Modelling isolated wave–turbulence interactions in the stable atmospheric boundary layer

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SUMMARY

The interaction between a large-scale isolated wave and turbulence is considered in the context of the stable atmospheric boundary layer. A two-dimensional model with a second-moment turbulence closure is used to simulate the interaction numerically. Results obtained using other closures are discussed. The initial form of the wave-like disturbance is suggested by observations, which are described in a companion paper. The turbulence is shown to exert only a small influence on the wave, while the turbulence itself remains close to production–dissipation equilibrium. Wave-like components of the turbulence fluxes vary roughly linearly with wave amplitude, but overall the wave is strongly nonlinear and depends heavily on the background flow. Only one term in the equations for the quadratic turbulence quantities varies in a simple way with wave amplitude; this term represents a net transfer of energy from the wave to the turbulence.

KEYWORDS: Numerical modelling  Solitary wave  Surface layer  Turbulence

1. INTRODUCTION

This paper presents an analysis of the interaction between atmospheric boundary-layer turbulence and a large-scale isolated disturbance in stable conditions. In this context, large-scale means that the disturbance occupies the full depth of the boundary layer and that buoyancy plays a major role in the dynamics. The disturbance has many of the features of solitary waves but it does not propagate exactly without change of form, even in the absence of turbulence. The specified structure of the disturbance is suggested by observational data from the Antarctic stable atmospheric boundary layer. We have simulated the interaction numerically using a two-dimensional model with a second-moment turbulence-closure scheme. The results of using simpler closures are briefly discussed. Analysis of data from the Antarctic, which originally motivated the study, is presented in a companion paper (Edwards and Mobbs (1997), hereafter referred to as EM).

Fundamental to any detailed study of wave–turbulence interaction is the existence of a three-way decomposition of any variable \(q\) of the form:

\[
q = \bar{q} + \tilde{q} + q'
\]

where \(\bar{q}\) represents the wave or disturbance part of \(q\), \(q'\) the turbulent part and \(\tilde{q}\) the background part. Unless otherwise stated the background part will be assumed to be one-dimensional. Such a decomposition can be defined by two separate averaging operators \(\langle \cdot \rangle\) and \(\langle \cdot \rangle\) as follows:

\[
\bar{q} = \langle q \rangle - \bar{q}
\]

\[
q' = q - \langle q \rangle .
\]

The sum of wave and background components is therefore \(\langle q \rangle\), which will be referred to as the mean part of \(q\). Using these operators, rate equations for each of the components of the decomposition can be derived from the governing equations. In doing so, it is normally assumed that the operators are linear and that the following properties also hold:

\[
\langle \langle a \rangle \ b \rangle = \langle a \rangle \ \langle b \rangle
\]

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\[
\overline{ab} = \overline{a} \overline{b}
\]
\[
\langle \overline{ab} \rangle = \overline{a} \langle b \rangle
\]
\[
\overline{\langle a \rangle} = \overline{a}.
\]

Equation (4) effectively asserts that mean and turbulent parts of variables are uncorrelated, that is \(\langle (a) b' \rangle = 0\). Equation (5), in conjunction with Eq. (7), similarly ensures that background and wave quantities are independent. (Note that this does not mean that \(q\) and higher-order quantities such as \(q'^2\) are uncorrelated.)

In practice the most appropriate definitions for the decomposition operators vary according to the form of the data. Einaudi and Finncigur (1981), Finnigur et al. (1984) and Finnigur (1988) considered the interaction of nearly monochromatic, periodic waves with turbulence. In these studies the operator \(\langle \cdot \rangle\) could be defined by a phase-averaging process:

\[
\langle q \rangle = \frac{1}{N} \sum_{j=1}^{N} q(t + j\,P)
\]

where \(P\) is the period of the wave. Departures from exact periodicity and errors in estimating the precise period, however, were a significant source of difficulty in analysis and interpretation. These problems arise from slow modulation of wavelength and amplitude.

Phase averaging clearly cannot be used to analyse time series of isolated waves in which each disturbance appears only once. If a large number of similar events have been observed, it may be possible to define an ensemble average. Failing this, techniques based on the assumption of a scale separation between wave and turbulence can be used. In EM, time series recorded in the Antarctic are decomposed using Fourier analysis. This approach suffers from possible problems caused by the representation of a solitary waveform by periodic functions. In fact the Fourier technique has proved successful, but other basis functions could be used, even ones with useful orthogonality properties. Principal component analysis might provide such a technique (see EM).

A more serious but related problem is that, while there is a degree of separation of time-scales visible in the data, the separation does not constitute a wide spectral gap. This means that Eq. (4) may not hold. In fact a time-scale separation on its own is only sufficient to ensure Eq. (4) holds in the limit as the ratio of turbulent to wave time-scales approaches zero. In practice there may not exist a sensible decomposition which satisfies Eq. (4). The physical processes of wave and turbulence may not be sufficiently decoupled. Evidence to support this hypothesis can be gained, once a decomposition has been defined, by calculating the terms which would normally be neglected using Eq. (4). In EM these terms are shown to be of a similar order of magnitude to those retained in the case of the stable atmospheric boundary layer data.

