Smooth quasi-homogeneous gridding of the sphere

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SUMMARY

We describe a variational technique for generating smooth quasi-homogeneous numerical grids covering the sphere. This work extends our earlier studies of non-standard global grids by relaxing the requirement of conformality to admit a greater degree of homogeneity. The motivation for this extension is to increase the minimum grid distance and thus to enable a larger time step to be used within an Eulerian grid-point model. The grids described here are specified by balancing terms in a variational principle that penalizes departures of grid-point density from smoothness and homogeneity. Grids generated by the new procedure are tested in both the cubic and octagonal configurations we have described in our earlier studies. The time-step restriction is successfully alleviated without significant loss of solution accuracy.

KEYWORDS: Global model Grid generation Shallow water

1. INTRODUCTION

The generation of convenient numerical grids for computational fluid mechanics has evolved into a separate subdiscipline of the field, and has already been extensively used in aerospace applications (e.g. Thompson et al. 1985). Development of massively parallel processing (MPP) architectures for large-scale computational problems, and the consequent need to split the domains of such problems into several distinct pieces, provides an incentive to reconsider the grid geometry of the models used for numerical weather prediction (NWP) and to see whether the grid generation techniques developed over the years might also be used in this context.

The use of exotic grid geometries for weather simulation on the sphere is not new; Sadourny et al. (1968), Williamson (1968, 1970) and Cullen (1974) investigated the feasibility of employing triangular grids based on a regular icosahedral partitioning of the domain while, retaining the more familiar and convenient rectangular-grid arrangement, Sadourny (1972) used a construction based on the central (‘gnomonic’) map projection between the sphere and the surface of a concentric cube. The emergence of MPP architectures has provided a fresh impetus to studies employing grids of these kinds (Ronchi et al. 1996; Rančić et al. 1996b), in which the two ‘strong’ singularities at the poles of a latitude–longitude arrangement are replaced by a greater number of considerably weaker grid singularities (twelve singularities for triangular grids, eight for square grids). It is then possible to forego the application of artificial filtering of the kind required by an Eulerian grid-point model near polar singularities; such filters are exceptionally difficult to code efficiently for MPP machines, owing to the need for distant processors to communicate at each application of the filters. But, as McGregor (1997) has shown in the case of the Sadourny arrangement, even a semi-Lagrangian model can benefit from the greater homogeneity of resolution that these grids provide.

One potential defect of the aforementioned non-standard grid constructions is that, in each case, the passage from one face to another of the fundamental polyhedron lacks continuity in the grid-relative gradients. Speculating that this might have been the source of some numerical truncation difficulties encountered in the earlier studies of non-standard grids, Rančić et al. (1996a) compared the performance of three versions of a shallow-water

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model possessing Arakawa-style conservative finite-differencing (Arakawa 1966; Janjić 1977, 1984). Two versions employed grids based on the gnomonic projection, while the third was applied to the grid formed by adopting a 'conformal' version of the cubic arrangement, which is, in some sense, the smoothest mapping possible between cube and sphere. The numerical improvements we found, associated with the smoother mapping for the Eulerian model, were corroborated in the semi-Lagrangian context by McGregor (1996). An alternative disposition of the eight grid singularities, obtained by placing them around the boundary of a circular (sub-hemispheric) region of interest, leads to another useful conformal mapping—this time between the sphere and the two sides of a pair of back-to-back octagons (Purser and Rančić 1997). In either arrangement, the full sweep of 360° about each map-singularity in the geographical domain is shoe-horned by the mapping into an angle of merely 270° in the map domain. The consequent distortions, tempered by the requirement of conformality (except at the singularities themselves), causes there to be an excessive accumulation of grid points in the vicinity of each singularity. While the extra resolution is beneficial to the extent that it compensates for the increased distortion of the grid near each singularity, and therefore helps to control the local truncation errors, the enhanced gridpoint density also has the undesirable side-effect of significantly curtailing the time step of the Eulerian method permitted by the Courant–Friedrichs–Lewy (CFL) condition. In order to generalise the conformal grid construction, and enable a more equitably distributed pattern of grid points less inclined to restrict the time step unnecessarily, we consider here the numerical techniques of grid generation. Of course, for global NWP we are not concerned so much with fitting coordinates to a complicated boundary (as is often the case in aerospace applications), but rather with optimizing the compromise between formal smoothness and grid-point homogeneity, in a computational grid with a predetermined connectivity (i.e. topology).

As in the case of well-established grid generation techniques, we shall find the most elegant expression of these optimizing criteria in the form of simple variational statements. These lead to solutions satisfying systems of elliptic equations.

