A simple theoretical model for the intensification of tropical cyclones and polar lows

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SUMMARY

A simple theoretical model for the intensification of tropical cyclones and polar lows is developed using a minimal set of physical assumptions. These disturbances are assumed to be balanced systems intensifying through the WISHE (Wind-Induced Surface Heat Exchange) intensification mechanism, driven by surface fluxes of heat and moisture into an atmosphere which is neutral to moist convection. The equation set is linearized about a resting basic state and solved as an initial-value problem. A system is predicted to intensify with an exponential perturbation growth rate scaled by the radial gradient of an efficiency parameter which crudely represents the effects of unsaturated processes. The form of this efficiency parameter is assumed to be defined by initial conditions, dependent on the nature of a pre-existing vortex required to precondition the atmosphere to a state in which the vortex can intensify. Evaluation of the simple model using a primitive-equation, nonlinear numerical model provides support for the prediction of exponential perturbation growth. Good agreement is found between the simple and numerical models for the sensitivities of the measured growth rate to various parameters, including surface roughness, the rate of transfer of heat and moisture from the ocean surface, and the scale for the growing vortex.

Keywords: Air–sea interaction Growth rate Surface fluxes WISHE

1. Introduction

The tropical cyclone is a complex system. With increases in computer speed and memory, there has been a tendency in computer models towards the representation of ever more physical processes and interactions in an attempt to simulate observed systems accurately. As a fairly recent example, Tripoli (1992) demonstrated a three-dimensional, nonhydrostatic multiple-nested grid model incorporating a five phase explicit microphysics prediction scheme. This complexity may, however, be deceptive. If the physical mechanism underpinning tropical-cyclone intensification is sufficiently simple, it should be possible to construct a very simple model. It can be argued that there is still a place for models at this end of the complexity scale, and that such a model can, as a consequence of its simplicity, provide insight into the essential processes that govern tropical-cyclone growth and indeed provide testable predictions for the dependence of the growth rate of systems on factors which may change in nature. The aim of this work is to develop an analytical model for the intensification of tropical-cyclone-type systems using the minimal set of physical assumptions and equations necessary. The phrase ‘tropical-cyclone-type systems’ will be used to refer to a group of warm-cored disturbances, occurring at a variety of latitudes, that have structures similar to those of tropical cyclones and have been proposed to intensify through a similar mechanism. This includes some polar lows and Mediterranean cyclones (Rasmussen and Zick 1987; Rasmussen 1989; Rasmussen and Nordeng 1992). The relative importance of the neglected physical interactions and the strength of the assumptions made will determine the correlation between the predictions of this model and systems in nature.

The simplest representation of a tropical-cyclone-type system is the so-called balanced model which was first developed by Eliassen (1951). The motion in an intensifying vortex is considered, at a given instant, to be quasi-balanced, i.e. the vortex is considered

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to be in approximate thermal-wind balance (the combination of hydrostatic balance and gradient-wind balance). A secondary circulation in the vortex can be forced by sources of heat and/or angular momentum (for example the latent heating due to convection, or the sink of angular momentum arising through surface friction).

Such models have been studied previously (for example by Charney and Eliassen (1964) and Ooyama (1964)). The different aspect of the model presented in this paper is in the parametrization scheme for convection, the crucial closure for a balanced model. This closure assumption places the dynamics and thermodynamics of moist convection on the implicit side, and the dynamics of the balanced flow on the explicit side, of a scale division. The intensification mechanism of the vortex is the mechanism for transferring between quasi-balanced states and is controlled by the cumulus parametrization scheme. These mechanisms have been termed ‘cooperative intensification mechanisms’ in the sense discussed by Ooyama (1982). The balanced models of Charney and Eliassen, and Ooyama, incorporated (in different forms) an intensification mechanism which was named ‘conditional instability of the second kind’ (later abbreviated to CISK) by Charney and Eliassen. Both papers argue that a low-level, frictionally induced, inflow supplies the energy for the vortex intensification by transporting either moist (Charney and Eliassen 1964) or conditionally unstable (Ooyama 1964) air. The speed of this inflow increases with the intensity of the vortex, creating the possibility of a positive feedback mechanism.

Recent numerical simulations (Craig and Gray 1996) were found, however, to be inconsistent with this theory but consistent with an alternative intensification mechanism known as WISHE (Wind-Induced Surface Heat Exchange). This mechanism, previously termed Air--Sea Interaction Instability (ASII, Emanuel 1986), is discussed more recently by Emanuel et al. (1994). It is based on the idea that the virtual temperature of a convecting atmosphere is directly linked to the equivalent potential temperature in the boundary layer which is increased by the wind-speed dependent heat and moisture fluxes from the ocean surface. A feedback arises in which increased surface fluxes increase the intensity of the vortex (through gradient adjustment to the warmed core) which in turn further increases the surface fluxes. In the most extreme form of this theory, it can be assumed that there is zero Convective Available Potential Energy (CAPE), i.e. it is assumed that convection acts on a much faster time-scale than that of the large-scale processes which generate it. This implies that the atmosphere is conditionally neutral to slantwise moist convection (along the sloping constant absolute angular-momentum surfaces). Emanuel (1989) constructed a numerical, balanced, axisymmetric model incorporating this assumption to investigate the threshold for tropical cyclogenesis.

This paper describes a highly simplified balanced model for the intensification of tropical-cyclone-type systems incorporating a minimal parametrization for convection through which the WISHE intensification mechanism can operate. It is in its quest for absolute simplicity that this model differs from that described by Emanuel (1989). The main motivation for this work is to provide a theoretical model to demonstrate the minimal physics and to determine the predictions of such a model. Evaluation of the model will indicate its usefulness (applicability to systems intensifying in nature) by comparison with a more complex numerical model. In addition, the predictions of this simple model, in particular the predicted growth-rate dependences, can be compared with those of other balanced models which incorporate cumulus parametrizations used in previous CISK theories.

The simple model is developed in section 2. Section 3 contains a description of the determination of the solution of the equation set and the predictions derived from it, and section 4 the evaluation of the simple model by comparison with a primitive-equation, nonlinear numerical model. Section 5 contains some conclusions.
2. Development of the Simple Model

(a) Methodology

As discussed in the introduction, the model developed and evaluated in this paper is a highly simplified balanced model for the intensification of tropical-cyclone-type vortices through the WISHE intensification mechanism. The secondary circulation in a balanced vortex can be forced by heat and/or momentum sources but it is in the parametrization of the heat source, through its dependence on the vortex-scale balanced flow, that the assumptions which characterize this mechanism are invoked. The mechanism assumes that the atmosphere is approximately conditionally neutral to slantwise moist convection (along sloping angular-momentum surfaces). It will be assumed here that the atmosphere is exactly conditionally neutral and that it is saturated throughout. This implies that the equivalent potential temperature is independent of height along absolute angular-momentum surfaces. This assumption, consistent with the chosen intensification mechanism, dramatically simplifies the determination of the solution of the model equations. In the absence of any inhibition to convection the heat source can be assumed to be directly related to the fluxes of heat and moisture from the ocean surface into the atmosphere which are controlled by the balanced flow through their wind-speed dependence. Although it is assumed for simplicity that the atmosphere is saturated, it was found to be important to represent the effects of unsaturated processes on the subcloud layer. This will be attempted through a crude parametrization using an ‘efficiency parameter’.

There are two possible transfers of momentum capable of forcing the secondary circulation. The first of these, interior turbulence, is neglected for simplicity, i.e. the atmosphere is assumed to be inviscid. The second possible momentum source, surface friction, is assumed to be confined to a thin boundary layer which acts as a boundary condition to the free atmosphere above it. Experiments with a primitive-equation numerical model (Craig and Gray 1996) have demonstrated that, consistent with the predictions of the WISHE mechanism, the growth rate of tropical-cyclone-type vortices is relatively insensitive to surface friction. Hence, in the quest for a minimal model, surface friction could be neglected. However, since the dependence of the growth of the simple model on surface friction acts as an easily performed, quantitative test of the model, surface friction will be represented despite it being a non-essential process. An Ekman layer-type parametrization will be used to represent the boundary layer. Linear balanced models incorporating cumulus parametrizations used in previous CISK theories (Charney and Eliassen 1964; Ooyama 1964; Bratseth 1985) predict a linear increase in the growth rate of tropical-cyclone-type disturbances with a frictional drag coefficient. Thus, representation of surface friction in the balanced model presented in this paper, which incorporates a cumulus parametrization that only allows the WISHE mechanism to operate, also allows a direct comparison between the growth-rate dependences predicted by the two parametrizations to be made.