Numerical studies of wave–turbulence interaction are not beset by the same problems. Standard single-point turbulence-closure models solve the Reynolds-averaged Navier–Stokes equation for a mean flow \(\langle u \rangle\) and incorporate some kind of parametrization of the turbulence correlation terms (e.g. \(\langle u'u' \rangle\)) but there are still at least two different approaches to defining a decomposition of the flow fields.

- In the first approach the mean flow is considered to represent the wave and background flow in a temporal, spatial or ensemble-averaged sense, while the effect of all turbulent eddies on this flow is parametrized by the closure scheme. There is no restriction on the size or form of the eddies parametrized except that they must satisfy Eq. (4), which implies that turbulent and mean fields are completely uncorrelated. Note that this is necessarily the approach taken in a one-dimensional study of a turbulent flow.
In two or three dimensions another approach is possible, in which at least some of the turbulent eddies are explicitly resolved. This is usually referred to as large-eddy simulation. In this case the turbulence closure represents only the effects of the subgrid-scale motions, and some of the resolved motions may also be classified as turbulent. A three-way decomposition of the resolved flow is then needed, which may involve performing an ensemble average of many runs.

The second approach was taken by Frederiksen and Bell (1983) and Shen and Holloway (1986) although in neither case was any attempt made to model waves which travel without change of form. The distinction between waves and turbulence in these papers was introduced by defining a transitional wave number separating motions in which buoyancy or inertial effects dominate. Interaction was considered in the sense of the interaction of different wave numbers, although time-averaged results were presented. These two studies both use spectral or pseudo-spectral (Shen and Holloway 1986) methods to integrate the governing equations. Both studies relate in particular to the ocean thermocline, rather than the atmospheric boundary layer where the presence of the lower boundary causes additional difficulties. Large-eddy simulations of the stable atmospheric boundary layer have been carried out, for example by Mason and Derbyshire (1990), although these have not been primarily concerned with large-scale disturbances such as gravity waves.

In this paper the first approach is adopted, so that the wave is fully resolved but the turbulence is modelled by a second-order closure scheme.

2. WAVE–TURBULENCE EQUATIONS

For the purposes of analysing the results of the model, the resolved mean flow is decomposed into wave and background parts. The background flow is considered to be fixed to simplify the analysis. Thus any wave-induced changes to the time- or space-averaged one-dimensional state will be considered here to be residual values of wave variables. Following Finnigan and Einaudi (1981) the wave parts of the Reynolds stresses are denoted by $r_{ij}$:

$$ r_{ij} = \langle u'_i u'_j \rangle - \overline{u'_i u'_j}, \tag{9} $$

and the wave parts of the turbulent heat fluxes are denoted here by $r_i$:

$$ r_i = \langle u'_i \theta' \rangle - \overline{u'_i \theta'}, \tag{10} $$

where $\theta$ is potential temperature. Mean and turbulent processes both contribute to the wave component of the kinetic energy $\widetilde{E}$ so we make the decomposition

$$ \widetilde{E} = \frac{1}{2} \langle u_i^2 \rangle - \frac{1}{2} \overline{u_i^2} = E_m + \frac{1}{2} r_{ii}, \tag{11} $$

where the mean part of the wave energy is given by

$$ E_m = \frac{1}{2} \overline{u_i^2} + \overline{u_i \widetilde{u}_i} + \frac{1}{2} \overline{u_i \widetilde{u}_i \widetilde{u}_i}. \tag{12} $$

In this case the last term is zero because the background flow is defined to be fixed. It is straightforward to derive rate equations for $E_m$ and $r_{ii}$. The equation for $E_m$ can be written as

$$ \frac{\partial E_m}{\partial t} = U dW + W dU + W dW + \Pi + B + D + \phi, \tag{13} $$
where the symbols on the right hand side of Eq. (13) have the following interpretation:

\[ U dW = - \langle u_i \rangle \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j}, \quad W dU = - \langle u_i \rangle \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \]

(14)

\[ W dW = - \langle u_i \rangle \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \]

(15)

\[ \Pi = - \langle u_i \rangle \frac{\partial \bar{p}}{\partial x_i} - \bar{u}_i \frac{\partial \bar{p}}{\partial x_i} \]

(16)

\[ B = \langle u_i \rangle \frac{g}{\theta_0} \delta_{i3} + \bar{u}_i \frac{g}{\theta_0} \tilde{\delta}_{i3} \]

(17)

\[ D = - \langle u_i \rangle \frac{\partial r_{ij}}{\partial x_j} - \bar{u}_i \frac{\partial u_i^* u_j^*}{\partial x_j} \]

(18)

\[ \phi = - \bar{u} \cdot f \mathbf{k} \times \mathbf{u}_g. \]

(19)

