On the concept of exergy and available potential energy

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SUMMARY

A systematic derivation of the concepts of available potential energy of Lorenz and Van Mieghem from an extended concept of exergy is performed. This procedure leads to an extension of Van Mieghem’s global theory to a local formulation which can be used in limited-area models. In the limit of small amplitude motion it is shown that the extended exergy reduces to the available potential energy of Boussinesq-approximated models. Moreover, Shepherd’s concept of pseudoenergy is extended in order to deal with diabatic and irreversible processes.

KEYWORDS: Available potential energy  Entropy  Exergy

1. INTRODUCTION

The energy of a meteorological system is an important quantity to characterize its state, but there are fundamentally different types of states bearing the same value of energy. For example, a fixed value of energy could be present in the form of macroscopic momentum within a system, or through the heat content in a basin of uniform temperature. The first form—kinetic energy—is a useful form and the second is a useless form of energy. The expression useful energy means that we are, in principle, able to transform it into any different type of energy. A basin of heat with a contrast in temperature is an example where the energy content may be partially useful. This example suggests that the usefulness depends on the distance of the system from a stable reference state (in this case the basin with uniform temperature). The question of what portion of the atmospheric energy content is useful is a classical one. Lorenz (1955) investigated what portion of total potential energy could be transformed into kinetic energy under the constraint of quasi-hydrostatic and adiabatic processes. He proposed a quantity called the available potential energy (APE), which is defined as the difference between a system’s (mass integrated) total potential energy and the total potential energy of a hydrostatic reference state. Van Mieghem (1956) extended the analysis of Lorenz to deal with non-hydrostatic states of the atmosphere. A disadvantage of these approaches is that they cannot be used in limited-area models, because there is no local equivalent of APE.

In technical thermodynamics there is a related quantity which measures the useful portion of a system’s energy with respect to its environment: the exergy of the system. Recently the concept of exergy has been applied to the atmosphere in order to deal with forms of energy available for conversion into kinetic energy (Marquet 1991, Karlsson 1990, Pearce 1978). The connection with the APE-theory of Lorenz (1955) has been investigated by Pearce (1978) and Marquet (1991), but not yet the connection with the APE-theory of Van Mieghem (1956).

Another approach to investigate the available energy was suggested by Andrews (1981) as well as by Holliday and McIntyre (1981), who proposed a possible unification of different concepts of available potential energy under the constraint of adiabatic-reversible processes. Andrews showed that pure gravity-wave energy in an incompressible flow, as well as Lorenz’s APE, can be derived from an extended available potential energy under the constraint of adiabatic and reversible processes. Shepherd (1993) used the adiabatic-reversible Hamiltonian formalism of hydrodynamics in order to derive the same quantity found by Andrews (called pseudoenergy by Shepherd). The connection between this theory

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and the concept of exergy has not been investigated systematically, although Marquet (1995) showed some remarkable correspondences.

The purpose of this contribution is to present a systematic derivation of Lorenz’s (1955) and Van Mieghem’s (1956) global APE-theories from an extended version of the concept of exergy, and to extend these theories to local formulations so that they can be used in limited-area models. It is shown that the local APE-formulations in Boussinesq-approximated models can be derived in a straightforward way from the extended exergy. Furthermore, Shepherd’s (1993) concept of pseudoenergy is extended in order to deal with diabatic and irreversible processes.

2. THE CLASSICAL CONCEPT OF EXERGY

The concept of exergy has its origins in technical thermodynamics. The word exergy was created by Rant (1956) for a quantity which can be defined as the available work of a system in connection with its environment in thermodynamic equilibrium. The state of thermodynamic equilibrium is characterized by constant values of temperature and pressure. When applying the concept of exergy to meteorological systems it is convenient to choose a reference state, analogous to the environment in technical thermodynamics, but to modify the equilibrium so that the pressure is in mechanical and, as a consequence, in hydrostatic equilibrium (Karlsson 1990). Dutton (1973) derived the isothermal and hydrostatic state by a maximization of entropy, indicating the property of thermodynamic equilibrium in the presence of gravity. Exergy can be determined from measurements of the system’s energy and mass. It is a positive quantity which is locally defined at $x$, given a reference state $x_0$, by the expression

$$e_{ex}(x) := e(x) - e(x_0) - \nabla_x e(x_0) \cdot (x - x_0)$$

(1)

at a fixed spatial location $r$ and time $t$. $e(x)$ is the specific energy of the system (for meteorological applications the sum of kinetic, internal and potential energy) which is a convex function of its variables. Convexity is equivalent to the condition of positive definiteness of the matrix of second derivatives of energy, and so coincides with the stability-condition of the local thermodynamic equilibrium (Falk 1968). The vector $x := (v, s, \alpha)$ represents the natural variables of energy (specific momentum, specific entropy and specific volume) in the special case of a one-component system. The subscript $x$ at the Nabla operator denotes derivatives according to

$$\nabla_x e(x) := \left( \frac{\partial e}{\partial v}_{|s, \alpha}, \frac{\partial e}{\partial s}_{|v, \alpha}, \frac{\partial e}{\partial \alpha}_{|s, v} \right) = (v, T, -p),$$

where $v$ is the specific momentum, $T$ the temperature and $-p$ the negative pressure, in phase space. $x_0$ is the isothermal and hydrostatic equilibrium state. Expression (1) has a simple geometric interpretation: the function of state $e_{ex}(x)$ is defined by subtraction of the tangent in point $x_0$ from the convex function $e(x)$. Convexity of $e$ means (Barner and Flohr 1989):