The equation set for the model contains only two prognostic equations; the equation for the conservation of absolute angular momentum and the thermodynamic equation (forced by heat and moisture fluxes just above the top of the boundary layer). The following subsections describe the development of the model, starting with the balanced vortex equations and boundary conditions in dry variables (the simplest form which neglects the effects of moisture). The equations are then converted to moist variables after which the assumptions derived from the chosen cumulus parametrization can be easily implemented.

(b) Balanced Vortex Equations and Boundary Conditions

Following Eliassen (1951), we consider thermally and frictionally forced flow in an axisymmetric vortex in thermal-wind balance. The equations are for an inviscid atmosphere
on an $f$-plane, with frictional forces confined to a thin boundary layer such that they act as a lower-boundary condition to the free atmosphere above.

Employing the notion of Schubert and Hack (1983), in cylindrical, axisymmetric coordinates, using dry variables (i.e. neglecting the effects of moisture), the equations for gradient-wind balance, conservation of absolute angular momentum, hydrostatic balance, mass continuity and the thermodynamic equation can be written as

\[
\frac{v^2}{r} + f v = \frac{\partial \phi}{\partial r},
\]

\[
\frac{DM}{Dt} = 0,
\]

\[
\frac{\partial \phi}{\partial z} = \left( \frac{g}{\theta_0} \right) \theta,
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w) = 0,
\]

\[
\frac{D\theta}{Dt} = Q,
\]

where the coordinate system is such that $r$ is the radius and, to simplify the hydrostatic equation, $z$ is the pseudo-height coordinate described by Hoskins and Bretherton (1972):

\[
z = \left( 1 - \left( \frac{P}{P_0} \right)^{(\gamma-1)/\gamma} \right) \frac{\gamma}{\gamma-1} H_s,
\]

where $\gamma$ is the ratio of specific heats, $p$ is pressure and the scale height, $H_s$, is given by $H_s = P_0/(\theta_0 g)$, with $g$ the acceleration due to gravity and $l$ the atmospheric density. The subscript zero denotes a ‘top of the boundary layer’ value. Associated with pseudo-height, an expression for pseudo-density, $\rho$, given by $\rho(z) = l_0 (p/p_0) \gamma/\gamma$ has also been introduced, which for an adiabatic atmosphere equals the true density. The dependent variables are the radial, azimuthal and vertical velocity components $u$, $v$ and $w$, respectively, the geopotential, $\phi$, and the potential temperature, $\theta$. $M$ is the absolute angular momentum, $M = (1/2) f r^2 + rv$, where $f$ is the Coriolis parameter and $Q = (1 - gz/(c_p \theta_0))^{-1}(2/c_p)$, where $Q$ is the diabatic heating rate and $c_p$ is the specific heat at constant pressure. The Lagrangian time derivative can be expanded to $D/Dt = \partial/\partial t + u(\partial/\partial r) + \rho(\partial/\partial z)$.

To obtain the lower-boundary condition to this system of equations it is assumed that the boundary layer is so shallow that the pseudo-density can be considered constant and that the Eulerian time derivative, $\partial/\partial t$, can be neglected. The equations for conservation of absolute angular momentum and mass continuity (integrated through the depth of the boundary layer) are then

\[
\zeta_0 v_0 = -\frac{C_D U v_0}{h_B},
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{w_0}{h_B} = 0,
\]

where $h_B$ is the depth of the boundary layer, $C_D$ is a nondimensional frictional drag coefficient, $U$ is the absolute wind speed at the sea surface ($U = (u_0^2 + v_0^2)^{1/2}$) and $\zeta_0$ is the absolute vorticity at the top of the boundary layer given by $\zeta_0 = \{ f + \partial (ru_0)/\partial r \}$. Elim-
inatig \( h_B u_0 \) from these equations gives the lower-boundary condition as the following expression for the frictionally forced vertical velocity out of the boundary layer:

\[
\begin{equation}
  w_0 = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{\xi_0} C_D U v_0 r \right).
\end{equation}
\]

This and the remaining boundary conditions are more conveniently expressed in terms of a stream function for the secondary circulation, \( \psi \). Using the continuity equation, the secondary circulation velocity components, \( u \) and \( w \) can be written as

\[
\begin{equation}
  (\rho u, \rho w) = \left( -\frac{\partial \psi}{\partial z}, \frac{1}{r} \frac{\partial}{\partial r} \rho r \psi \right).
\end{equation}
\]

The complete set of boundary conditions is then

\[
\begin{align*}
\psi(0, z) &= \psi(r, z_T) = 0, \\
\psi r &\to 0 \quad \text{as} \quad r \to \infty, \\
\psi_0 &= \frac{\rho_0}{\xi_0} C_D U v_0
\end{align*}
\]

where \( z_T \) is the height of the top of the system.

\((c)\) **Conversion to moist variables**

The assumption of conditional neutrality to moist convection (made in the following subsection) is most easily expressed by transforming the relevant equations from functions of the potential temperature, \( \theta \), to functions of equivalent potential temperature, \( \theta_e \). For saturated air \( \theta_e \) is defined by

\[
\begin{equation}
  \theta_e \approx \theta \exp \left( \frac{L_c q_s}{c_p T} \right),
\end{equation}
\]

where \( L_c \) is the latent heat of condensation and \( q_s \) is the saturation mixing ratio. During three-dimensional motion in a moist adiabatic atmosphere \( \theta_e \) is changed only by diabatic processes other than internal latent-heat release. Hence transforming the thermodynamic equation, (5), to its moist form gives

\[
\begin{equation}
  \frac{D\theta_e}{Dt} = Q_e,
\end{equation}
\]

where \( Q_e = \{\theta_e/(c_p T)\}(\mathcal{Q} + L_c \mathcal{R}) \) with \( \mathcal{Q} \) the diabatic heating rate due to sensible-heating sources external to the system (as previously found in the definition of \( Q \)) and \( L_c \mathcal{R} \) an equivalent term due to latent-heat sources external to the system.

The equation for thermal-wind balance is formed by combining the assumptions of hydrostatic balance and gradient-wind balance. Mathematically it can be obtained by adding \( \partial/\partial z \) (1) and \( \partial/\partial r \) (3) to obtain

\[
\begin{equation}
  \left( \frac{2v}{r} + f \right) \frac{\partial v}{\partial z} = \frac{g}{\theta_0} \frac{\partial \theta}{\partial r}.
\end{equation}
\]
It can be shown (as described in appendix A) that, assuming a saturated atmosphere and neglecting small terms which depend on the total water content, this equation may be written in moist variables as

\[
\left( \frac{2v}{r} + f \right) \frac{\partial v}{\partial z} = \frac{g}{\theta_0} \frac{\Gamma_m}{\Gamma_d} \frac{\partial \theta_e}{\partial r},
\]

(15)

where \( \Gamma_d \) and \( \Gamma_m \) are the dry and moist adiabatic lapse rates respectively.

\((d)\) Cumulus parametrization

Several simplifying assumptions can be made by specifying the cumulus parametrization. It will be assumed that the atmosphere is conditionally neutral to slantwise moist convection (along the sloping absolute angular-momentum surfaces). This condition permits the operation of only the WISHE intensification mechanism. In a saturated atmosphere this implies that \( \theta_e \) is constant along these surfaces. The heating rate, \( Q_e \), can then be related directly to fluxes of sensible and latent heat from the ocean surface into the boundary layer arising from the thermodynamic disequilibrium between the two,

\[
Q_e = \frac{C_{E,T} U (\theta_{es} - \theta_e)}{h_B},
\]

(16)

where \( C_{E,T} \) is a nondimensional coefficient for the transfer of heat and moisture and \( \theta_{es} \) is the sea surface equivalent potential temperature. The assumption of conditional neutrality links the boundary layer to the rest of the troposphere such that an increase in boundary-layer \( \theta_e \) increases \( \theta_e \) in the whole of the convecting layer above it. Thus to obtain the rate of temperature change in the system, the energy flux must be divided by the amount of air that the heating is spread over. For this reason the height of the boundary layer, \( h_B \), in the heating-rate equation (16), is more properly replaced by the length of the surfaces of constant absolute angular momentum between the surface and the top of convection. This will be approximated by the entire depth of the system, \( z_T \) (note that in the linearized system the length of the absolute angular-momentum surfaces exactly equals \( z_T \)).