In the above, \( \theta_0 \) is a reference potential temperature; \( f \) is the Coriolis parameter; \( \mathbf{k} \) is the unit vertical vector; and \( \mathbf{u}_g \) is the geostrophic wind vector. Other symbols are standard. The first three terms in Eq. (13) correspond to advection. The terms \( \Pi, B, D, \) and \( \phi \) represent pressure, buoyancy, diffusion and Coriolis effects respectively. For the turbulence part we get

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} r_{ii} \right) = \frac{\partial}{\partial r} (U dr + W dW + W dq + r dU + r dW + q dW) + b + d - \bar{\zeta}, \]

(20)

where

\[ U dr = - \bar{u}_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} r_{ii} \right), \quad W dr = - \bar{u}_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} r_{ii} \right) \]

(21)

\[ W dq = - \bar{u}_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} u_i^2 \right) \]

(22)

\[ r dU = -r_{ij} \frac{\partial \bar{u}_i}{\partial x_j}, \quad r dW = -r_{ij} \frac{\partial \bar{u}_i}{\partial x_j}, \quad q dW = -u_i^* u_j^* \frac{\partial \bar{u}_i}{\partial x_j} \]

(23)

\[ b = \frac{g}{\theta_0} r_3 \]

(24)

\[ d = - \frac{\partial}{\partial x_j} \left( \left( \frac{1}{2} u_i^* u_i^* + p' u_j^* \right) - \left( \frac{1}{2} u_i^* u_i^* + p' u_j^* \right) \right) \]

(25)

The first three terms in Eq. (20) represent advection and the next three terms represent shear production. \( b \) and \( d \) represent buoyancy and diffusion respectively and \( \bar{\zeta} \) is the wave part of the dissipation rate \( \varepsilon \).

These equations can be used to consider the important question of whether the wave induces transfer of energy between mean flow and turbulence. With the background flow considered fixed this involves transfer of energy between \( E_m \) and \( r_{ii} \), the mean and turbulent parts of the wave energy. This transfer occurs via the terms \( D \) in Eq. (13) and \( r du \) in Eq. (20), which are related by the following equation:

\[ D + r du = - \frac{\partial F_i}{\partial x_i} \]

(26)
where

\[ F_i = \langle u_j \rangle r_{ij} + \ddot{u}_j \dot{u}_i. \]  

(27)

The picture is greatly simplified (Finnigan 1988) by considering the equation formed by applying the operator \( \overline{\cdot} \) to Eq. (20) and integrating the result vertically. \( \overline{\cdot} \) in that case denotes a long-time average. The result is that only one term contributes to the energy transfer in the interior of the atmosphere, namely \( r dW \). An almost analogous procedure in the case of the present numerical model is to integrate the equations over the entire region of the disturbance, \( V \). The result of this process is discussed later, but note that there are more contributory terms even from the interior in the case of a solitary wave. This is because the terms \( r dU \) and \( q dW \) in Eq. (20) vanish in the integrated form only if the following integrals vanish:

\[ \int_V r_{ij} dV \quad \text{and} \quad \int_V \ddot{u}_j dV. \]

This occurs for linear periodic waves but not for general solitary waves.

3. THE NUMERICAL MODEL

A series of experiments have been performed with the aim of investigating the interaction of a plane (two dimensional) isolated wave with turbulence in a stable atmospheric boundary layer, under varying conditions of background stability and mean shear. Our numerical model solves standard Reynolds-averaged equations for the mean velocity and potential temperature using a finite-difference method on a uniform rectangular grid in two dimensions. The turbulence closure is essentially that of Wyngaard (1975).

At the lower boundary the mean horizontal velocity is matched to a log-linear surface layer, and the potential temperature is prescribed with a constant cooling rate \( C \) which was set to \( 10^{-5} \text{ K s}^{-1} \). A zero vertical derivative condition was used at the lower boundary for all turbulence quantities except dissipation, which was assumed to be inversely proportional to height. For practical reasons we must assume that the detailed structure of the surface layer is not significant to the wave–turbulence interaction. (The surface layer in the model extends to 2.5 m.)

At the upper boundary, stress-free conditions were imposed on the mean flow variables while the turbulence quantities were all held fixed at small non-zero values. These conditions are appropriate to the observations described in EM where the waves were trapped in a thin boundary layer. At the top of the boundary layer strong stability inhibits turbulence and vertical motion, hence there is no vertically propagating wave activity. The height of the upper boundary in the model was set to 100 m.

The equations were solved in a frame moving at a constant speed \( u_F \) in the \( x \) direction such that \( \langle u \rangle < 0 \) at all heights at both vertical boundaries. Thus these boundaries can be properly defined as upstream and downstream respectively. Note that \( u_F \) is not the wave speed and in general there is a critical level \( z_c \) somewhere in the domain.

At the upstream boundary the mean potential temperature \( \langle \theta \rangle \) was made to decrease at the constant rate \( C \), while the other variables were held fixed. The values used corresponded to a one-dimensional boundary layer which was evolving slowly in relation to a time-scale for the waves. This background state was found by integrating the equations forward in time, starting with a quadratic and unidirectional mean velocity profile. The potential temperature was initially proportional to the stream function for the mean velocity.