- $e_{ex}(x) > 0 \quad \forall \ x \neq x_0$
- $e_{ex}(x_0) = 0$
- $\nabla_x e_{ex}(x = x_0) = 0$
- $H_{ex} = H_e$ is positive definite,
where $H_{ex}$ and $H_x$ are the matrices of the second derivatives of exergy and energy, respectively. So, $e_{ex}$ fulfills a local variational principle and, in the vicinity of the reference state $x = x_0$, $e_{ex}(x)$ can be approximated to lowest order

$$e_{ex}(x) = \frac{1}{2}(x - x_0) \cdot H_e(x_0) \cdot (x - x_0) \quad .$$

(2)

Gibbs equation for exergy can be derived by varying (1) with respect to the natural variables of energy. Gibbs equation for energy has the form (Falk 1968)

$$\delta e(x) = \nabla_x e(x) \cdot \delta x + \delta \phi \quad ,$$

(3)

where $\delta$ is an arbitrary variation. $\phi$ is the potential energy which is not dependent on the phase space variables $x$ but on spacial location $r$. Because the reference state $x_0$ is a special state of the system, Eq. (3) also holds for $x = x_0$. From insertion of Eq. (3) in the variation of Eq. (1) follows

$$\delta e_{ex}(x) = \{\nabla_x e(x) - \nabla_x e(x_0)\} \cdot \delta x - (x - x_0) \cdot \delta \nabla_x e(x_0) \quad .$$

(4)

Interpretation of the variations $\delta$ as individual derivatives, and insertion of the balance equation of mass, leads, for a given isothermal and hydrostatic reference state, to

$$\frac{de_{ex}}{dt} = \alpha \nabla \cdot (v p_0) + \frac{de}{dt} - T_0 \frac{ds}{dt} \quad .$$

From consideration of energy conservation it follows that, after integration over the mass of an insulated system, the balance

$$\frac{d}{dt} E_{ex} = -T_0 \frac{d}{dt} S \quad .$$

(5)

The mass-integrated balance of exergy for an insulated system is equivalent to the negative balance of entropy multiplied by the temperature of the thermodynamic equilibrium. This result is the same as Dutton (1973) obtains for a quantity called entropic energy, showing the equivalence of this concept to the concept of exergy. The balance (5) in addition to the positivity of exergy shows that $E_{ex}$ can be interpreted as a Lyapunov-functional of the system because of the second law of thermodynamics (e.g. Jetschke 1989). Therefore, the isothermal and hydrostatic reference state is a stable one.

3. Extension of the concept of exergy

In the classical concept of exergy derived in section 2 no restriction on the actual state of the system is made except for the convexity of energy. But the state of the atmosphere is in many cases—especially in large scales—restricted to the condition of convective stability. In this case there is a portion of exergy which can be identified as useless in respect of its availability for kinetic energy. Van Mieghem (1956) showed that to be sure that there is no portion of energy available for conversion into kinetic energy in the case of stable stratification, the mechanical, and so the hydrostatic, equilibrium condition is sufficient, rather than a thermodynamic equilibrium with constant temperature as suggested by the classical concept of exergy according to Karlsson (1990).

Gibbs equation for the sum of internal energy, $e_{inn}$, and potential energy (here called total potential energy) for a one-component system reads

$$\frac{d}{dt} (e_{inn} + \phi) = T \frac{ds}{dt} - p \frac{d\alpha}{dt} + \frac{d\phi}{dt} \quad .$$
In order to decompose the total potential energy in state functions whose balances are divided by the entropy balance, a temperature $T_0(s)$ is defined depending solely on entropy and the temperature decomposed into

$$T = \hat{T} + T_0(s) .$$

The idea is that most of the vertical structure of temperature can be represented by $T_0(s)$ in the sense that $T_0(s) = T_0(s_R + \Delta s)$, where $T_0(s_R) = T_R(z)$ is the reference temperature profile associated with the mean state of the atmosphere, and $\Delta s$ is a small entropy deviation. Now we can separate a total differential in the Gibbs equation of total potential energy

$$\frac{d}{dt} (e_{in} + \phi) = \hat{T} \frac{ds}{dt} - p \frac{d\alpha}{dt} + \frac{d\phi}{dt} + \frac{d}{dt} q_0 .$$