The penalty-function of the problem is then an integral over the map-domain of terms of the following three types:

(i) terms inhibiting departures from local smoothness;
(ii) terms inhibiting departures from homogeneity of the distribution of grid points;
(iii) a 'Lagrange-multiplier' term to constrain the vector-valued mapping function to the intended spherical domain.

The third term is rendered obsolete if the map-function, instead of mapping to the three earth-centered cartesian coordinates of the spherical earth, maps to a planar projection of this domain. The second term is not present when we care only about smoothness and not homogeneity. However, we shall find that the first term is always needed, if only to formally regularize the variational problem.

2. Variational grid generation criteria

Before we proceed, it is helpful to list the definitions of the frequently-used quantities.

(a) Definitions

Let $X = (X, Y, Z)^T$ be earth-centered Cartesian coordinates for points on the unit sphere and let $x = (x, y)^T$ be the corresponding map coordinates. Let $\nabla$ and $\nabla^2$ denote the
map-space gradient and Laplacian operators respectively. \( J(s, t) \) will denote the Jacobian bilinear operator with respect to map coordinates. Thus,

\[
J(s, t) = \frac{\partial s}{\partial x} \frac{\partial t}{\partial y} - \frac{\partial s}{\partial y} \frac{\partial t}{\partial x}.
\]

The two tangent basis vectors,

\[
V_x = \frac{\partial X}{\partial x}, \quad V_y = \frac{\partial X}{\partial y},
\]

are defined to possess the (Cartesian-space) cross-product,

\[
Q = V_x \times V_y,
\]

a vector parallel to \( X \) and of absolute magnitude

\[
q = Q \cdot X.
\]

We shall employ \( \Lambda(x) \) as a Lagrange-multiplier function.

\( (b) \) Variational principle

The variational formulation of the problem of optimizing the distribution of grid points according to a criterion that balances both smoothness and homogeneity is stated simply as ‘Extremize \( \mathcal{L}(X, \Lambda) \),’ where

\[
\mathcal{L} = \frac{1}{2} \int \int \left[ (|V_x|^2 + |V_y|^2) + a q^2 + \Lambda(|X|^2 - 1) \right] \, dx \, dy. \tag{1}
\]

Here, the terms in the first pair of curved brackets are the ones responsible for smoothness—explicitly, they suppress variations in the map-space gradient of absolute position. The second term, \( a q^2 \), suppresses variations in the absolute Jacobian, and hence in the spatial density of grid points. The constant, \( a \), in this term is a weight determining the relative degree to which inhomogeneities in the absolute Jacobian, \( q \), are penalized. (The integral of \( q \) itself over the unit sphere is, of course, simply \( 4\pi \), so, given this constraint, it is easy to verify that the integral of the second term in (1) is minimized when \( q \) is uniform—implying a uniform density of map grid points over the spherical surface.) The third term is simply the Lagrange-multiplier constraint, to ensure the possibility of conformity of Cartesian position vector \( X \) to the surface of the unit sphere in the solution to the variational principle, even while all three coordinates of this position vector are free to vary. The Lagrange multiplier \( \Lambda \) must be a function of position to effect this conformity over the entire sphere.

If, in (1), we replace the second term, \( a q^2 \), by the equivalent, \( a |Q|^2 \), and allow simultaneous variations of the vector function \( X \) and of the scalar function \( \Lambda \) of \( x \) and \( y \), we find directly that stationarity of functional \( \mathcal{L} \) implies the satisfaction of the following nonlinear Euler–Lagrange elliptic equations:

\[
\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} - a \left( \frac{\partial X}{\partial x} \times \frac{\partial Q}{\partial y} - \frac{\partial X}{\partial y} \times \frac{\partial Q}{\partial x} \right) = \Lambda \cdot X, \tag{2}
\]
\[ |X|^2 = 1. \] (3)

An equivalent, but more concise and elegant restatement of (2) is:
\[
\nabla^2 X_i - \alpha \epsilon_{ijk} J(X_j, Q_k) = \Lambda X_i,
\] (4)

(where the summation convention is assumed for the Cartesian components). Suppose we select the point \( X = (0, 0, 1) \) (or orient the Cartesian system to make these the coordinates of some selected geographical point). Then, since \( q = Q_z \equiv J(X, Y) \), \( \nabla q = \nabla Q_z \), \( \nabla Z = 0 \), the local distributions of \( X \) and \( Y \) around this point satisfy the tangent-plane representation of (2) at this point:
\[
\nabla^2 X - \alpha J(Y, q) = 0,
\]
\[
\nabla^2 Y + \alpha J(X, q) = 0,
\]
or equivalently:
\[
\nabla^2 X + \frac{\alpha}{2} \frac{\partial (q^2)}{\partial X} = 0, \quad (5)
\]
\[
\nabla^2 Y + \frac{\alpha}{2} \frac{\partial (q^2)}{\partial Y} = 0. \quad (6)
\]

In the special limiting case, \( \alpha = 0 \), (5) and (6) reduce to a pair of Laplace-equations for which, symmetry permitting, a possible solution consists of a conformal mapping between map domain and sphere. In this respect, the variational method with parameter \( \alpha \) can be regarded as a direct generalisation of the conformal mapping methods of Rančić et al. (1996a) and Purser and Rančić (1997).

