the order of 5 m s⁻¹ (the precise value is not known because of poor data from the remote masts on some occasions).

![Figure 18](image_url)

Figure 18. Mean variables for the steady-state run of the numerical model. (a) Horizontal velocity components, \( \langle u \rangle \) and \( \langle v \rangle \); and temperature, \( \langle T \rangle \); (b) vertical velocity, \( \langle w \rangle \).
One of the most important results of EM was that the effect of the wave-induced turbulence \( r_{ij} \) and \( r_i \) on the mean wave was relatively small. To assess whether this is also the case for the observed waves, estimates of components of the Reynolds stress divergence term in the mean momentum equations can be made and compared with the rate-of-change term \( \partial \langle u \rangle / \partial t \). Using a value for the wave speed \( c \) of 5 m s\(^{-1}\) found by averaging the observed wave speeds, \( \partial \langle u'^2 \rangle / \partial x \) can be estimated by:

\[
\frac{\partial}{\partial x} = \frac{1}{c} \frac{\partial}{\partial t}.
\]

In the data the scales for horizontal and vertical variation of mean fields appear to be similar, so it is likely that this also holds for the turbulence. Also \( \partial \langle u'^2 \rangle / \partial x \) is of a similar magnitude to the other turbulence term in the equation for \( \langle u \rangle \), \( \partial \langle u u' \rangle / \partial z \). Comparing estimates of \( \partial \langle u'^2 \rangle / \partial x \) with \( \partial \langle u \rangle / \partial t \) gives a ratio of about 0.01, which indicates that, as in the numerical models, the effect of the wave-induced turbulence terms on the development of the mean wave is small. Recall that in the one case in which \( \partial \langle u' w' \rangle / \partial z \) could be estimated directly (wave 6), a similar ratio was obtained. It should be noted, however, that the effect of the turbulence would be significant if the waves were to propagate over large distances without change of form.

(b) The turbulence

One of the most important comparisons between data and models concerns the levels of turbulent energy produced. Graphs of diagonal stress components for the model runs are shown in Fig. 20. These should be compared with Fig. 2(a) remembering to multiply the results in that case by a factor of 18. Several points should be made relating to this comparison. Firstly the energy levels produced by the numerical models depend heavily on the values used for various constants, in particular the diffusion coefficients; this is explained and quantified by Edwards (1992). Secondly the energy levels depend heavily on the strength of the mean shear and the stability. This explains the discrepancy between the background values in the present case. Note that the reduced background stability in the models also leads to reduced anisotropy there, the factor in Eq. (6) for the model results is closer to 1/2 than 1/6. The values of turbulence energy from the data analysis have been shown to depend on the value of \( p_e \). However, a value of 24 s was chosen as the most appropriate and it is the results obtained using this value which are considered throughout this section. Bearing in mind that the stabilities are significantly different in the models and the data, the amplitudes of the turbulence energy disturbances are surprisingly similar.
In both models the disturbance energies are positive at 16 m with a maximum close to the centre of the mean wave. The value of \( \langle u'^2 \rangle \) in the steady-state run could be misleading because the steady-state model has no wave in \( \langle u \rangle \). While this also holds in the data run s18, in that case it is probably due to the averaging procedure and not representative of a real wave with the same turbulence distribution.

The values of the off-diagonal stresses and heat fluxes for the steady-state model are shown in Fig. 21. These can be compared with Figs. 2(b) and 3(b), again after multiplication by 18. As with the diagonal stresses, the amplitudes of the disturbances are within a factor of two of those in the data run s18. An important feature of the averaged data run is a large wave-induced downward heat flux but this does not occur in the model results. The modelled momentum fluxes show a mainly downward turbulent flux of wave-induced momentum whereas in the data the opposite trend is more common.

(c) The importance of production

From the point of view of the effect of the mean flow on the turbulence, the main result of the numerical modelling of EM was that the turbulence structure was essentially production dominated. Edwards (1992) showed this to be partly a result of time-scales inherent in the models. Although randomness and lack of spatial resolution in the data make it difficult to estimate the production tensor itself, the results do suggest that production is also dominant in practice in this case. The principal evidence for this is that the main
wave-induced variations in the turbulence correlations occur during the period of the mean wave where the mean shear perturbations are also concentrated, rather than later on. In some cases particular peaks appeared to be related to variations in production, such as the maximum in $\langle v'^2 \rangle$ in wave 2, as described in section 4. In the model it was production due to vertical shear that was most important, but in the data the horizontal and vertical shear components are equally as important. In fact the effects of horizontal shear are accentuated in conditions of strong anisotropy by the effect of the relative sizes of coefficients in the production tensor. A neat explanation of this is given by Lauder (1990). It is this which leads to the disproportionate effect of streamline curvature on turbulent flows noted by Bradshaw (see Lauder et al. 1975). An implication of this is that if the relative importance of horizontal and vertical scales in the model had been the same as those observed in the data, there might have been more significant differences between the results of the different closure schemes, as second-moment schemes are better at predicting these horizontal shear effects. Vertical-shear components for the two numerical runs are shown in Fig. 22. The wave variation in $\partial \langle u \rangle / \partial z$ is similar to that in the data run s18, although the estimated background shear in that case is negative, so that the positive perturbation represents a reduction in magnitude. This estimate is not very reliable, however, owing to the lack of vertical resolution, and the effect of different background flow directions on the averaging procedure.