The function $q_0(s)$ is defined through the integral

$$q_0(s) := \int_{s_B}^{s} T_0(s') \, ds' + q_0(s_B) . \quad (6)$$

$s_B$ is a constant value of entropy. $q_0(s)$ contains the part of energy solely expressible by the entropy. Therefore, the balance of $q_0(s)$,

$$\frac{d}{dt} q_0(s) = T_0(s) \frac{ds}{dt} , \quad (7)$$

shows a strong link with the balance of entropy, and $q_0(s)$ can only be changed by irreversible and diabatic processes. $q_0(s)$ according to Eq. (6) is similar to a function defined by Andrews (1981) (Eq. (2.15) in Andrews's paper). The difference from Andrews’s definition is that the upper integration boundary of Eq. (6) is not necessarily the individually constant entropy of adiabatic-reversible processes as in his paper. It is important to notice that $T_0(s)$ contains no new information about the system if the reference state and the distribution of entropy are known. To see this, $T_0(s)$ is expanded in a Taylor series around the hydrostatic reference state (assuming the convergence of the series)

$$T_0(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n T_0}{ds^n} (s_R)(s - s_R)^n , \quad (8)$$

where the subscript R denotes the reference state. The reference temperature is defined to be a specialization of $T_0(s)$, e.g. $T_0(s_R) = T_R$. The needed derivatives can be calculated successively and as a function of the reference state only

$$\frac{d^n T_0}{ds^n} (s_R) = \nabla S_R \cdot \nabla \left\{ \frac{d^{n-1} T_0}{ds^{n-1}} (s_R) \right\} (\nabla S_R \cdot \nabla S_R)^{-1} . \quad (9)$$

The available work, $e_w$, is defined as the mechanical part of the total potential energy through subtraction of the function (6) from the total potential energy

$$e_w := e_{in} + \phi - q_0(s) \quad (10)$$

for which Gibbs equation reads

$$\frac{d}{dt} e_w = \hat{T} \frac{ds}{dt} - p \frac{d\alpha}{dt} + \frac{d\phi}{dt} . \quad (11)$$
The sum of available work and kinetic energy can be expressed using Eq. (10) as
\[ e_w + e_{\text{kin}} = e_{\text{in}} + e_w + e_{\text{kin}} - q_0(s) = e - q_0(s) \, . \]

Therefore, it follows from the energy conservation of an insulated system that the sum of the (mass integrated) available work, \( E_w \), and kinetic energy, \( E_{\text{kin}} \), is conserved for adiabatic-reversible (ad, rev) flows, where specific entropy, \( s \), is constant along trajectories
\[ \frac{d}{dt}(E_w + E_{\text{kin}}) \bigg|_{\text{ad,rev}} = 0 \, . \]

But the available work is not necessarily a positive quantity. In order to guarantee the positivity of an exergy-like quantity related to \( e_w \), the positive definiteness of the matrix of second derivatives of the available work is required. This can be expressed using its definition (10) as
\[ \frac{\partial^2(e_{\text{in}} - q_0(s))}{\partial s^2} > 0, \quad \frac{\partial^2 e_{\text{in}}}{\partial \alpha^2} > 0, \]
\[ \left[ \frac{\partial^2(e_{\text{in}} - q_0(s))}{\partial s^2} \right] \left( \frac{\partial^2 e_{\text{in}}}{\partial \alpha^2} \right) > \left( \frac{\partial^2 e_{\text{in}}}{\partial s \partial \alpha} \right)^2. \]  
(12)

Because of the stability of the local thermodynamic equilibrium the internal energy \( e_{\text{in}} \) fulfills the conditions (12) (Falk 1968), so that according to Eqs. (6) and (9) the condition
\[ \frac{dT_0(s)}{ds} \approx \frac{dT_0}{ds}(s_R) = \frac{dT_R(z)}{dz} \left\{ \frac{ds_R(z)}{dz} \right\}^{-1} \leq 0 \]
guarantees (12), if the reference state is a function of \( z \) only. Therefore, a stable reference state with temperature decreasing with height fulfills conditions (12). As shown in subsection 3(b) the condition of temperature decreasing with height can be dropped in the case of a perfect gas.

Now when the same transformation is applied to the available work it leads to the same form as when it is applied to the system’s energy, which is the classical concept of exergy (see Eq. (1))
\[ e_{\text{APE}}(x) := e_w(x) - e_w(x_R) - \nabla_x e_w(x_R) \cdot (x - x_R) \, . \]  
(13)

The vector \( x := (s, \alpha) \) contains the natural variables of available work, and the subscript \( x \) at the Nabla operator characterizes the derivatives in phase space
\[ \nabla_x e_w(x) := \left( \frac{\partial e_w}{\partial s} \bigg|_{q}, \frac{\partial e_w}{\partial \alpha} \bigg|_{s} \right) = \left( \bar{T}, -p \right) \, . \]

\( x_R \) denotes a hydrostatic but not isothermal reference state. With the conditions (12) fulfilled we have the same conclusions as for the exergy \( e_{\text{ex}} \):

- \( e_{\text{APE}}(x) > 0 \, \, \forall \, x \neq x_R \)
- \( e_{\text{APE}}(x_R) = 0 \)
- \( \nabla_x e_{\text{APE}}(x = x_R) = 0 \)
- \( H_{e_{\text{APE}}} = H_{e_w} \) is positive definite,
where $\mathbf{H}_{eape}$ and $\mathbf{H}_{ew}$ are the matrices of second derivatives of $e_{ape}$ and available work, respectively. An expansion of $e_{ape}$ to the lowest order

$$e_{ape}(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{x}_R) \cdot \mathbf{H}_{ew} (\mathbf{x}_R) \cdot (\mathbf{x} - \mathbf{x}_R)$$  \hspace{1cm} (14)

shows that $e_{ape}$ can be interpreted as a variance–energy. The result for the Gibbs equation of exergy (Eq. (4)) can also be assigned to give:

$$\delta e_{ape}(\mathbf{x}) = \{\nabla_x e_w (\mathbf{x}) - \nabla_x e_w (\mathbf{x}_R)\} \cdot \delta \mathbf{x} - (\mathbf{x} - \mathbf{x}_R) \cdot \delta \nabla_x e_w (\mathbf{x}_R) \hspace{1cm} (15)$$