The downstream boundary condition for all quantities was \( \partial / \partial x = 0 \), the domain being arranged to be sufficiently long for horizontal variations due to the wave to have become small by this point.
The initial mean flow for the two-dimensional runs was formed by adding a wave-like disturbance to the background one-dimensional flow. The same one-dimensional flow was used as the upstream boundary condition. For the purposes of analysis this background flow was considered to be fixed, leading to small error terms in Eqs. (13) and (20). The initial direction of the background flow was at an angle of $\pi/6$ to the direction of propagation of the wave (the $x$ direction) so as to avoid the case in which the wave and the mean flow move in the same direction. Observations presented in EM show that the two directions are frequently different. The wave itself was specified by a stream function $\tilde{\Psi}$ such that

\[ \tilde{\mathbf{u}} = \nabla \times (0, -\tilde{\Psi}, 0). \]  

(28)

$\tilde{\Psi}$ was given by

\[ \tilde{\Psi} = A \text{sech}^2(lx) \left( 1 - \cos \frac{\pi z}{H} \right), \]  

(29)

where $A$ is an amplitude; $l$ is an inverse horizontal length-scale set to 0.009 m$^{-1}$; and $2H$ is the height of the boundary layer, which is set to 100 m. This form of $\tilde{\mathbf{u}}$ was suggested qualitatively by observations of waves in the stable atmospheric boundary layer which are presented in EM. The resulting distribution of mean vertical velocity $\langle w \rangle$ is symmetrical about $z = H$ and antisymmetrical about $x = 0$ but is otherwise very similar to the $\langle w \rangle$ distribution shown later in Fig. 8. The initial distribution of $\langle \theta \rangle$ was chosen to satisfy

\[ \langle u \rangle \cdot \nabla \langle \theta \rangle = 0. \]  

(30)

Since turbulence terms typically turned out to be small compared with advective terms, this represents a $\langle \theta \rangle$ distribution which is almost steady in a frame of reference moving at speed $u_T$. Turbulence correlations were initially set to their upstream one-dimensional values throughout the model domain.

See Edwards (1992) for more details of the numerical procedures.

4. OUTLINE OF NUMERICAL EXPERIMENTS

A set of four runs of the numerical model is now described. The purpose of the runs was to study numerically the interaction of a plane (i.e. two-dimensional) isolated wave with atmospheric turbulence in a stable boundary layer. The initial conditions used for the runs were chosen to show the effects of varying background shear and buoyancy as well as different wave amplitudes. In order to do this, three different one-dimensional background states were produced, to each of which a wave-like perturbation in the mean flow was added. For one of the three one-dimensional background states, two runs with different wave amplitudes were considered, giving a total of four runs. The background states consisted of a low stability, low shear state and two more stable states which differed from each other principally in the strength of the mean shear. The one with lower shear (run 1) is used as a standard for comparison and is analysed in greater detail.

The background states are designed to show the effects of varying background flow conditions on the wave, rather than to represent realistic one-dimensional simulations. The initial states are characterized by the global averages of the gradient Richardson number $\langle Ri_g \rangle$ and shear $\langle S^2 \rangle$. These are shown in Table 1. (Throughout this section a global average over the spatial domain is denoted by $\langle \cdot \rangle$.)

Figures 1–3 show the one-dimensional (background) profiles of horizontal velocity and temperature for each of runs 1–3 respectively.

All the results of the two-dimensional runs in this section relate to the output of the model after a fixed time interval $T_f$, of 100 s. $T_f$ has been chosen such that the distance
travelled by the wave is greater than its horizontal wavelength (i.e. the distance over which significant wave perturbation occurs). This allows the mean flow to react to the presence of the wave. By this time the effect of the initial condition on the turbulence has become negligible. At values of \( t \) much larger than \( T_t \) sharp gradients develop at the centre of the wave and the wave structure no longer resembles the initial state. Also, the initial wave ultimately decays to insignificance. (This is largely because the initial condition is not an exact travelling-wave solution of the governing equations.) Thus the choice of \( T_t \) is a compromise between allowing the flow to develop fully, and maintaining a wave which resembles those which are observed. Because of the restricted range of appropriate values of \( T_t \), the global effect of the wave is best considered by integrating the results over the spatial domain at a fixed time rather than averaging over time at a fixed horizontal position.

\[
\begin{array}{c|c|c|c}
\text{Run} & \{R_i\} & \{2S^2\} & A \\
\hline
1 & 0.104 & 0.0121 & 15 \\
2 & 4.62 \times 10^{-3} & 0.0963 & 15 \\
3 & 0.0962 & 0.0471 & 15 \\
4 & 0.104 & 0.0121 & 7.5 \\
\end{array}
\]

Note that most of the shear in run 2 is concentrated in the surface layer, outside of which there is significantly less shear than in the standard run, run 1. Run 4 has the same background state as the other runs but reduced wave amplitude \( A \).
Figure 2. One-dimensional background flow state for run 2. The surface value of potential temperature $\theta$ has been subtracted.