(d) Linearity and $rdW$

EM analysed the results of numerical modelling using a triple decomposition of variables of the form:

$$ q = \bar{q} + \tilde{q} + q', $$

(10)

where $\tilde{q}$ represents the wave or disturbance part of $q$, $q'$ the turbulent part and $\bar{q}$ the background or one-dimensional part. The decomposition was defined by two separate averaging operators $\bar{\phantom{a}}$ and $\langle \phantom{a} \rangle$ as follows:

$$ \tilde{q} = \langle q \rangle - \bar{q}, $$

(11)

$$ q' = q - \langle q \rangle. $$

(12)

The wave parts of the turbulence correlations were denoted by $r_{ij}$ and $r_i$, that is:

$$ r_{ij} = \langle u'_i u'_j \rangle - \bar{u'_i u'_j}, $$

(13)

Figure 22. Mean vertical-shear components at 16 m from the numerical model runs: (a) steady-state run, and (b) time-dependent run.
\[ r_i = (u'_i \theta') - \bar{u}'_i \bar{\theta}'. \]  

(14) \[ \frac{1}{2} r_{ii} \] is then the turbulent contribution to the wave kinetic energy. The production term in the equation for \( \frac{1}{2} r_{ii} \) was denoted by \( rdu \) and given by:

\[ rdu = rDU + rDW + qdW \]  

(15)

where:

\[ rDU = -r_{ij} \frac{\partial \bar{u}_i}{\partial x_j}, \quad rDW = -r_{ij} \frac{\partial \bar{u}_i}{\partial x_j}, \quad qdW = -u'_i u'_j \frac{\partial \bar{u}_i}{\partial x_j}. \]  

(16)

One way to assess the overall effect of the passage of the waves on the boundary layer is to integrate the governing equations over the entire region of the disturbance. Applying this procedure to the numerical results, EM suggested that the instantaneous (or local) response of the turbulence was dominated by terms which varied linearly with wave amplitude but had negligible global average. Thus the overall energy transfers were strongly nonlinear and dependent on the background flow. \( rDW \) however, was found to respond in an approximately quadratic way and thus to have a more predictable spatially averaged effect than the other energy transfer terms.

For the data it is difficult to assess accurately whether the turbulence reponds linearly to the wave. However, the limited investigation of a smaller wave in section 4 failed to provide any evidence that the instantaneous, local response was strongly nonlinear. Although estimating the term \( rDW \) would be even more uncertain than estimating the production tensor it is significant that this term would be expected to be of greater relative importance than it is in the models. This is because the much weaker background stability in the models leads to much stronger background turbulence. Values of wave and background shear and of wave-induced turbulence, on the other hand, are of similar magnitude in the models and data. Hence, \( qdW \) is likely to be of much greater relative importance in the models than in reality. A consequence of this is that, for the observed waves, \( rDW \) is likely to have a greater influence on the production tensor than for the model waves. In the model results the globally averaged contributions of each of the three terms in the production tensor were of similar magnitude; hence for the observed waves only \( rDU \) and \( rDW \) are likely to be important. When globally averaged, it is to be expected that the wave causes positive transfer of energy from mean to turbulent parts of the flow and that the amount of energy transferred is quadratic in the wave amplitude. This is still consistent with the observation of an approximately linear local response of the turbulence to the wave because when spatially averaged, the linear contribution will cancel, leaving only a quadratic residual.

6. DISCUSSION

Without a very large number of good cases the data are not suitable for defining values of coefficients in closure schemes. Standard closure schemes rely in any case on the existence of a wider spectral gap than that observed. However, the observed values of wave-induced perturbations in the turbulence correlations were similar to those predicted by the Wyngaard (1975) second-moment closure used in EM. In general, turbulence fluxes seemed to be largely governed by mean shear production, as in the model. Both of these results depend on the constants of the model and thus constitute a verification of the suitability of the model used. It should be noted though, that the background stability in the numerical model was widely different to that in the data, and levels of turbulence energy depend strongly on this parameter.
An interesting feature of some of the observed waves is a low- (vertical) level burst of turbulence occurring after the wave in the mean flow. However, this does not happen sufficiently often to appear in the averaged data run. The only sign of the burst in the averaged wave is an enhanced turbulent downward flux of momentum. With regard to the choice of type of closure scheme relevant to the modelling of this data, this burst provides the only very clear evidence of history effects. It is the ability to model such effects which makes dynamic closure schemes (i.e. those which contain dynamical equations for turbulence quantities) superior to algebraic closures. Another advantage of dynamic closures is that they can allow for the diffusion of turbulence. It is difficult to estimate the importance of diffusion from these data. However, the relative uniformity of turbulence energy levels between different heights contrasts with the strong vertical variations of turbulent viscosity that would be predicted by a zero-equation model responding only to the local mean shear.