Defining $\Delta \psi := \psi - \psi_R$ and choosing individual variations $\delta$ gives explicitly:

$$\frac{d}{dt}e_{ape} = \widehat{T} \frac{ds}{dt} - \Delta p \frac{d\alpha}{dt} + \Delta \alpha \frac{dp_R}{dt}$$ \hspace{1cm} (16)

with

$$e_{ape} := e_{ion} - \Delta q_0 (s; s_R) + p_R \Delta \alpha \hspace{1cm} \text{and} \hspace{1cm} \Delta q_0 (s; s_R) := q_0 (s) - q_0 (s_R) \hspace{1cm} (17)$$

where $T_0 (s_R) = T_R$ has been used. From Eqs. (16) and (17) it can be deduced that in the special case $T_0 (s) = T_0 = \text{constant}$ we get the classical concept of exergy when adding the kinetic energy to $e_{ape}$.

The quantity (17) was first derived by Andrews (1981) as a measure of energy available for conversion into kinetic energy. But the derivation of Andrews assumes adiabatic–reversible processes, so that this derivation can be interpreted as an extension of his results in order to treat irreversible and diabatic effects in the formalism. Shepherd (1993) called $e_{ape}$ (non-kinetic) pseudoenergy. His derivation uses the Hamiltonian formalism (likewise adiabatic–reversible).

In order to get a closed energy-cycle a function is defined to be the difference between the total potential energy and $e_{ape}$

$$e_{an} := e_{in} + \phi - e_{ape} = e_{in,R} + \phi + \Delta q_0 (s; s_R) - p_R \Delta \alpha$$ \hspace{1cm} (18)

which is called anergy in technical thermodynamics, obeying the Gibbs equation

$$\frac{d}{dt}e_{an} = T_0 \frac{ds}{dt} - p_R \frac{d\alpha}{dt} + \frac{d\phi}{dt} - \Delta \alpha \frac{dp_R}{dt}$$ \hspace{1cm} (19)

(a) Determination of the reference state

It was shown that the knowledge of the reference temperature profile is sufficient to calculate $T_0 (s)$ (see Eq. (8)). This profile could be determined by a horizontal average of the temperature field, or of a different thermodynamic field by means of an integration of the hydrostatic equation. But the question arises which field is the most suitable for applying the averaging procedure? To answer this question recall that the extended exergy concept was designed to find the useful energy under the constraint of a stable stratification. Every vertical profile of temperature leads, according to Eqs. (8), (6) and (13), to a special amount of (mass integrated) $e_{ape}$. If the profile is determined by the horizontal average of a thermodynamic field the values of $E_{ape}$ will not differ significantly. Nevertheless, that hydrostatic reference state is being sought which minimizes the mass integral of extended exergy

$$E_{ape} := \int_V \rho e_{ape} dV = \min$$ \hspace{1cm} (20)
The volume integral will be performed over a closed system with fixed boundaries. So, if a limited-area model with open boundaries is considered then it has to be extended over the system’s environment. The variation of \( E_{\text{APE}} \) is performed at a constant thermodynamic state. Using Eq. (15) leads to

\[
\delta_R E_{\text{APE}} = - \int_V \alpha_R (\rho - \rho_R) \; \delta_R p_R \; dV .
\]  

(21)

Note that, because of the hydrostatic reference state in connection with the constant mass, the variation \( \delta_R p_R \) is arbitrary inside the volume and zero at the boundaries. To exploit this expression it is assumed that the reference state depends only on the vertical coordinate \( z \), then

\[
\delta_R E_{\text{APE}} = - \int_A \alpha_R (\bar{\rho} - \rho_R) \; \delta_R p_R \; dz \quad \text{with} \quad A := \int_A \int dxdy \quad \text{and} \quad \bar{\rho} := A^{-1} \int_A \rho \; dxdy ,
\]

where the horizontal integration has been performed over areas outside orography. The condition \( \rho_R = \bar{\rho} \) leads to a vanishing variation and \( E_{\text{APE}} \) reaches an extremal value. To be sure that we have a minimum it is necessary to calculate the second variation

\[
\delta_R^2 E_{\text{APE}} = - A \int_z \bar{\rho} \delta_R \alpha_R \; \delta_R p_R \; dz ,
\]

because the second variation of the independent variable \( \delta_R p_R \) vanishes. The variation of the specific volume is calculated by using the hydrostatic equation

\[
\alpha_R^{-1} = - g^{-1} \frac{d p_R}{dz} .
\]

A partial integration with respect to the vertical coordinate leads to

\[
\delta_R^2 E_{\text{APE}} (\rho_R = \bar{\rho}) = A \int_z \frac{d}{dz} \left\{ (2g \bar{\rho})^{-1} \right\} \left( \delta_R p_R \right)^2 dz
\]

because the variation at the boundaries vanishes due to the constraint of constant mass. The conclusion is that \( E_{\text{APE}} (\rho_R = \bar{\rho}) \) reaches a minimum because the horizontally averaged density decreases with height under stable conditions. The reference state in the extended exergy concept has to be calculated by a horizontal average of the density field of a closed system. In the case of a limited-area model with open boundaries the averaging has to be extended over the environment of the system. The remaining thermodynamic reference variables can be calculated by integration of the hydrostatic equation and utilization of a gas law \( p_R (\rho_R, T_R) \).