Figure 3. One-dimensional background flow state for run 3. The surface value of potential temperature $\theta$ has been subtracted.

5. RESULTS FOR THE STANDARD CASE, RUN 1

(a) Global averages

(i) Mean wave-energy equation. The results of integrating Eq. (13) over the entire solution domain term by term at time $T_i$ are shown in Fig. 4. Both the total advective term $udu$ and the pressure term $\Pi$ can be written as boundary integrals which differ from zero only as a result of differences between the flows at the upstream and downstream boundaries. Assuming that the two ends are sufficiently far from the wave for horizontal gradients
to be negligible, differences between the flows at the upstream and downstream ends of the domain can be considered to constitute wave-induced changes in the background one-dimensional state (in our terminology such changes are considered as residual values of wave variables). Thus interaction with the background flow is the dominant effect on the rate of change of mean wave energy, with losses due to buoyancy and turbulence of secondary importance.

(ii) **Turbulent wave-energy equation.** Terms in the equivalent turbulent equation, which is the average of Eq. (20) taken at time $T_t$, are shown in Fig. 5. Here the most significant terms are the elements of the net production $rdu$ and the dissipation $\varepsilon$. Note that the largest positive term is $rdW$. As described in section 2 this would be the only non-zero production term in the interior of the domain for the case of a linear periodic wave.

(iii) **Energy transfer.** Transfer of wave energy between mean and turbulent components is governed by Eq. (26) which states that $D + rdu = -\nabla \cdot F$. In this case only about half of the average diffusive loss of mean wave energy $D$ appears as a gain of turbulent wave energy, the remainder being absorbed by the flux term. In the averaged equation this takes

![Figure 4](image1.png)  
**Figure 4.** Terms in the global average of Eq. (13). The correspondence between the numbers against the bars and the terms in the equation is as follows: $1 = dtU$, $2 = Nt$, $3 = B$, $4 = D$, $5 = \phi$, $6 = UdW$, $7 = WdU$, $8 = WdW$, $9 = rdW$. $dtU$ is an error term arising from development of the background flow.

![Figure 5](image2.png)  
**Figure 5.** Terms in the global average of Eq. (20). The correspondence between the numbers against the bars and the terms in the equation is as follows: $1 = dtq$, $2 = b$, $3 = d$, $4 = \tau$, $5 = Ud\tau$, $6 = Wd\tau$, $7 = Wdq$, $8 = rdU$, $9 = rdW$, $10 = qdW$, $11 = dr_t$. $dtq$ is an error term arising from development of the background flow.
the form of a boundary integral $I$ where

$$ I = \frac{1}{V} \oint_{\partial V} F_i n_i \, dS $$

$$ F_i = \langle u_i \rangle r_{ij} + \bar{u}_j u_i' u_j'. $$

$V$ is the area of the domain and $n$ is the unit vector normal to the boundary $\partial V$. The relatively large value of $I$ indicates that the values of the wave variables $\tilde{u}_i$ or $r_{ij}$ on the boundary must be significant. This is a further indication that wave-induced changes to the background flow have occurred.

(iv) **Buoyancy.** Although the buoyancy terms in both energy equations are fairly small it is interesting to note that they constitute an overall loss of energy at large scales and a gain at small scales. The same trend was found in the wave–turbulence study of Shen and Holloway (1986).

(b) **Energy exchange at fixed heights**

In the equation for $\frac{1}{2} r_{ij}$, net production $r du$ and dissipation $\tilde{c}$ are roughly in balance at all heights. There is also a balance between advection $u dr$ and rate of change $dtr$. These four terms are shown as a function of $x$ at a height of 50 m at time $T_i$ in Fig. 6. The breakdown of the production term into its three components is also shown at this height in Fig. 7. $rdu$ is essentially positive for all $x$ at all heights. This fact can be related to the production tensor $P_{ij}$ (see section 7), although $rdu$ does not dominate the total production $r du$. Buoyancy $b$ and diffusion $d$ are smaller than the other terms and only show a significant net change of energy lower down: at 25 m $b$ is a gain and $d$ a loss term.

The proportionality of production and dissipation is related to the close balance between the turbulent kinetic energy and dissipation. This is a characteristic of the turbulence closure and is discussed by Edwards (1992) where a time-scale relating to the balance is derived. For this model the time-scale is about 6 s which corresponds to a distance of about 30 m, so a lag of dissipation behind production on this time-scale would not show up on these plots.
Figure 7. A breakdown of the production term in the equation for $\frac{1}{2} r_{ii}$ at time $T_i$ (see text) as a function of $x$ at a height of 50 m.

Figure 8. Mean vertical velocity $\langle w \rangle$ (m s$^{-1}$) for run 1 at time $T_i$ (see text).

Similar plots of terms in the mean energy equation show strong bimodal signals due to pressure and advection, and smaller contributions from turbulence and buoyancy.