3. NUMERICAL PROCEDURE FOR GRID GENERATION

The symmetries of the gridding are exploited wherever possible to enable the solution to the grid generation problem to be worked out on the smallest non-redundant fragment of the map. Thus, for a cube possessing a group of symmetries of order 48, it is sufficient to solve the gridding problem for the fragment corresponding to a single triangular octant of one of the six fundamental squares of the cube. Symmetries are invoked to constrain the locations of points along the three edges of this triangular octant and mirror symmetry, where needed, to infer Cartesian locations of the points of the map grid just outside this triangle. For the octagon, the grid generation is always performed in the symmetrical configuration (the interface between octagons being a great circle) so that a group of symmetries of order 16 can be employed, with the representative fragment now being one octant of one of the two octagonal maps.

Equations (2) and (3) can be solved within the minimal non-redundant octant of the cubic or octagonal grid by various iterative methods. In essence, the character of the problem is that of solving an elliptic equation. However, it is non-standard in the following respects: (i) the solution is a vector field; (ii) the nonlinearity is significant; (iii) the problem is one that is continuously constrained. Consequently, some special care is needed in order to guarantee a successful numerical solution. Perhaps the simplest viable method is to apply the standard second-order discretization of (2) under constraint (3) and employ a vector valued successive over-relaxation (SOR). The sensitivity of the residual of (2) at a given grid point can be directly estimated by finite variations of the three components of the solution vector there. An ordinary “relaxation” of the vector \( X \) changes it by an amount
calculated to annihilate the residual of (2) locally (within the degree of approximation inherent in the estimated sensitivity of that residual), while satisfying the constraint (3) exactly (for some new \( \Lambda \)). An 'over-relaxation' merely extrapolates this change in \( X \) by the pre-determined over-relaxation factor, in order to speed the overall convergence of such repeated iterations cycled over all the grid points of the representative octant. Unfortunately, SOR is notoriously slow when applied to finely discretized problems, such as an accurate computation of our mapping function, \( X \). We have therefore adopted a multigrid refinement of this technique, in which the variational problem is simultaneously represented at a hierarchy of resolutions progressively coarsened by a factor of two; roughly speaking, each scale of residual in the problem is reduced using predominantly the grid for which this reduction is effected most efficiently. (For a thorough description of the principles and techniques of multigrid, see Brandt 1977; McCormick 1988).

Figure 1 shows a comparison of cubic grids generated by the variational method. Figure 1(a) shows the conformal case \( (a = 0) \), while Fig. 1(b) shows the more evenly distributed arrangement of grid points around a singularity obtained by choosing \( a = 10 \). However, we see in Fig. 1(b) that the grid is distinctly non-orthogonal in the immediate vicinity of the map singularity.

For the symmetric octagon, Fig 2(a) shows the arrangement of grid points in the conformal case \( (a = 0) \), which we compare with the more homogeneous arrangement of Fig. 2(b) in which the value taken by the homogeneity parameter is \( a = 10 \).

In the case of an 'enhanced octagon' (Purser and Rančić 1997) where one octagonal map targets a region of interest with enhanced resolution, the application of the Schmidt (1977) conformal transformation to bring about this enhancement occurs after the basic grid generation for the symmetric octagon has been completed.

4. Test Results

In order to test how smoothing affects the numerical results, we ran our shallow-water model, described in Rančić et al. (1996a) and Purser and Rančić (1997), on the smoothed versions of cubic and octagonal grids, in the standard test with Rossby–Haurwitz wavenumber 4 (cf. Williamson et al. 1992). We compared the derived results against the reference solution, described in Jakob et al. (1993).
Figure 2. Comparison of symmetric octagon grids; (a) conformal with \( a = 0 \) (see text); (b) quasi-homogeneous with \( a = 10 \).