The condition of a time-independent reference state corresponds with the condition of a sufficiently small horizontally averaged vertical mass-flux as can be deduced from the time derivative of the vertically integrated hydrostatic equation

\[
\frac{\partial p_R}{\partial t} = - g \int_z^{\infty} (\nabla \cdot \rho v) \; dz' = g \bar{\rho} w ,
\]

where \( w \) is the vertical velocity.
(b) Application to perfect gases

Formally, if \( T_0(s) \) as well as the substance under consideration are known Eq. (17) can be used to compute \( e_{\text{APE}} \). But in practice \( T_0(s) \) is not known analytically, but through a Taylor series expansion (8) up to any accuracy. If the function (6) is expanded directly in a Taylor series around the hydrostatic reference state

\[
\Delta q_0 = \frac{dq_0}{ds}(s_R) \Delta s + \frac{1}{2} \frac{d^2q_0}{ds^2}(s_R)(\Delta s)^2 + \sum_{n=3}^{\infty} \frac{1}{n!} \frac{d^nq_0}{ds^n}(s_R)(\Delta s)^n,
\]

it can be seen that Eq. (9) can be used to calculate the needed coefficients successively. However, by restricting the considerations to the lowest order, it will be shown that this represents the APE formulation proposed by Van Mieghem (1956) as an extension of Lorenz's APE to non-hydrostatic states of the atmosphere. Inserting Eq. (22) in Eq. (17) (with \( \Delta e_{\text{int}} = c_v \Delta T \), \( c_v \) being the specific heat at constant volume) and expanding the temperature deviation \( \Delta T(s, \alpha; s_R, \alpha_R) \) for a perfect gas in a Taylor series around the reference state, gives

\[
\Delta T = T_R \left( \frac{\alpha}{\alpha_R} \right)^{-R/c_v} e^{\Delta s/c_v} - 1 \right) \quad \text{with} \quad T_R = T_B \left( \frac{\alpha_R}{\alpha_B} \right)^{-R/c_v} e^{(s_R-s)}/c_v.
\]

The subscript B characterizes absolute constant thermodynamic reference values and \( R \) is the individual gas constant. The first order terms cancel each other. Insertion of the first order expression

\[
\frac{\Delta \alpha}{\alpha_R} = \frac{\Delta s}{c_p} - \frac{c_v}{c_p} \left( \frac{\Delta p}{p_R} \right),
\]

and elimination of specific volume leads to

\[
e_{\text{APE}} = \frac{1}{2} \left[ \left( \frac{T_R}{c_p} - \frac{dT_0}{ds}(s_R) \right)(\Delta s)^2 + R \frac{T_R c_v}{c_p} \left( \frac{\Delta p}{p_R} \right)^2 \right],
\]

in the lowest order. The factor multiplied with the entropy–variance can be transformed by consideration of Gibbs equation in the reference state for a perfect gas, in the form

\[
c_p \frac{dT_R(z)}{dz} = T_R(z) \frac{ds_R(z)}{dz} + \alpha_R(z) \frac{dp_R(z)}{dz},
\]

and the hydrostatic condition of the reference state. Using Eq. (9) it is found that

\[
\frac{T_R}{c_p} - \frac{dT_0}{ds}(s_R) = \frac{T_R}{c_p} - \frac{dT_R}{dz} \left( \frac{ds_R}{dz} \right)^{-1} = \frac{g}{c_p} \left( \frac{ds_R}{dz} \right)^{-1} = \frac{g^2}{c_p^2} N_R^{-2},
\]

where \( N_R \) is the Brunt–Väisälä frequency of the reference state. Thus, the conditions (12) have been verified for a stable reference state in the case of a perfect gas, and we have the positivity of extended exergy in the vicinity of the reference state.

Expression (24) with (25) was first derived by Van Mieghem (1956) in a mass-integrated form, in order to extend Lorenz's APE formulation to non-hydrostatic actual states of the atmosphere. Therefore, Eq. (24) can be interpreted as an extension of Van Mieghem's results to a local formulation.
From Eq. (24) it is immediately possible to derive the local form of APE in a Boussinesq-approximated model for shallow convection. Shallow convection is restricted to the lower few kilometres of the atmosphere and, as a consequence, relative pressure deviations from the reference state are negligible with respect to relative temperature deviations. The Taylor series expansion of \( s(T, p) \) for shallow convection, leads in first order to the approximation (Dutton and Fichtl 1969)

\[
\Delta s = c_p \frac{\Delta T}{T_R}
\]

and, with Eq. (25),

\[
\varepsilon_{APE, B} = \frac{g^2}{2 N_R^2} \left( \frac{\Delta T}{T_R} \right)^2.
\]

\( \varepsilon_{APE, B} \) is known as APE in a Boussinesq-approximated model for shallow convection (Dutton and Fichtl 1969) and is usually not derived from a superior principle like exergy, but from the approximated dynamic model equations.