In a saturated atmosphere there can be no contributions to \( Q_e \) from unsaturated downdraughts or from the entrainment of air across the top of the boundary layer. In an attempt to represent these effects, an 'efficiency parameter', \( \beta(r) \) (where \( 0 < \beta < 1 \)), is introduced as a scaling factor in the thermodynamic equation (following Emanuel (1997)) which then becomes

\[
Q_e = \beta \frac{C_{E,T} U (\theta_{es} - \theta_e)}{z_T}.
\]

(17)

This efficiency parameter attempts to represent, very crudely, the negative feedback effects of unsaturated processes occurring outside the eyewall (which, of course, are otherwise neglected in a saturated model). In the eyewall region \( \beta \) will thus be large (\( \approx 1 \)) whereas outside the eyewall \( \beta \) will be <1 since here the unsaturated processes act to reduce the heating rate.

3. Solution of the model equation set

\((a)\) Methodology

To determine analytically an expression for the growth rate of the system, the model equations will be linearized about a resting basic state. It is implicitly assumed that the genesis stage of the system has already occurred, leaving a balanced, small amplitude,
vortex weak enough that the tropical-cyclone-type system can be considered to be a small perturbation from a state of relative rest. Emanuel (1989) argues that the troposphere must become nearly saturated on the mesoscale in the tropical-cyclone core before amplification can begin, otherwise any disturbance will decay owing to low \( \theta_e \) downdraughts associated with convection induced by Ekman pumping. This implies that a pre-existing disturbance of finite strength must exist in order to bring the core to saturation, and indeed this is generally observed in nature where tropical cyclones are known to develop from pre-existing disturbances such as easterly waves (Anthes 1982).

The assumption of a balanced vortex appears inconsistent when only small-amplitude perturbations exist, since balanced flow implies a strong control of organized convection in the eyewall region of a tropical cyclone by the balanced flow, and this cannot occur until the system reaches a relatively intense stage. This latter inconsistency in the linear perturbation analysis of balanced systems was discussed by Ooyama (1982). It must thus be assumed that, whilst this pre-existing disturbance is of sufficient strength to validate the balanced vortex assumption, its amplitude is still negligible in relation to that of the growing vortex (this assumption will be relaxed in section 4(e)). It is possible, however, that the pre-existing vortex may influence the growing vortex as in nature. Many authors believe that the development of tropical cyclones is an intrinsically nonlinear process (e.g. Schubert and Hack 1983) which must cast doubts over the usefulness of a linear model for predicting growth rates even early on in the intensification stage of a vortex. The reasons why the linear model does prove to be successful despite nonlinear effects will be discussed later (section 4(e)).

The following subsection describes the analytic solution of the model equation set. The equations are linearized about a resting basic state and then the partial-differential equation for the azimuthal velocity (just above the boundary layer) is determined. This is solved as an initial-value problem to give a predicted evolution for the system. A final subsection contains a discussion of the predictions of the model.

(b) Linear perturbation analysis

Using primes to denote perturbation quantities and overbars to denote undisturbed quantities, each dependent variable is assumed to equal the sum of an undisturbed and perturbation value, e.g. \( v(r, z, t) = \bar{v}(r, z) + v'(r, z, t) \). Linearizing about a state of rest implies, \( \bar{v}, \bar{u}, \bar{w}, \bar{\psi} = 0, \bar{\theta_e} = \text{constant} \) and \( \zeta = f \). When considering small perturbations from rest the momentum surfaces can be considered to be approximately vertical. Application of the WISHE concept implies that \( \theta_e \) is constant along momentum surfaces and hence \( \theta_e \) can be considered independent of height. It will also be assumed that, to a first approximation, the atmospheric pseudo-density, \( \rho \), is constant everywhere (the Boussinesq approximation).

The linearized equation set for the simple model is given below: these are the linearized forms of the equations for thermal-wind balance, (15), and conservation of absolute angular momentum, (2), the thermodynamic equation, ((13) using (17)), the stream-function equations, (10), and the equations for the boundary conditions, (11):

\[
\frac{f}{\partial z'} v' = \frac{g}{\bar{\theta}_e} \frac{\partial}{\partial r} \theta'_e, \quad (18)
\]
\[
\frac{\partial}{\partial t} v' + fu' = 0, \quad (19)
\]
\[
\frac{\partial}{\partial t} \theta'_e = \frac{\beta}{C_{\text{E,T}}} U'(\theta_{e0} - \bar{\theta}_e) \frac{z_T}{\bar{z}}, \quad (20)
\]
\[(\rho u', \rho w') = \left(-\frac{\partial}{\partial z} \psi', \frac{1}{r} \frac{\partial}{\partial r} (r \psi') \right), \quad (21)\]

\[\psi'(0, z) = \psi'(r, z_T) = 0\]

\[r \psi' \to 0 \quad \text{as} \quad r \to \infty\]

\[\psi'(r, 0) = \frac{\rho k v_0'}{f}. \quad (22)\]

In the equation for the boundary condition at the ocean surface the quadratic drag coefficient, \(C_D\), was replaced by a linear drag coefficient \(k\) (following Ooyama (1964) where \(k = C_D U\)) before linearization since the more realistic quadratic frictional stress would linearize to zero.

The solution of this equation set is achieved by calculation of a diagnostic equation for the secondary circulation defined by \((u', w')\) in terms of the perturbation stream function, \(\psi'\). The secondary circulation equation is obtained by combining the thermodynamic equation and the equation for conservation of absolute angular momentum using thermal-wind balance. Taking \(\partial (f \cdot (19)) / \partial z = \partial (\{g \Gamma_m / (\theta_e \Gamma_d) \cdot (20)\}) / \partial r\) and using the stream-function equation for \(u'\) gives

\[\frac{\partial^2}{\partial z^2} \psi' = \frac{2 \rho \alpha}{f z_T} \left( \beta \frac{\partial}{\partial r} v_0 + v_0' \frac{\partial \beta}{\partial r} \right), \quad (23)\]

where

\[\alpha = \frac{\Gamma_m}{\Gamma_d} \frac{C_{E,T} (\theta_e - \theta_e)}{2 f \theta_e}. \quad (24)\]

Cyclonic flow, dominated by the azimuthal velocity, has been assumed throughout the boundary layer, such that \(U' = v_0'\) in the surface flux term. Without the assumption that \(\theta_e\) is independent of \(z\), the secondary circulation equation is elliptic, requiring all four boundary conditions defined in (11) to solve it. However, Eq. (23) is of the parabolic form which requires only two boundary conditions (those along the upper and lower boundaries) to solve it. The remaining two boundary conditions (at the inner and outer boundaries) are no longer necessary (in fact, they overspecify the problem).