The superiority of second-moment closure schemes rests largely on their ability to predict some of the effects of anisotropy and streamline curvature. The data exhibit very strong anisotropy due to stability and also significant streamwise mean strains (equivalent in this case to streamline curvature), so for these reasons higher-order closure schemes are more appropriate. The effects of using different closure schemes are discussed briefly by EM, and in more depth by Edwards (1992).

A major problem in respect of the modelling of the data is that the spectral gap between wave and turbulence is not very large. This means that many terms which are normally neglected in the derivation of governing equations for models are in fact of a similar order to those retained. To get around this problem really requires either a completely different type of model, and only direct simulation would really solve the problem, or a different type of decomposition of the data; it may be that if there were enough similar events to use an ensemble average as the sole decomposition operator the results would show a sufficient degree of independence between the two components of the decomposition. On the other hand, it is also possible that the physical processes of wave and turbulence in this case are not sufficiently separate for any such operator to exist.

One of the most important conclusions of this study is that the effect of the wave-induced turbulence on the development of the mean wave is small. Another important conclusion is that there is no evidence that the instantaneous, local effect of the mean wave on the turbulence is strongly nonlinear. The conclusions of the numerical modelling with respect to the term $r dW$ in the wave-induced turbulent kinetic energy equation, are more likely than in the model to apply to the spatially averaged effect of the wave. This is because the wave perturbations are more dominant relative to background values of variables in the observations than in the model. Thus the wave is likely to cause a net transfer of energy from mean flow to turbulence; the magnitude of the transfer being locally linear but, when spatially averaged, quadratic in the wave amplitude.

7. **Conclusions**

1. The transport of wave-induced velocity by wave-induced turbulence has been found to be generally upwards.
2. Waves are usually accompanied by a strong downward turbulent heat flux.
3. Waves are often associated with a burst of low-level turbulence which occurs later than the perturbation in the mean shear.
4. The effect of the turbulence on the mean wave is small.
5. The instantaneous, local effect of the wave on the turbulence appears to be approximately linear in the wave amplitude. It is likely that, averaged over the whole wave,
this effect almost cancels, leaving a residual which varies quadratically in the wave amplitude, as in the numerical simulations of EM.

6. Comparisons between the data and numerical simulations confirm the need for at least second-order closure schemes in order to capture the principal effect of wave–turbulence interaction.

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APPENDIX

Effect of averaging on turbulence correlations

It is shown in this appendix that the double averaging procedure reduces the values of turbulence correlations by a factor $1/n$ where $n$ is the size of the ensemble. Further, $r$th order turbulence moments are reduced by $1/n^{r-1}$.

The first average $\langle . \rangle$ is taken to be a finite sum while the second $\langle . \rangle$ is assumed to satisfy $\langle q \rangle = E[q]$. That is the mean of a quantity $q$ is equal to its expectation when $q$ is considered to be a random variable. The turbulent parts $q - \langle q \rangle$ are then random variables with zero mean. Note that due to the linearity of the transform $\langle . \rangle$, the turbulent part of the double average is the average of the turbulent parts, that is:

$$\{ q \} - \langle \{ q \} \rangle = \{ q - \langle q \rangle \}. \quad \text{(A.1)}$$

Thus if the turbulent part $q - \langle q \rangle$ of a quantity $q$ in the $r$th realization is denoted $X_r$, then $X_r$, $r = 1, \ldots, n$ are independent random variables with expectation zero. The turbulent part of the average $\{ q \}' = \{ q \} - \langle \{ q \} \rangle$ is by virtue of Eq. (A.1):

$$X = \frac{1}{n} \sum_{r=1}^{n} X_r. \quad \text{(A.2)}$$

Let $Y_r$, $Y$ be defined similarly for another quantity $s$. For fixed $r$, $X_r$ and $Y_r$ may not be independent. Now consider $E[XY]$.

$$E[XY] = E \left[ \frac{1}{n} \sum_{r=1}^{n} X_r \frac{1}{n} \sum_{r=1}^{n} Y_r \right] \quad \text{(A.3)}$$

$$= \frac{1}{n^2} E \left[ \sum_{r=1}^{n} X_r \sum_{r=1}^{n} Y_r \right]$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} E [X, Y_r] + \frac{1}{n^2} \sum_{r \neq s} E [X, Y_s].$$

The first term is $1/n$ the average expectation of $X, Y_r$ while the second is zero due to the independence of the variables at different $r$. In the original notation the result is:

$$\langle \{ q \}' \{ s \}' \rangle = \frac{1}{n} \langle \{ q \}' \{ s \}' \rangle. \quad \text{(A.4)}$$
The extra averaging has therefore reduced the turbulence correlations by a factor $1/n$. The generalization of the argument to higher moments is straightforward.

The same result follows in the case where $\langle \cdot \rangle$ is a low-pass filter of a time series if $\langle X_r Y_s \rangle = 0$ for $r \neq s$.

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