4. LOCAL EXTENDED EXERGY-CYCLE

Now it is possible to calculate an extended exergy-cycle by inserting the balances of entropy and mass in the Gibbs equations (16) and (19) and considering the common kinetic energy balance resulting from the equation of motion

\[
\rho \frac{dv}{dt} = -\nabla p - \rho \nabla \phi - 2\rho \Omega \times v - \nabla \cdot F.
\]  

(26)

\( \Omega \) is the vector of the rotation of earth and \( F \) is the stress tensor. The balance of entropy of a one-component system including radiative effects reads

\[
\rho \frac{ds}{dt} = \frac{1}{T} \left( -\nabla \cdot J_q + J - \frac{F^T}{T} : \nabla v \right).
\]  

(27)

where \( J_q \) is the heat flux and \( J := -\nabla \cdot R \) is the convergence of radiative flux of energy (Dutton 1973). After some rearrangements the local cycle of energy follows:

\[
\rho \frac{d}{dt} e_{\text{kin}} = \underbrace{-\nabla \cdot (\Delta p v + F \cdot v)}_{\text{I}} + \underbrace{\Delta p \nabla \cdot v + \rho \frac{\Delta \alpha}{\alpha_R} v \cdot \nabla \phi + \frac{F^T}{T} : \nabla v}_{\text{II}}
\]

\[
\rho \frac{d}{dt} \varepsilon_{\text{APE}} = \underbrace{-\Delta p \nabla \cdot v - \rho \frac{\Delta \alpha}{\alpha_R} v \cdot \nabla \phi - \frac{F^T}{T} : \nabla v}_{\text{IV}}
\]

\[
\rho \frac{d}{dt} e_{\text{an}} = \underbrace{-\nabla \cdot (J_q + p_R v)}_{\text{I}} + \underbrace{\frac{F^T}{T} : \nabla v - \frac{F^T}{T} \cdot \nabla v}_{\text{III}}
\]

(28)

There are flux-divergence terms (I), the conversion terms between \( \varepsilon_{\text{APE}} \) and kinetic energy are reversible processes according to velocity-convergence effects and through buoyancy (II). Dissipation (III) destroys kinetic energy, while the irreversible terms (IV) do not have any definite sign. These effects are able to create APE.
Looking at the irreversible mechanisms which create $e_{APE}$ (term (IV) in Eq. (28))

$$\rho q_{APE} := \frac{\hat{T}}{T} \left( -\nabla \cdot \mathbf{J}_q + J - \mathbf{F}^T : \nabla \mathbf{v} \right) = \frac{\hat{T}}{T} \frac{\mathrm{d}s}{\mathrm{d}t},$$

it is obvious that from the supplied heat,

$$\rho q := -\nabla \cdot \mathbf{J}_q + J - \mathbf{F}^T : \nabla \mathbf{v},$$

only a fraction is available. The factor

$$\eta_c := \frac{\hat{T}}{T} = 1 - \frac{T_0(s)}{T}$$

is called the *Carnot factor* in technical thermodynamics (Baehr 1992). It states which portion of the supplied heat is used by the system to create $e_{APE}$. The possibility of negative Carnot factors enables the atmosphere to create extended exergy, although the supplied heat is zero when integrated over the volume of the system. In this case there must be a local heating at positive Carnot factors and a local cooling at negative Carnot factors. Lorenz (1955) called the factor $\eta_c$ the *local efficiency*. It takes values between −0.3 and 0.1 on average.

The local conservation law for pseudoenergy for *adiabatic-reversible* processes follows from the energy-cycle (28), by addition of the balances for $e_{kin}$ and $e_{APE}$, which can be found in Andrews (1981) and Shepherd (1993).

When adding $e_{APE}$ and kinetic energy and then integrating over the volume of a closed system it follows from cycle (28) that

$$\frac{\mathrm{d}}{\mathrm{d}t} (E_{kin} + E_{APE}) = \frac{1}{\int_{\partial V} (\mathbf{J}_q + \mathbf{R} + \mathbf{F} \cdot \mathbf{v}) \cdot \mathrm{dA} - \int_V T_0 \rho \frac{\mathrm{d}s}{\mathrm{d}t} \mathrm{dV}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{an} = \int_V T_0 \rho \frac{\mathrm{d}s}{\mathrm{d}t} \mathrm{dV}.$$  

(30)

Thus the time derivative of entropy provides a coupling between the balance of extended exergy plus kinetic energy, $E_{APE} + E_{kin}$, and of extended energy, $E_{an}$. The structure of Eq. (30) is similar to that of Eq. (5). The difference is that it is not permissible to take $T_0(s)$ out of the integral except if this temperature is constant, and so the concept of extended exergy is equivalent to the classical concept of exergy. The boundary integral (I) in Eq. (30) is vanishing when considering an insulated system. It follows that the *sum of $E_{APE}$ and $E_{kin}$ is conserved under the constraint of adiabatic-reversible processes*.