Now, to apply the boundary conditions to the secondary circulation equation, it must be integrated twice with respect to \(z\), and since \(\theta_e\) is independent of \(z\) integration gives

\[\psi' = \frac{2 \rho \alpha}{f z_T} \frac{z^2}{2} \left( \beta \frac{\partial}{\partial r} v_0 + v_0' \frac{\partial \beta}{\partial r} \right) + A(r, t)z + B(r, t), \quad (25)\]

where \(A\) and \(B\) are two constants of integration. Application of the boundary conditions at the lower and upper boundaries (\(z = 0\) and \(z = z_T\)) allows these constants to be determined, yielding the following final expression for the perturbation stream function:

\[\psi' = \frac{2 \rho \alpha}{f z_T} \left( \frac{z^2}{2} - \frac{z z_T}{2} \right) \left( \beta \frac{\partial}{\partial r} v_0 + v_0' \frac{\partial \beta}{\partial r} \right) + \frac{\rho k v_0'}{f} \left( 1 - \frac{z}{z_T} \right). \quad (26)\]

To obtain an expression for the growth rate of the system the prognostic equation for the conservation of absolute angular momentum must be solved. This can be applied just
above the boundary layer. The perturbation radial velocity, \( u' \), can be obtained from the secondary circulation equation using the stream-function expressions given by (21). Thus

\[
u' = -\frac{2\alpha}{f z_T} \left(\frac{2z_0}{2} - z_T\right) \left(\beta \frac{\partial}{\partial r} v_0' + v_0' \frac{\partial \beta}{\partial r}\right) - \frac{k v_0'}{f} \left(\frac{1}{z_T}\right), \tag{27}
\]

and applying this just above the boundary layer (at \( z = 0 \)) gives

\[
u_0' = \frac{\alpha}{f} \left(\beta \frac{\partial}{\partial r} v_0' + v_0' \frac{\partial \beta}{\partial r}\right) + \frac{k v_0'}{f z_T}. \tag{28}
\]

This can now be substituted into the equation for the conservation of absolute angular momentum applied just above the boundary layer to give

\[rac{\partial}{\partial t} v_0' + \alpha \left(\beta \frac{\partial}{\partial r} v_0' + v_0' \frac{\partial \beta}{\partial r}\right) + \frac{k v_0'}{z_T} = 0. \tag{29}
\]

Intensification of the vortex requires that \( \partial v_0'/\partial t > 0 \), hence \( \beta \) must have a negative gradient with radius at the radius of maximum azimuthal velocity (where \( \partial v_0'/\partial r = 0 \)). The same conclusion was reached by Emanuel (1997). This equation, (29), reveals the necessity of including a parametrization for unsaturated processes. If the parameter \( \beta \) had been neglected (i.e. set equal to 1) the solution to the resultant momentum equation (corresponding to (29)) would be that of a radially propagating, decaying wave.

The solution to this partial-differential equation can be determined by treating it as an initial-value problem and using the method of characteristics. It is found to be of the form

\[
v_0' = e^{\sigma t} h(R - \alpha t), \tag{30}
\]

where

\[
\sigma(r) = -\left(\alpha \frac{\partial \beta}{\partial r} + \frac{k}{z_T}\right), \tag{31}
\]

is the radially dependent exponential growth rate of the system,

\[
R(r) = \int \frac{dr}{\beta}, \tag{32}
\]

and is thus a stretched horizontal coordinate, and \( h \) is a function which is defined by the condition that at \( t = 0 \),

\[
h(R(r, 0)) = v_0'(r, 0). \tag{33}
\]

The validity of this solution can easily be confirmed by substitution into (29). The initial azimuthal velocity structure propagates radially outwards with time at a speed given by

\[
\frac{\partial r}{\partial t} = \alpha \beta. \tag{34}
\]

The form of \( \beta(r) \) is unknown apart from the bounds declared in subsection 2(d) where this parameter was introduced.

The simple model predicts that the structure of an intensifying system should propagate radially outwards with time. The predicted propagation rates (of the eyewall where \( \beta \sim 1 \)) are approximately 2 and 5 m s\(^{-1}\) for the polar-low and tropical-cyclone simulated systems respectively (where the terms in \( \alpha \) are quantified later). These values are of comparable magnitude with the radial velocities in the systems which implies that the latter
could not be neglected in calculations of the propagation rates. The propagation will not be investigated further in this paper.

The growth-rate equation given by (31) depends critically on the form of \( \partial \beta / \partial r \); however, some general comments can be made. The growth rate of a tropical-cyclone-type system is predicted to increase with the sensible and latent heating from the ocean surface but decrease with surface friction. These dependences are in agreement with the numerical experiments performed by Craig and Gray (1996) (that found a general increase in growth rate with the heat and moisture transfer coefficients but insensitivity to the frictional drag coefficient or a slight decrease in growth rate with increasing frictional drag coefficient) but contrary to the dependences predicted by growth-rate equations developed for balanced vortices incorporating a CISK-type parametrization for convection. These latter equations predict that the growth rate should increase linearly with frictional drag coefficient but be independent of the heat and moisture transfer coefficients unless changes in these parameters affect either the boundary-layer specific humidity (Charney and Eliassen 1964; Bratseth 1985) or the conditional instability of the atmosphere (Ooyama 1964).

The structure of the vertical-velocity perturbation field, \( w' \), will be discussed in the following subsection. An expression for this is simply obtained from the secondary circulation equation, (26), using the stream-function expressions, (21) giving

\[
\begin{align*}
    w' &= \frac{\alpha}{f} \left( \frac{z^2}{z_T} - z \right) \left\{ \beta \left( \frac{1}{r} \frac{\partial}{\partial r} v_0 + \frac{\partial^2}{\partial r^2} v_0 \right) + \frac{v_0}{r} \left( \frac{1}{r} \frac{\partial \beta}{\partial r} + \frac{\partial^2 \beta}{\partial r^2} \right) \right\} \\
    &\quad + \frac{k}{f} \left( 1 - \frac{z}{z_T} \right) \left( \frac{\partial}{\partial r} v'_0 + \frac{v'_0}{r} \right). \\
\end{align*}
\]

(35)

(c) Discussion

The structure and development of the disturbance predicted by the linearized equation set discussed in the previous subsection are physically reasonable (i.e. comparable with nature) in some aspects but not in others. The azimuthal wind field decays linearly with height (a consequence of the equation for thermal-wind balance, (18), since \( \theta_o \) is independent of height) such that an assumed low-level cyclonic circulation becomes anticyclonic above \( z_T / 2 \). The realism of the predicted outward expansion of the system and exponential growth of perturbations will be examined in the latter part of this paper.

The predicted secondary circulation (given by (28) and (35)) is dependent on the radial structures of the surface azimuthal velocity and the efficiency parameter. In tropical cyclones the azimuthal velocity increases from the centre of the system out to the radius at which it peaks (being such that the eye region is often close to being in solid-body rotation). Beyond this the azimuthal velocity decays, and for mathematical convenience a functional form \( v(r) \propto r^{-1/2} \) can be considered as reasonable. The radial structure of \( \beta \) is unknown; however, it is constrained such that it is always positive and it peaks in the eyewall but has a negative radial gradient at the radius of maximum azimuthal velocity. The physical processes occurring in the eye once the system has become reasonably intense (in particular the radial eddy diffusion of absolute angular momentum which is essential if the wind speed inside the eyewall is to amplify (Emanuel 1997)) have been neglected in this simple model. Thus, an analysis of the predicted secondary circulation will be restricted to the region outside the eye (or this could be considered as before the formation of an eye) i.e. the eyewall and the main body of the tropical-cyclone-type system.

Considering firstly the equation for the radial velocity just above the boundary layer, (28), the heating terms \( ((\alpha / f)(\beta \partial v'_0 / \partial r + v'_0 \partial \beta / \partial r)) \) force inflow (negative \( u'_0 \)) just above the boundary layer in the body of the system (where both \( v'_0 \) and \( \beta \) have negative radial
gradients), opposed by the frictional term. Inflow still occurs at the radius of maximum azimuthal velocity (where $\partial v'_0/\partial r = 0$) providing $-\alpha v'_0 \partial \beta / \partial r > k v'_0 / z_T$. Inflow at this point is necessary for amplification of $v'_0$.

Using the same assumptions, an analysis of the equation for the vertical velocity, (35), shows that both heating terms act to force descent throughout most of the domain with ascent in the eyewall region (since $(z^2/z_T - z) < 0$ for $z < z_T$). The final, frictional term in the equation forces ascent throughout the domain. Note that in the presence of a positive azimuthal velocity gradient near the centre of a system (such as might be found for an eye region close to solid-body rotation, common in tropical cyclones) the first heating term in (35) will force unphysical, infinitely strong descent unless $\beta(r \to 0) \to 0$. This is significant as it suggests that whilst the physical processes necessary to represent the nonconvectoring eye region properly are missing from this simple model, the subsidence that leads to its creation is a necessary consequence of the moist convective dynamics of the surrounding region.