(c) Contour plots of mean and turbulence quantities

(i) Mean variables. Contours of $\langle w \rangle$ and $\langle \theta \rangle$ are shown for the entire solution domain at time $T_i$ in Figs. 8 and 9 respectively. Advection has distorted the initially symmetrical wave pattern and advected the disturbance at roughly the background flow speed. The appearance of the mean fields is only weakly affected by running the model with the turbulence terms turned off.

(ii) Turbulence variables. The most obvious features of the distributions of the turbulence variables can all be explained by considering two factors. These are the contribution of mean vertical shear to the production tensor, and the return to isotropy caused by the
pressure–strain parametrization. The dominance of vertical shear results partly from the choice of parameters in the initial wave and partly from the contribution of the background flow which has no horizontal variation at all. The result is that the distributions of turbulence variables resemble those of the mean shear terms which contribute to their production. Variables which lack strong direct shear production such as \(\langle w^2 \rangle\) are driven by the return to isotropy to an intermediate form resembling the turbulent kinetic energy, which is shown in Fig. 10.

6. SUMMARY OF RESULTS FOR OTHER RUNS

(a) Reduced static stability run, run 2

The averaged values of terms in Eq. (20) for run 2 are shown in Fig. 11. Contours of \(\langle w \rangle\) and turbulent kinetic energy at time \(T_f\) are plotted in Figs. 12 and 13 respectively.

The main features of the development of the wave and the dependence of the turbulence structure on the mean flow are the same as in run 1. As before, the mean wave energy is increasing at time \(T_f\) due to the secondary wave which is visible in \(\langle \theta \rangle\). The maximum
Figure 11. Terms in the global average of Eq. (20) for run 2. The correspondence between the numbers against the bars and the terms in the equation is as follows: 1 = \text{dr}q, 2 = b, 3 = \text{d}q, 4 = \varepsilon, 5 = U\text{dr}, 6 = W\text{dr}, 7 = W\text{dq}, 8 = r\text{d}U, 9 = r\text{d}W, 10 = q\text{d}W, 11 = d\text{r}. \text{dr}q \text{ is an error term arising from development of the background flow.}

Figure 12. Mean vertical velocity \(w\) (m s\(^{-1}\)) for run 2 at time \(T_f\) (see text).

Figure 13. Turbulent kinetic energy (m\(^2\)s\(^{-2}\)) for run 2 at time \(T_f\) (see text).
vertical displacement caused by the wave is very similar in the two cases. Again it is the principal shear components of the mean flow which govern the turbulence structure. In the $r_{ii}$ equation the same balances between $rdu$ and $\bar{e}$ and between $udr$ and $dr\bar{r}$ are apparent. $r dW$ is again positive at all points despite the fact that the average value of the total production term $r du$ is negative.

The main differences between the two runs can be easily related to the differences between the background flows. For instance, mean shear structure leads to a positive mean wave energy, whereas the mean wave energy in run 1 is negative (because in run 1 the wave causes a reduction in mean kinetic energy compared with the background flow). The vastly reduced background temperature gradient results in the buoyancy terms becoming negligible, while the reduced stability leads to larger values of all turbulence quantities including the turbulent kinetic energy. The reduced stability evident in the smaller gradient Richardson number also leads to a significantly different mean pressure distribution, showing the greater importance of momentum relative to buoyancy forces. The average transfer of wave energy between its two components is also different. In this run the transfer terms in both the mean and turbulent wave-energy equations are negative on average, showing in both cases a net loss of energy to the background flow.

(b) High shear run, run 3

The values of terms in Eq. (20) averaged over the whole domain at time $T_f$ are shown in Fig. 14. Contour plots of $\langle u \rangle$ and the turbulent kinetic energy at time $T_f$ are given in Figs. 15 and 16 respectively. The appearances of the mean and turbulent fields are very similar to the standard case. In the $r_{ii}$ equation, production $rdu$ and dissipation $\bar{e}$ are again in balance as are the advection $udr$ and rate of change $dr\bar{r}$. $r dW$ is again positive despite the reversal in sign of both the average value of $r_{ii}$ and that of $\bar{e}$.

The main differences are directly related to the increased shear in the background flow. This leads to a larger value of average mean wave energy $\langle E_m \rangle$. There is also a larger average turbulent kinetic energy despite the similar values of $R_i$. The larger shear causes a more rapid decay of the initial wave, the vertical amplitude being significantly less in this run than in the standard run. The contour plots of $\langle u^2 \rangle$ and $\langle v^2 \rangle$ show less similarity than they do in the standard case, leading to more complicated forms in the graphs of terms in the $r_{ii}$ equation. As in the low stability run, the energy-transfer terms in both the wave-energy equations are negative on average, so that these terms represent a loss of wave energy to the background flow.

(c) Reduced wave amplitude run, run 4

Run 1 was repeated with the initial wave amplitude $A$ of Eq. (29) reduced from 15 m$^2$s$^{-1}$ to 7.5 m$^2$s$^{-1}$. Examination of the contour plots at time $T_f$ showed only small differences in the forms of the variables other than a decrease in the wave amplitude. The maximum vertical amplitude of the secondary wave in $\langle \theta \rangle$ was 15 m using $A = 7.5$ m$^2$s$^{-1}$ as compared with 27.5 m with $A = 15$ m$^2$s$^{-1}$. A plot of the turbulent kinetic energy is shown in Fig. 17.