Similar to exergy, the sum of $e_{APE}$ and kinetic energy represents a Liapunov-functional when dissipation is the sole irreversible process within the system and the velocity vanishes at the systems boundary. Thus, the hydrostatic equilibrium is a stable one. Consider a situation in which $E_{kin} = 0$ and $E_{APE} > 0$ at an initial time in an insulated system. Then, if dissipation is the sole irreversible process, all of $E_{APE}$ is transformed into kinetic energy and dissipated, until all motion is removed and the system is in the reference state.

(a) *Lorenz's balance of available potential energy*

In the analysis of Lorenz (1955) the assumption of quasi-hydrostatic processes in the atmosphere is made. So far in this contribution no such approximations have been imposed. In a quasi-hydrostatic atmosphere it can be shown by partial integration that the
potential energy of a column of air is proportional to internal energy plus a term due to orographic effects

\[ P := \int_{A} \int_{z_s}^{\infty} \rho g z \, dz \, dxdy = \frac{R}{g} \int_{A} \int_{\lambda_0}^{\lambda_0} T \, dp \, dxdy + \int_{A} \phi_s p_s \, dxdy, \tag{32} \]

where the subscript \( s \) characterizes surface values. In order to incorporate the quasi-hydrostatic approximation in the extended exergy balance, the exact balance equation from cycle (28) is integrated over the system’s (closed) volume

\[ \int_{\tau} \frac{d e_{APE}}{dt} \, d\tau = \int_{\tau} \left\{ -p \frac{d \alpha}{dt} + p_R \alpha \nabla \cdot v - \frac{\alpha}{\alpha_R} v \cdot \nabla \phi + \frac{d}{dt} \phi + \frac{T - T_0(s)}{T} q \right\} \, d\tau. \]

d\( \tau := \rho \, dV \) is an element of mass, and the irreversible effects are represented by the rate of heating

\[ q = T \frac{ds}{dt}. \]

The second and third term on the right-hand side of the \( e_{APE} \) balance add to make a divergence term which vanishes after integration over the mass. The reversible work term is reformulated in the following way

\[ p \frac{d \alpha}{dt} = \frac{d}{dt} (p \, \alpha) - \alpha \frac{dp}{dt}. \]

Because of the quasi-hydrostatic condition of a column (Eq. (32)) it follows that for a perfect gas

\[ \frac{d}{dt} \int_{\tau} e_{APE} \, d\tau = \int_{\tau} \frac{R}{p} T \omega + \frac{T - T_0(s)}{T} q \right\} \, d\tau + \frac{d}{dt} \int_{A} \frac{\phi_s}{g} p_s \, dxdy, \]

where the vertical velocity \( \omega := \frac{dp}{dt} \) in pressure-coordinates has been used.

In order to reproduce Lorenz’s APE-balance a pressure associated with \( T_0(s) \) is defined

\[ T_0(s) = T_B \left\{ \frac{p_0(s)}{p_B} \right\}^\kappa \quad \text{with} \quad \kappa = \frac{R}{c_p}. \tag{33} \]

Eq. (33) represents the classical \( T(s, p) \) dependence for a perfect gas, in which \( T \) is replaced by \( T_0(s) \) and \( p \) by \( p_B \). By elimination of entropy between Eq. (33) and \( s(T, p) \) the Carnot factor (= local efficiency) can be recalculated in the form

\[ \eta_c = \frac{T - T_0(s)}{T} = 1 - \left\{ \frac{p_0(s)}{p} \right\}^\kappa. \tag{34} \]

Except for the orographic term this is the exact APE-balance for a quasi-hydrostatic atmosphere according to Lorenz, which can be found in Van Mieghem (1973), for example. In general, orographic effects are ignored when considering the balance of APE. Here it is possible to deal with such terms which may be important in regions with strong orography.

5. Example: Polytropic reference state

A typical example for a reference state is the classical polytropic atmosphere with a constant negative temperature gradient \( \gamma_R < g/c_p \) (Ertel 1938). In this case the exact
Figure 1. The underlying global (idealized) temperature and pressure distributions for exergy comparison—see text.

analytical specification of $T_0(s)$ and $e_{APE}$ as functions of temperature and pressure is possible. Using the temperature profile

$$T_R(z) = T_s - \gamma_R z$$

an integration of the hydrostatic equation for a perfect gas leads to the entropy distribution

$$s_R(z) = s_B + \ln \left\{ \left( \frac{T_s}{T_B} \right)^{c_p} \left( \frac{p_s}{p_B} \right)^{-R} \right\} + c_x \ln \left( 1 - \frac{z}{H_R} \right).$$

$H_R = T_s/\gamma_R$ is the height of the reference atmosphere, and the coefficient $c_x = c_p(\gamma_t/\gamma_R - 1)$ with $\gamma_t = g/c_p$ is a stratification-modified specific heat. $T_s$ and $p_s$ are constant surface values of reference temperature and pressure, respectively. The temperature $T_0(s)$ according to Eqs. (8) and (9) is

$$T_0(s) = T_R \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \left( \frac{\Delta s}{c_x} \right)^n \right\}.$$ 