The greatest difficulty in relating this simple model to nature is in the determination of the form of $\partial \beta / \partial r$. This will depend on factors such as the relative humidity of the air in the system and details of the microphysics. The form of $\partial \beta / \partial r$ will be assumed to be set by the structure of a pre-existing vortex. This simple model, by nature of its linearity, is designed to simulate only the initial intensification stages of what will eventually become a tropical-cyclone-type vortex in which nonlinear effects are assumed to dominate. These initial intensification stages often occurred before a visible eye could be observed and, it is assumed, eye dynamics affect the intensification of the system in nature. It is known that in nature tropical-cyclone-type systems develop from pre-existing disturbances and that these disturbances have their own characteristic growth scales. It is hypothesized that in the initial stages of vortex intensification, from a pre-existing disturbance, the form of $\partial \beta / \partial r$ is determined by this disturbance, i.e. by initial conditions. This is consistent with the large diversity of scales on which tropical cyclones form (Merrill 1984). The assumption that this gradient is actually an initial condition implies that it can be treated as an additional external parameter. The growth-rate equation, (31), now predicts a growth rate dependent only on the external (initial and basic-state) conditions.

A final note of caution is that if the growth rate is dependent on the nature of the pre-existing vortex through the form of $\partial \beta / \partial r$ then it is possible that this vortex will also affect the growth rate of the developing system in other ways. This may occur as a result of the relative angular momentum associated with the pre-existing vortex, and suggests that the growth rate may be dependent on the azimuthal velocity structure of a pre-existing vortex required to precondition the atmosphere such that a tropical-cyclone-type disturbance can intensify.

4. Assessment of the simple model

(a) Methodology

The usefulness of the simple model presented in sections 2 and 3 may only be assessed by its ability to represent aspects of tropical-cyclone growth which occur in nature. A convenient way of performing this assessment is by using a fully nonlinear, primitive-equation numerical model which, since it attempts to model virtually all the physical processes and feedbacks occurring in nature (within the constraints of the model), represents our closest approximation to a ‘real life’ system.

There are two main aspects of our simple model that will be assessed using the numerical model. The first is the prediction of the exponential growth of perturbations. Although this is commonly assumed in linear perturbation analysis it is by no means
obvious that tropical-cyclone-type systems actually grow exponentially, and thus this will be examined. The second, the relevance of the growth-rate equation, (31), to systems in nature will be evaluated. A simple scaling will be used to parametrize \( \frac{\partial \beta}{\partial r} \). It is assumed that \( -\frac{\partial \beta}{\partial r} \) scales as \( \beta_0/L \) at the eyewall where \( \beta_0 \) is the difference in \( \beta \) between the eye-wall and the environment and \( L \) is a typical radial length-scale for this change. The growth-rate equation can then be written as

\[
\sigma = \beta_0 \frac{\alpha}{L} - \frac{k}{z_T}. \tag{36}
\]

The linear model thus predicts a relationship between the growth rate of the vortex and the length-scale \( L \) scaled by the factor \( \beta_0 \). It is assumed that the initial length-scale for growth of a tropical-cyclone-type system, \( L \), is dependent on a pre-existing vortex required to precondition the atmosphere such that intensification can begin. In the numerical model this is equivalent to the weak imposed vortex with which convection is initiated or, to be more precise, to the state of this vortex just before the start of its intensification. A value for \( L \) can thus be extracted from the imposed vortex just before intensification begins, and the sensitivity of the simulations to \( L \) can be investigated by varying the size or amplitude of the initial vortex whilst still maintaining its structural form.

There is no unique method of extracting an appropriate length-scale from the numerical model that is relevant to the simple model. It is assumed that the various length-scales in tropical-cyclone-type systems vary together. Hence \( L \), which is defined as a radial decay length-scale for the efficiency parameter, scales with the radius of maximum azimuthal velocity in the numerical and linear models. For comparison between the predicted and measured growth rates, \( L \) in the linear model is redefined as the radius of the maximum azimuthal velocity in the numerical model. The ratio of the actual to the measured length-scale will be incorporated into the efficiency factor, \( \beta_0 \).

The growth-rate equation, (36), predicts the growth rates of disturbances to within a factor of a constant. With \( L \) defined as the radius of maximum winds, the value of this constant, \( \beta_0 \), will now be influenced by a number of simplifications and approximations made in the model. Such simplifications include: the neglect of the azimuthal velocity structure of the imposed initial vortex, the very crude attempt to include the damping effect of unsaturated processes on the system through an efficiency parameter, the assumption that the ratio of the moist to the dry adiabatic lapse rates is constant throughout the depth of the atmosphere, and the neglect of the buoyancy effects of water vapour. Another potential source of error is in the method chosen for extracting \( L \) from the numerical model.

(b) Numerical model

The main prerequisite of the chosen numerical model is that it can represent the complex interactions in a tropical-cyclone system as fully as possible and that the constraints that must be imposed for computational expediency (considering the large number of simulations to be performed) do not strongly inhibit the assessment of the simple model. In particular, the numerical model must represent convection explicitly since, as the simple model employs a drastically simplified cumulus parametrization, it would be wrong to compromise its evaluation by comparing it against a numerical model also employing a cumulus parametrization. To assess fully the usefulness of the simple model, ideally the numerical model chosen for comparison should be able to simulate tropical-cyclone-type disturbances over a range of latitudes.

The experiments were performed using modified versions of the nonhydrostatic, axisymmetric tropical-cyclone model originally written by Rotunno and Emanuel (1987) and
used to simulate polar-low development by Emanuel and Rotunno (1989). The versions used here include the modifications described by Craig (1995, 1996). Details of the configuration are given in Table 1. The axisymmetry of the numerical model is a constraint on the assessment of the simple model in that the strength of the axisymmetric assumption made in the simple model cannot be evaluated. However, the WISHE intensification mechanism is not dependent on the presence of asymmetric effects, and the successful simulation of tropical cyclones and polar lows using axisymmetric models supports the assumption that vortex asymmetries are not essential for their growth.

The simulations are initialized with either the mean hurricane season sounding of Jordan (1958), or a modified version of the 12 GMT 13 December 1982 sounding from Bear Island in the Norwegian sea as used by Emanuel and Rotunno (1989) and with low-level vortices of the type described by Emanuel and Rotunno (1989). In the control simulations the initial tropical-cyclone vortex was given a maximum azimuthal wind speed of 15 m s⁻¹ at 75 km radius and the initial polar-low vortex had a maximum wind speed of 10 m s⁻¹ at 50 km radius. The initial conditions are described in greater detail by Craig (1995, 1996).

In some of the simulations to be discussed, changes have been made to the fluxes of heat and moisture or momentum from the ocean surface. These are represented simply in the numerical model using bulk aerodynamic formulae and are given by

\[
F_q = C_T U (\theta_s - \theta),
\]

\[
F_q = C_B U (q_s - q),
\]

\[
F_m = \rho C_D U^2,
\]

respectively, where \( \rho \) is density, \( U \) is the wind speed at the sea surface (lowest model level), \( \theta_s - \theta \) is the potential-temperature difference between the sea surface and the lowest model level, and \( q_s - q \) is the corresponding difference for the water vapour mixing ratio. The momentum drag coefficient is given by Deacon’s formula

\[
C_D = 1.1 \times 10^{-3} + 4 \times 10^{-5} U,
\]

(following Rotunno and Emanuel (1987)) whereas the heat and moisture transfer coefficients, \( C_T \) and \( C_B \), are assumed to be independent of wind speed. Further details of this representation are given by Craig and Gray (1996).

(c) Examination of the perturbation growth

Investigation, using the numerical model, into the realism of the prediction that perturbations grow exponentially requires: the choice of appropriate perturbation quantities to
study, the definition of the basic-state values of the chosen perturbation quantities, and the choice of an appropriate spatial position at which to study the growth of the perturbation quantity.