Graphs of terms in the wave-energy equations as functions of $x$ at fixed height were also very similar in form to those for the standard case, the principal difference being a reduction in amplitude. In general, the maximum values attained by the terms in these graphs for run 4 are about 0.4 times the values attained in the corresponding graphs for run 1. Thus halving the initial wave amplitude has had a nearly linear effect on each individual term in the wave-energy equations, even after 100 s of run time. An important exception is $r dW$ which is quadratic in the wave amplitude. The amplitude of this term
Figure 14. Terms in the global average of Eq. (20) for run 3. The correspondence between the numbers against the bars and the terms in the equation is as follows: 1 = d\eta, 2 = b, 3 = d, 4 = \bar{e}, 5 = U\,dr, 6 = W\,dr, 7 = W\,d\eta, 8 = r\,dU, 9 = r\,dW, 10 = q\,dW, 11 = d\,dr. d\eta is an error term arising from development of the background flow.

Figure 15. Mean vertical velocity (\langle u \rangle) (m s\(^{-1}\)) for run 3 at time \(T_f\) (see text).

Figure 16. Turbulent kinetic energy (m\(^2\) s\(^{-2}\)) for run 3 at time \(T_f\) (see text).
Figure 17. Turbulent kinetic energy $(m^2 s^{-2})$ for run 4 at time $T_f$ (see text).

Figure 18. A breakdown of the production term in the equation for $\frac{1}{2} r_{ii}$ for run 4 at time $T_f$ (see text) as a function of $x$ at a height of 50 m.

(see Fig. 18) is about a quarter of that for run 1 shown in Fig. 7. The other term whose parametrization is nonlinear in the wave amplitude is the diffusion $d$. The variation in this term is intermediate in the sense that the maximum value attained in run 4 at 25 m is about 0.3 times the appropriate value for run 1. However, the form is slightly different so that the peak-to-peak amplitude varies by a factor close to 0.4, as it does for most of the other terms.

Plots of the global averages of terms in the wave-energy equations for the reduced wave-amplitude runs paint a much more complicated picture. These are shown for the case of run 4 in Figs. 19 and 20, the corresponding figures being 4 and 5. In the equation for $\{r_{ii}\}$, the value of the term $rdW$ in run 4 is between a third and a quarter of its value in run 1. Amongst the other terms some are decreased by approximately a factor of two, such as $\varepsilon$ and $b$, while others are wildly different, in some cases taking the opposite sign. Repeating
Figure 19. Terms in the global average of Eq. (13) for run 4. The correspondence between the numbers against the bars and the terms in the equation is as follows: 1 = dtU, 2 = Π, 3 = B, 4 = D, 5 = φ, 6 = UdW, 7 = WdU, 8 = WdW, 9 = drU. drU is an error term arising from development of the background flow.

Figure 20. Terms in the global average of Eq. (20) for run 4. The correspondence between the numbers against the bars and the terms in the equation is as follows: 1 = drq, 2 = b, 3 = d, 4 = ε, 5 = Udr, 6 = Wdr, 7 = Wdg, 8 = rdlU, 9 = rdlW, 10 = qdgW, 11 = drq. drq is an error term arising from development of the background flow.

runs 2 and 3 with reduced wave amplitude again showed that \( rdW \) is the only term in either equation which always varies by a similar factor when the initial wave amplitude is halved.

7. Discussion

An important conclusion which can be drawn from the results of the previous section is that the dominant terms in both mean and turbulent equations respond roughly linearly to variations in wave amplitude, for waves of the type studied. (Note that the vertical displacement amplitude of the waves is significant compared with the depth of the domain.) On the other hand, the global changes caused by the waves do not vary linearly with wave amplitude. Now in the case of the wave-energy equations, the net energy exchange caused by each term is small in relation to the maximum instantaneous value of the term. The global changes appear to be the result of small nonlinear effects which depend on the interaction between the wave and the background flow. The exception to this rule is \( rdW \) which is the nonlinear part of the turbulent wave-energy production term. This term is
always positive and behaves in a roughly quadratic way with respect to variations in wave amplitude. \( r dW \) is nonlinear in the sense that it represents production of turbulent kinetic energy by the action of wave-induced turbulence acting on the gradients of the wave in the mean flow. The positivity of \( r dW \) would be expected from the simple gradient-transport hypothesis:

\[
 r_{ij} = -2\nu_r \tilde{S}_{ij} \quad (33)
\]

where \( \nu_r \) is the turbulent viscosity and \( S_{ij} \) is the symmetric part of the rate-of-strain tensor. Equation (33) implies that

\[
 r dW = -r_{ij} \frac{\partial \tilde{u}_i}{\partial x_j} = +2\nu_r \tilde{S}^2 > 0. \quad (34)
\]