(a) Cubic grid

The weight of the penalty contribution, \( a \), is successively given values 0, 10, 20 and 30 in these experiments. Figure 3 shows the minimum grid spacing of the smoothed cube as a function of resolution for these different values of the weighting parameter. The allowable maximum time step of an Eulerian model is closely proportional to this measure, which therefore indicates its computational efficiency. The minimum grid distance clearly increases with the increase of the weighting parameter: the lowest curve corresponds to the conformal cube (\( a = 0 \)), and the uppermost (\( a = 30 \)) approaches the curve that characterizes the gnomonic cube. For the sake of comparison, we include Fig. 4, which presents the effective minimum grid distance on the standard longitude–latitude grid, when polar filtering starts from 55, 60, 65, 70, 75 and 80 deg. We see that with larger values of parameter \( a \), the cubic grid allows time steps competitive with those on the standard longitude–latitude grid, but without any filtering.

Figure 3. Minimum grid spacing on the cubic grid covering the unit sphere as a function of grid resolution, for several values of smoothing parameter, \( a \). The gnomonic cube is the one defined in Rančić et al. (1996).
Figure 4. Effective minimum grid spacing on the standard longitude–latitude grids covering the unit sphere as a function of grid resolution, for several critical latitudes beyond which polar filtering starts. The case without polar filtering is denoted by \( x_{min} \).

Experiments were done with \((18 \times n) \times (18 \times n) \times 6\) grid boxes (i.e. velocity points on the Arakawa B-grid) over the sphere, with \( n \) taking values 1, 2 and 4. The time step in the respective series was 240/n sec. Figure 5 presents the evolution of \( l_2 \) error and Fig. 6 the evolution of \( l_\infty \) error derived in these series (see Williamson et al. 1992 for the definitions of these two diagnostics). The penalty of the introduced grid smoothing becomes obvious from these figures, where the measured diagnostics are increasing (though not severely) with the increase of weighting parameter. The exception is in the case of the highest-resolution run, where the \( l_\infty \) error is actually the smallest for the largest value of the weighting parameter.

(b) Octagonal grid

Similar tests were performed on the octagonal grid. However, in this case smaller values of the weighting parameter \( a \) (i.e. 0, 1, 2 and 5) were sufficient to provide for a comparable increase of the minimum grid distance. Figure 7 shows these minimum grid distances for the octagonal grid, as a function of resolution. For brevity, we present here only the \( l_2 \) diagnostics (Fig. 8) derived from the test at the highest resolution with \((12 \times 4) \times (12 \times 4) \times 14\) grid boxes over the sphere. The time step in these experiments was 60 sec.

The curves \( l_2 \) initially behave similarly to those for the cube. However, as time progresses, the more homogeneous grids exhibit greater accuracy, which we attribute to the better resolution they provide at high latitudes. Yet, even though improving homogeneity appears to improve the performance on the octagonal grid, it still cannot compete for accuracy with the cubic grid (compare with Fig. 5(c)).

5. Discussion

We have proposed a systematic variational method enabling one to control the homogeneity of the spherical grids that generalise the conformal cubic and octagonal configurations described in Rančić et al. (1996) and Purser and Rančić (1997). The new method
does not destroy the smoothness of those mappings, merely their unused property of conformality. Neither does it alter the cost of each time step in a forecast. However, a more homogeneous grid distribution has a more lenient implied CFL limit, and hence a greater computational efficiency for the forecast as a whole. This relative efficiency improves progressively as the spatial resolution increases. In the octagonal configuration, we also find that a small improvement in formal accuracy results from choosing a more homogeneous grid.

Conformal grids admit numerically efficient transformation between earth-centred Cartesian coordinates and the map domain, but, unless supplemented by a separate interpolation procedure, our quasi-homogeneous mappings are restricted to a pre-calculated table of fixed points on a grid. This does not impede an Eulerian model, but in a semi-Lagrangian model, where regional inhomogeneities of grid resolution matter less, conformal grids remain more appropriate.

The variational generation of smooth mappings is here exemplified for the geometries accommodating grids with quadrilateral elements. The same technique clearly applies equally well to triangular-element grids. The icosahedral arrangement (Williamson, 1968), being most symmetrical, is the counterpart to our cube, while a double-sided dodecagon is the obvious counterpart to the octagon.
Figure 6. As Fig. 5, but for the $l_\infty$ diagnostics.

Figure 7. Minimum grid spacing on the octagonal grid covering the unit sphere as a function of grid resolution, for several values of the homogeneity parameter, $a$; see text.
Figure 8. Evolution of the $l_2$ diagnostics (see text) in the Rossby–Haurwitz wavenumber-4 experiment of Williamson et al. (1992), for different values of homogeneity parameter, $a$. The resolution in the experiment is 37 856 grid boxes.

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