Pressure-depression data were used to evaluate the prediction of exponential perturbation growth. Although the alternative measure of intensification, maximum azimuthal velocity, was found to grow similarly with time (with the same intensification rate), the pressure-depression data contained less random noise (due to individual convective elements) and was hence preferable. Strictly, perturbation quantities should be measured at the radius of maximum azimuthal winds in the numerical model since this is the scale of fastest velocity perturbation growth. Inside this radius, physical processes not represented by the simple model affect perturbation growth. For the polar-low simulations, the surface pressure data were taken at a set radius determined as the length-scale for the growing vortex (defined in the following subsection). The same set radius was used for all polar-low simulations except those in which the radius or magnitude of maximum azimuthal velocity in the imposed initial vortex was altered. For simulations performed with the tropical-cyclone version of the numerical model, the variation of surface pressure with time was found to be approximately independent of the radius (relatively near the vortex centre) at which it was measured in the early stages of intensification. Hence, the pressure-perturbation data were taken at the vortex centre for convenience and consistency (the data were taken at the centre for all of the tropical-cyclone simulations).

To calculate the pressure depressions, the resting basic-state surface pressures must be defined for the tropical and polar environments. Since these should be background values, independent of the imposed initial vortices, they were defined as the surface pressures at the outer edge of the domains (at the last grid point before the outer absorbing ‘sponge’ layer was applied) at 20 hours into the control tropical-cyclone and polar-low simulations (at about the time intensification of the vortex began). The pressure varies only very slowly with time at this point and is virtually constant over the outer domain region (range of variation $<2$ mb throughout the tropical-cyclone control simulation within 500 km of the domain edge).

Figure 1 shows how the natural logarithm of the surface pressure depression varies with time for a set of polar-low and tropical-cyclone simulations in which the heat and moisture transfer coefficients, $C_T$ and $C_E$, have been varied. Regression lines have been drawn through the approximately linear portions of the curves, the limits of which were determined objectively using the criteria described in appendix B. For the polar-low data sets (Fig. 1(a)) the plotted curves are approximately linear over a significant time period from $\sim 15$ hours until the time at which the model system tends to maturity and the growth rate slows (at least $\sim 30$ hours later).

The tropical-cyclone data (Fig. 1(b)) is far noisier than the polar-low data, and smoothed data is plotted for all the simulations with the unsmoothed data shown additionally for the control simulation. Although the curves appear approximately linear throughout most of the simulation, there is a kink in the curves (i.e. change in the growth rate measured as the gradient of a regression line) during the simulation. It is possible that this kink is associated with the increasing importance of the effects of eye dynamics in the system as it intensifies. This is consistent with the effect generally becoming apparent at central surface pressure depressions of around 15 mb, since satellite observations of tropical cyclones (Dvorak 1975) indicate that a clear eye is only visible in relatively intense systems. The periods of the initial growth rates in the tropical-cyclone data, of durations exceeding 20 hours, are generally shorter than for the corresponding polar-low data. These simulations also show that for both tropical-cyclone and polar-low vortices there is a general increase in growth rate with increasing values of $C_T$ and $C_E$ in agreement.
with the prediction of the simple model. It is trends such as these that we hope to quantify using the linear solution of the model equations.

To conclude, the prediction of exponential perturbation growth is, perhaps surprisingly, applicable to the initial vortex growth in the nonlinear, primitive-equation numerical model. The simulated polar lows appear to grow exponentially throughout their intensification period whereas, although the same is also true of the simulated tropical cyclones, the actual value of the exponential growth rate changes sharply during the simulation.

\[(d) \quad \text{Examination of the growth-rate equation}\]

(i) Method. The final part of the evaluation of the simple model is the analysis of the applicability of the growth-rate equation (31) to tropical-cyclone-type systems in nature using
series of polar-low and tropical-cyclone simulations. Comparison between the numerical and simple models requires: an appropriate choice of numerical-model experiments, definition of the length-scale for the growing vortex and the basic state parameters, and a consistent method for calculating the growth rates of the numerical model and gauging estimates for the error in both the measured and predicted growth rates.

The growth-rate equation was explored as fully as possible (whilst staying consistent with the laws of physics) by performing a large number of tropical-cyclone and polar-low simulations with the numerical model. All of the relevant parameters in the linearized model equations were varied and the growth-rate values and sensitivities obtained from the numerical model were compared with those predicted by the simple model. Simulations were performed varying $C_T$, $C_E$, $C_D$, $f$, the sea surface temperature (SST), and the radius, $r(v_{\text{max}})$, and magnitude, $v_{\text{max}}$, of the imposed initial maximum azimuthal velocity. In general the chosen model parameter was varied from its control value before beginning the new simulation. The only exception to this was for the simulations in which $C_D$ was varied as described below.

To distinguish between the proposed intensification mechanisms, CISK and WISHE, experiments were performed using this model in which $C_D$ was altered from its control value at the start of the simulations (Craig and Gray 1996). These experiments found that the intensification rate of both the tropical-cyclone and polar-low simulations was approximately insensitive to $C_D$ (although a slight decrease in the intensification rate with increasing $C_D$ was apparent in the polar-low simulations). Since a change in surface friction affects the initiation of convection in the model (through its effect on Ekman pumping), experiments were also performed in which $C_D$ was not altered from its control value until after this had occurred and the vortex had begun to intensify (at 30 hours). This clarified the slight negative dependence of growth rate on $C_D$ in both types of simulations and, for this reason, data from such simulations will be used here. A side effect of this delayed change in $C_D$ is that the time period of the exponential growth in the simulations once $C_D$ has been modified is relatively short. Comparison between the measured and predicted growth-rate dependence on $C_D$ will only be shown for the polar-low simulations, since a significant period of exponential growth did not occur after changing $C_D$ in the tropical-cyclone simulations.

The majority of the basic-state parameters required to calculate the predicted growth rates can be obtained simply from the numerical models. The control values for these, with a few additional control parameter values and explanations for them (where necessary), are given in Table 2. A full list of the changes in model parameters made for the various experiments is given in Table 3.

The length-scale for growth, $L$, was defined as the radius of maximum azimuthal velocity at the time at which intensification of the model vortex began. Before this time the simulated system is in a state of adjustment which has no correspondence in nature. This length-scale was generally measured from data obtained 20 hours after the model simulation was initiated, although occasionally data obtained at 10 or 30 hours were more appropriate. The length-scale estimated from the control tropical-cyclone or polar-low simulation was found to be unchanged for all other similar simulations except for those in which the radius or magnitude of the maximum azimuthal velocity in the imposed vortex used to initiate the numerical model was varied. Although $L$ does not equal the initial $r(v_{\text{max}})$, since the model vortex structure is modified by, amongst other effects, the application of surface friction before convection is initiated and intensification begins, it may be influenced by it. In fact, in the polar-low simulations $L$ was found to increase

* Note that these first experiments were also performed (with a different objective) by Craig and Gray (1996).
TABLE 2. VARIOUS CONTROL-PARAMETER VALUES AND BASIC-STATE PARAMETER VALUES USED IN THE EVALUATION OF THE SIMPLE MODEL USING THE NUMERICAL MODEL

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Polar low</th>
<th>Tropical cyclone</th>
<th>Explanation (where necessary)</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>9.81 m s⁻²</td>
<td>9.81 m s⁻²</td>
<td>Typical values for the appropriate cold, dry or warm, moist atmosphere</td>
</tr>
<tr>
<td>Γₘ</td>
<td>7 K km⁻¹</td>
<td>4 K km⁻¹</td>
<td></td>
</tr>
<tr>
<td>Γₐ</td>
<td>10 K km⁻¹</td>
<td>10 K km⁻¹</td>
<td>Height of tropopause judged from initial model temperature soundings used</td>
</tr>
<tr>
<td>zₜ</td>
<td>7 km</td>
<td>15 km</td>
<td></td>
</tr>
<tr>
<td>Cₑ,ₜ</td>
<td>1.166E⁻³</td>
<td>1.2E⁻³</td>
<td>Control values</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Polar low: Mean of control Cₜ and Cₑ values</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Tropical cyclone: Control Cₑ value since fluxes are mainly latent heat</td>
</tr>
<tr>
<td>f</td>
<td>0.000136 s⁻¹</td>
<td>0.0000614 s⁻¹</td>
<td>Control values</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>As used in the numerical simulations</td>
</tr>
<tr>
<td>SST</td>
<td>279 K</td>
<td>302.15 K</td>
<td>As above</td>
</tr>
<tr>
<td>Initial r(vₘₚₓ)</td>
<td>50 km</td>
<td>75 km</td>
<td>As above</td>
</tr>
<tr>
<td>Initial vₖₓ</td>
<td>10 m s⁻¹</td>
<td>15 m s⁻¹</td>
<td>As above</td>
</tr>
</tbody>
</table>

See text for explanation of symbols.