This series converges for any stable stratification ($c_x > 0$) and we get the temperature $T_0(s)$ after elimination of the reference temperature

$$T_0(s) = T_s \left( \frac{T_s}{T_B} \right)^{c_p/c_x} \left( \frac{p_s}{p_B} \right)^{-R/c_x} e^{-(s-s_B)/c_x}.$$ 

According to the conditions (12) $e_{APE}$ of a polytropic reference state is positive for arbitrary large deviations from the reference state because $dT_0(s)/ds$ is negative. The deviation $\Delta q_0(s(T, p); s_R(T_R, p_R))$ is calculated using the definition (6)

$$\Delta q_0 = -c_x T_s \left( \frac{T_R}{T_s} \right)^{-c_p/c_x} \left( \frac{p_R}{p_s} \right)^{R/c_x} \left( \frac{T}{T_R} \right)^{-c_p/c_x} \left( \frac{p}{p_R} \right)^{R/c_x} - 1$$

as a function of temperature and pressure. Now the extended exergy can be computed with its definition (17).
Figure 2. The distributions of extended and classical exergy resulting from the temperature and pressure fields shown in Fig. 1, and their approximations calculated using Eq. (24)—see text.

(a) Typical classical and extended exergy distributions

In this subsection the comparison will be made of a typical global extended exergy distribution with the classical exergy according to Karlsson (1990). The idealized underlying temperature distribution as well as the pressure field (resulting from an integration of the quasi-hydrostatic equation) are shown in Fig. 1. The reference state, calculated by a horizontal average of the density field, is to a good approximation a polytropic state characterized by Eqs. (35) and (36).

Therefore, the extended exergy is calculated by insertion of Eq. (38) in its definition (17). The exergy according to Karlsson (1990) is calculated in the same way by considering the constant temperature $T_0(s) = T_0 = 255 \, K$. The exact results are compared with approximations calculated using Eq. (24). Figure 2 shows the distribution of the resulting exergies.

The classical exergy varies essentially in the vertical direction (maximum values are 3000 J kg$^{-1}$ at the surface and 2000 J kg$^{-1}$ at 10 km); the extended exergy varies more in the horizontal (maximum values are 1500 J kg$^{-1}$ at the pole and 400 J kg$^{-1}$ at the equator). The distribution of classical exergy is essentially that of the static-stability component of available enthalpy (consisting of the variance of horizontally averaged temperature deviations from a constant equilibrium temperature) defined by Marquet (1991). Although the extended exergy distribution shows some analogies to the field of the baroclinicity-energy component of available enthalpy (consisting of the variance of temperature deviations from a horizontal average) defined by Marquet, there is an important difference. As depicted by Marquet (1991) the derivative of the reference temperature profile does not affect directly the baroclinicity-energy component of available enthalpy, but according to Eq. (24) influ-
ences the distribution of extended exergy. This shows the closer relationship of extended exergy with Lorenz's APE than with available enthalpy.

The approximate versions of classical and extended exergy according to Eq. (24) do not show significant deviations from the exact representations (relative deviations are less than 10%), so the quadratic approximations of classical and extended exergy seem to be quite accurate.

Figure 3 presents the fields of Carnot factors resulting from the temperature field of Fig. 1, calculated by the insertion of Eq. (37) in Eq. (29). Analogous to the distribution of extended and classical exergy the corresponding Carnot factors vary essentially in the horizontal and vertical direction, respectively. Given a rate of heating, the regions of production of classical and extended exergy are thus distributed differently. Although maximum absolute values of extended exergy are generally less than those of classical exergy, absolute values of the Carnot factor of extended exergy (0.21) exceed those of classical exergy (0.12).

6. Conclusions

It has been shown that the results from the APE theories of Lorenz (1955) and Van Mieghem (1956) can be deduced systematically from an extended concept of exergy. Beyond that, these results have been reformulated in a local form, so that they can be used in limited-area models of the atmosphere. The APE of Boussinesq-approximated models for shallow convection has been derived from the extended exergy in the limit of small-amplitude motion.

Furthermore, the concept of pseudoenergy introduced by Shepherd (1993), who constructed a conservation law in the framework of Hamiltonian formalism, is included in the extended-exergy concept in the special case of adiabatic-reversible processes.

The generality of the extended-exergy concept allows the extension to multi-component systems, although in this contribution—for simplicity—a one-component system has been considered exclusively.

Nevertheless, it should not be forgotten that there is always an ambiguity in the determination of the reference state in meteorological applications of the concept of exergy.
In technical thermodynamics there is a system and its environment. The environment can always be considered as a reservoir for the system, and the reference state is given by the state of the reservoir which is controlled to be in thermodynamic equilibrium. This is different in meteorological systems. Consider a limited-area model with open boundaries; this model has an environment which might be a reservoir for the system but, unlike the situation in technical thermodynamics, it is neither in thermodynamic nor in mechanical equilibrium and is therefore not suitable for a reference state. Hence, the reference state will always be an artificial state which exists nowhere inside or outside the system. However, it has been shown that a horizontal average of the density field of a closed system leads to the best reference state, in the sense that it minimizes the volume integral of extended exergy. For an open limited-area model the reference state is also defined through the system’s environment. Therefore, the extended exergy of the limited-area model can be interpreted as the local contribution to the global extended exergy.

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