However, the same conclusion can be reached without the assumption of isotropic diffusion inherent in Eq. (33) by considering the production tensor \( P_{ij} \). Assume that the background flow is one-dimensional and that the disturbance is production dominated, as it is in this case, so that

\[
 r_{ij} = \lambda_{(ij)} \tilde{P}_{ij}, \quad (35)
\]

where \( \lambda_{(ij)} \) are positive numbers. Then consider separately the response of \( P_{ij} \) to wave-induced variations in mean shear. In each case \( r dW \) can be found by simple algebraic manipulation. The result for \( \partial \tilde{u}/\partial z \) is:

\[
 r dW = -r_{13} \frac{\partial \tilde{u}}{\partial z} = \lambda_{13} \overline{w}^{\prime 2} \left( \frac{\partial \tilde{u}}{\partial z} \right)^2 > 0. \quad (36)
\]

For \( \partial \tilde{w}/\partial z \) or \( \partial \tilde{w}/\partial x \) non-zero we get a similar result although for \( \partial \tilde{u}/\partial x \), which equals \(-\partial \tilde{w}/\partial z \), and for \( \partial \tilde{w}/\partial x \) the results are not so conclusive.

Note that the result that \( r dW \) is consistently positive contrasts with the findings in respect of periodic waves observed by Finnigan et al. (1984), Finnigan and Einaudi (1981) and Finnigan (1988) where \( r dW \) was oscillatory. Unless a strong temperature wave existed, the average value of the term was small due to a phase lag of \( \pi/2 \) between the components of \( r_{ij} \) and the wave shear. This result was shown by Finnigan (1988) to agree with the predictions of rapid distortion theory. The difference between the two cases is that for the solitary waves studied here, the turbulence is close to an equilibrium state in which the wave-stress tensor \( r_{ij} \) is roughly proportional to the wave part of the production tensor \( \tilde{P}_{ij} \).

In the rapid distortion regime, which is used to explain the observations of periodic waves, the assumption is that the turbulence is in the initial stage of straining by the mean flow, far from equilibrium. Thus in that case the rate of change of \( r_{ij} \) is proportional to \( \tilde{P}_{ij} \).

The standard run was repeated using both zero- and two-equation turbulence closures. The lack of advective, diffusive and history effects in the zero-equation closure make it distinctly inadequate for wave–turbulence problems. Of the two more complicated closures, the eleven-equation form has some advantages in its ability to model the effects of anisotropy arising both from the wave and from the background stability. Some of the most important differences between the two dynamic closures were actually caused by the choice of constants in the closures rather than the number of modelled quantities. This applies to the energy levels and the time-scale for lag between the mean flow and turbulence. Edwards (1992) derives an expression for this time-scale in terms of the model closure constants. If the details of the turbulence are not of interest then the closure scheme may be of little importance, while if the development of the mean flow is not of interest then it is probably unnecessary to use a fully time-dependent model. If only the turbulence is
being modelled dynamically, good solutions can be obtained very quickly by solving along characteristics neglecting only the comparatively small diffusion term.

Perhaps the most important result of the comparison of different closure schemes was to reinforce the result that the effect of the turbulence on the mean flow is small relative to advective and pressure effects.

8. SUMMARY AND CONCLUSIONS

This paper has considered the interaction of a particular form of plane solitary wave with atmospheric turbulence in a stable boundary layer. The turbulence was modelled by a widely used form of second-moment closure. A number of conclusions can be drawn from the results:

1. The effect of the turbulence on the wave is small.
2. The turbulence itself remains close to equilibrium, in the sense that production balances dissipation. Thus the turbulence structure is closely related to the mean shear structure. For the parameters chosen, it is sufficient to consider the principal vertical shear components, while for those variables for which the production term does not contain principal shear components, the return-to-isotropy part of the pressure–strain parametrisation tends to dominate. This state of affairs means that the distributions of many of the turbulence variables are very alike, suggesting that a simpler model could produce similar results.
3. The dependence of the wave-like component of the turbulence on the amplitude of the wave in the mean flow is very roughly linear.
4. The overall effect of the wave on the background flow, however, is nonlinear and depends strongly on the structure of the background flow. This point in particular makes it difficult to draw conclusions about the effect of real waves on the atmospheric boundary layer, especially as the initial mean disturbance used is not a travelling wave solution of the Navier–Stokes equation.
5. The only term in the turbulence equations which responds in a fairly predictable way to changes in wave amplitude is $rdW$, which is roughly quadratic in the wave amplitude. $(rdW$ is the nonlinear part of the wave-like turbulent kinetic-energy production term $rd\omega$.) This term, which is always positive, is one of those representing the transfer of energy from wave to turbulence.
6. Buoyancy is not an important term in mean or turbulence equations despite the fairly large temperature gradients present in some cases. However, changes in the background stability, or in the background shear, can cause large changes in the overall level of turbulent kinetic energy. The average level of turbulent kinetic energy varied by a factor of four between the runs considered.

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