TABLE 3. NEW VALUES OF MODEL PARAMETERS IN EXPERIMENTS TO EVALUATE THE GROWTH-RATE EXPRESSION

<table>
<thead>
<tr>
<th>Varied Parameter</th>
<th>New value of parameter</th>
<th>New value of parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Polar low</td>
<td>Tropical cyclone</td>
</tr>
<tr>
<td>Cₑ,ₜ</td>
<td>+40%</td>
<td>+40%</td>
</tr>
<tr>
<td>Control values ±</td>
<td>+20%</td>
<td>+20%</td>
</tr>
<tr>
<td></td>
<td>-20%</td>
<td>-20%</td>
</tr>
<tr>
<td></td>
<td>-40%</td>
<td>-40%</td>
</tr>
<tr>
<td></td>
<td>-60%</td>
<td>-60%</td>
</tr>
<tr>
<td>C₀</td>
<td>+100%</td>
<td></td>
</tr>
<tr>
<td>Control values ±</td>
<td>+40%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-40%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>zero</td>
<td></td>
</tr>
<tr>
<td>SST</td>
<td>284 K</td>
<td>304.15 K</td>
</tr>
<tr>
<td></td>
<td>274 K</td>
<td>303.15 K</td>
</tr>
<tr>
<td></td>
<td></td>
<td>301.15 K</td>
</tr>
<tr>
<td></td>
<td></td>
<td>300.15 K</td>
</tr>
</tbody>
</table>

See text for explanation of symbols.

with the initial r(vₘₚₓ) whereas L for the tropical-cyclone simulations was found to be relatively insensitive to it. L was also sensitive to the magnitude of the initial vₖₓ such that, in general, an increase in this leads to a decrease in L. The grid-point radii extracted from the numerical-model simulations as the appropriate length-scales for growth are given in Table 4 together with their assumed error which is taken to be the width of a model grid box.

The value of θₑ in the simple model is the basic-state equivalent potential temperature which, together with the value of θₑₛ, represents the thermodynamic disequilibrium between
the atmosphere and ocean, and thus controls the energy fluxes between the two. The equivalent quantity in the numerical model is thus the $\theta_e$ value in the lowest model level in regions of strong convection, after convection has been initiated and the vortex has begun to intensify, i.e. after the boundary layer has recovered from any unsaturated downdraughts caused by the initial atmospheric overturning in the model. Values for $\theta_e$ and $\theta_{es}$ have been estimated from time series of these values (not shown) and are listed in Table 5 for the two model atmospheric soundings (tropical and polar) and for the respective given imposed SSTs. The error in values of $\theta_{es}$ is negligible compared with that in $\theta_e$ and thus, since the latent- and sensible-heat fluxes from the ocean are dependent on the difference between these values, estimated errors are only given for $\theta_e$.

It is important to devise a consistent method for the calculation of growth rates and their associated error from time series of the natural logarithm of surface pressure depression. Regression lines were calculated over a defined range of data points and the gradient (regression coefficient) of these lines together with the calculated standard error provide values for the model vortex growth rate and its 95% confidence interval. The details
of the calculation procedure are described in appendix B. The predicted growth rates from the simple model were calculated using (36) with the parameter values extracted from the numerical model as described previously. The parameters with the most significant errors were the length-scale for growth, $L$, and the basic-state equivalent potential temperature, $\overline{\theta}_e$. The estimated errors in these quantities (given in Table 4 and Table 5 respectively) were used to provide estimated errors in the predicted growth-rate values.

(ii) Results. Figures 2, 3 and 4 show comparisons between the measured growth rates and the values of $\alpha/L$ (calculated using model parameters) for polar-low and tropical-cyclone simulations in which $\alpha$ and $L$ were varied. These include all the experiments listed in Table 3 except those in which $C_D$ was varied. The growth rates of the simulated tropical cyclones vary in the predicted senses with some of the model parameters, specifically the heat and moisture transfer coefficients, SST and, perhaps to a lesser extent, $f$. In addition, the simulated polar lows also show good agreement for the growth-rate dependence on the initial vortex through variation of the magnitude and radius of its maximum azimuthal velocity. This is indicated in the figures by data points from a given series of experiments.
lying on or about a straight line. In fact, most of the data points in Figs. 2 and 3 fall approximately on the same straight line, which suggests that the simple model is predicting an essential scaling in the growth rates measured from the simulations. The sensitivity of the growth rate of the tropical-cyclone simulations to the initial vortex, i.e. to the initial values of $r(v_{\text{max}})$ and $v_{\text{max}}$ (shown in Fig. 4) will be discussed at the end of this section.

In Figs. 2 and 3, regression lines have been drawn, calculated from all the data points in a particular figure except those obviously failing to fall on the common line. These are the outlying points from some of the experiments in which the Coriolis parameter was varied (two such points from the polar-low simulations and one such point from the tropical-cyclone simulations) and their neglect will be justified in subsection (e). The gradients of the regression lines give an indication of the magnitude of the difference in the efficiency factor between the eyewall and the environment, $\beta_0$. For the polar-low and the tropical-cyclone simulations these (dimensionless) gradients are $0.16 \pm 0.05$ and $0.05 \pm 0.02$ respectively (95% confidence interval). It should be remembered, however, that errors resulting from the simplifications made in the analytical model, and the difficulties associated with trying to compare predictions from it with a more complicated
Figure 4. As Fig. 2 but for the tropical-cyclone simulations and for the initial \( r(v_{\text{max}}) \) and \( v_{\text{max}} \) experiments only.

The numerical model, may be incorporated into the determined \( \beta_0 \). In particular, the length-scale taken from the numerical model is expected to scale with, but unlikely to equal, that defined in the simple model. The remainder of this results section will discuss, in turn, the individual series of experiments, beginning with a closer analysis of the dependence of the growth rate on \( C_T \) and \( C_E \), indicated in Fig. 1, and concluding with an examination of the sensitivity of the growth rates of the polar-low simulations to \( C_D \).

Figure 5 is a plot of the measured growth rates as a function of the proportion of the control values of \( C_T \) and \( C_E \). In general, for both the tropical-cyclone and polar-low simulations, there is an approximately linear increase in growth rate with \( C_T \) and \( C_E \) as predicted by the simple model (assuming of course that the remaining terms in the growth-rate equation are independent of \( C_T \) and \( C_E \)). This dependence appears to break down for high values of the heat and moisture transfer coefficients (i.e. 40% increase above the control value). This is not presently understood. It is possible, however, that if these transfer coefficients are perturbed by too great an amount then the effects of nonlinear feedbacks may become important.

The simple model predicts that the growth rate of a tropical-cyclone-type vortex should depend inversely on the length-scale of the growing vortex, \( L \), (i.e. faster growth
Figure 5. Measured exponential growth rates as a function of the proportion of the control values for the heat and moisture transfer coefficients, $C_T$ and $C_h$, respectively, for (a) polar-low and (b) tropical-cyclone simulations.

for smaller length-scales). As discussed in the previous subsection, the length-scale for growth extracted from the numerical model is the radius of maximum surface azimuthal velocity at about the time intensification begins. In the tropical-cyclone simulations this value appears almost independent of the radius of maximum velocity in the initial imposed vortex. In the polar-low simulations, however, $L$ appears to have an approximately linear dependence on the radius of maximum winds in the imposed, initial vortex (and indeed an exact linear relationship is assumed as can be seen from Table 4) and hence the relationship between the growth rate of the vortex and the length-scale for growth can be examined using these simulations. Figure 6 shows the dependence of the growth rate on length-scale for the polar-low simulations. It is evident that the growth rate of the system decreases with increasing $L$ (as predicted) although there is some scatter in the data. It is difficult to draw a conclusion from the single tropical-cyclone simulation in which $L$ was modified (plotted
in Fig. 4). The simple model predicts that this reduced $L$ should lead to an increased intensification rate. This does occur; however, the increase is greater than that predicted (from extrapolation of the calculated regression line in Fig. 3).

The remaining series of experiments from which data points are plotted in Figs. 2, 3 and 4 are those in which SST, $f$, and $v_{\text{max}}$ were varied. Examination of the appropriate figure shows that the data points from the SST experiments appear consistent with the growth-rate equation predicted by the simple model, although there is some scatter for the tropical-cyclone simulations. A similar examination, looking at the experiments in which $f$ was varied, reveals that although the variation of measured growth rate with $f$ is generally in the predicted sense (although again there is some scatter for the tropical-cyclone simulations) and the data points for high $f$ simulations tend to lie near the relevant line, the measured growth rates appear to become insensitive to $f$ as $f$ becomes small. The data points from the polar-low simulations in which the initial $v_{\text{max}}$ was varied seem to be consistent with the other experiments when adjusted for the decrease in the length-scale for growth found with increasing initial $v_{\text{max}}$. In the tropical-cyclone simulations, however, the variation of the growth rate with initial $v_{\text{max}}$ appears not to follow a systematic pattern (Fig. 4). Explanations for the observed effects of varying $f$ and the initial $v_{\text{max}}$ are proposed in the following subsection.

Finally, the sensitivity of the numerical simulations to $C_D$ is examined. Figure 7(a) shows the natural logarithm of the pressure perturbation as a function of time for a series of polar-low simulations in which $C_D$ was varied (where the start of the regression lines is the time when $C_D$ is changed). Measured growth rates are plotted as a function of $C_D$ and decrease approximately linearly with increasing $C_D$ (Fig. 7(b)). The growth-rate equation from the simple model, (36), predicts

$$\sigma \propto -k,$$

where $\sigma$ is the growth rate and $k$ is a linear frictional drag coefficient. This can be related to the nondimensional quadratic frictional drag coefficient, $C_D\nu_s$, by assuming $k \sim C_D\nu_s$,
Figure 7. (a) Natural logarithm of the pressure depression as a function of time for 90 hours of polar-low simulations for a series of runs in which the frictional drag coefficient, \( C_D \), is varied from its control value after 30 hours. The plotted regression lines are calculated as described in appendix B. (b) Measured exponential growth rates as a function of the proportion of the control value for the frictional drag coefficient, \( C_D \).

where \( v_s \) is a reference surface wind speed that has a constant value and is independent of surface friction (following Ooyama (1964)). The simple model then predicts

\[ \sigma \propto -C_D, \]  

in agreement with the observed results from the polar-low simulations.

An estimate of \( v_s \) can be calculated from the gradient of a regression line plotted through the points in Fig. 7(b). On comparing this gradient with that predicted by the simple model (including the wind-speed dependence of \( C_D \) through Deacon's formula, (40), and assuming that the surface azimuthal velocity component far exceeds the radial component), \( v_s \) is found to lie between 6.9 and 10.0 m s\(^{-1}\) (95% confidence interval on
regression-line gradient). This is of the same order of magnitude as the measured maximum azimuthal velocity (10 to 20 m s\(^{-1}\)) during the rapid intensification period of the polar-low vortex. This value for \(v_0\) is also consistent with that estimated from the y-intercept of the regression line calculated from the data shown in Fig. 2 (although the error in the latter value is considerable). Assuming this intercept represents the term \(-C_D v_0/z_T\) yields a predicted value of \(-0.006\) h\(^{-1}\) (for a typical wind speed of 8 m s\(^{-1}\)) which compares with a measured value of \(-0.001 \pm 0.007\) h\(^{-1}\) (95\% confidence interval in the y-intercept).

To conclude, these figures provide considerable support for the assertion that the growth-rate expression determined from the simple model is predicting the growth rates of simulated tropical cyclones and polar lows within a constant factor which is at least partially controlled by the effects of unsaturated processes.

(e) Discussion

The success of an expression for growth rate predicted by a linear model is perhaps surprising considering the increasing dominance of nonlinear effects with intensification. This success can be explored by linearizing the simple nonlinear model about an ‘average’ vortex representing some average structure of the intensifying system during the early intensification stages. A parallel is the manner in which the dependence of growth rate on a quadratic frictional drag coefficient was determined by relating it to the linear frictional drag coefficient used in the simple model using a reference surface wind speed, \(v_s\).

It is not possible to perform this analytically for a realistic model vortex; however, suggestive results can be obtained from a calculation where the simple model is linearized about a basic state which is independent of height, i.e. \(\bar{v} = \bar{v}(r)\). Assuming that no secondary circulation is forced by sources of heat or momentum as a result of the basic-state vortex then (29) becomes

\[
\frac{\partial}{\partial t} v'_0 + \frac{\alpha}{\{1 + (2\bar{v}/fr)\}} \left( \frac{\beta}{\partial r} v'_0 + v'_0 \frac{\partial \beta}{\partial r} \right) + \frac{k v'_0}{z_T} = 0. \tag{43}
\]

The solution to this is

\[
v'_0 = e^{\alpha t} h(R^* - \alpha t), \tag{44}
\]

where \(R^*(r) = \int [1 + 2\bar{v}/(fr)]\ dr/\beta\), the propagation speed, \(c^*\), is given by

\[
c^* = \frac{\alpha \beta}{\{1 + (2\bar{v}/fr)\}}, \tag{45}
\]

and the growth rate, \(\sigma^*\), is given by

\[
\sigma^* = \frac{\beta \alpha}{\{1 + (2\bar{v}/fr)\}} L - \frac{k}{z_T}. \tag{46}
\]

This analysis shows that the growth rate of a system will depend on the azimuthal velocity structure. The observed sensitivity of the measured growth rate (at the radius of maximum azimuthal velocity) to the value of \(f\) and the initial \(v_{\text{max}}\) will thus depend on their relative magnitudes. In the limit \(f \gg v_{\text{max}}/r(v_{\text{max}})\) the predicted growth-rate expression becomes that found for a resting basic state, i.e. (36). From Fig. 2 it appears that the polar-low simulations generally satisfy this criteria except for the simulations with small values of \(f\). In the opposite limit, \(v_{\text{max}} / r(v_{\text{max}}) \gg f\), the growth rate is predicted to become insensitive to \(f\) (remembering \(\alpha \propto 1/f\)). This is in agreement with the measured growth rates of both the polar-low and tropical-cyclone simulations which were observed to become
insensitive to $f$ when it became small. This expression can also suggest an explanation for the observed dependence of growth rate on $v_{\text{max}}$ in the tropical-cyclone simulations which appear to be in a regime which is sensitive to the azimuthal velocity structure of the vortex (Fig. 4). This dependence appears to be the combination of an increase in predicted growth rate with $v_{\text{max}}$ due to a decrease in $L$ and the inverse dependence on $v_{\text{max}}/r(v_{\text{max}})$ predicted by (46) (assuming that the comparative reduction in $r(v_{\text{max}})$, at a given intensity or $v_{\text{max}}$, with increased initial $v_{\text{max}}$ persists throughout the simulations).

5. CONCLUSIONS

A simple model for the intensification of tropical-cyclone-type systems has been developed using a minimal set of physical assumptions. Tropical-cyclone-type systems are assumed to be balanced systems intensifying through the WISHE (Wind-Induced Surface Heat Exchange) intensification mechanism, driven by surface fluxes of heat and moisture into an atmosphere which is neutral to moist convection. The equation set was linearized about a resting basic state and solved as an initial-value problem. A system was predicted to expand outwards with time as it intensifies which it does with an exponential perturbation growth rate scaled by the radial gradient of an efficiency parameter which crudely represents the effects of unsaturated processes. The form of this efficiency parameter was assumed to be dependent on the nature of a pre-existing vortex required to precondition the atmosphere to a state in which the vortex can intensify. The introduction of an efficiency parameter was necessary to give growing solutions to the model equation set. Its somewhat arbitrary nature is disconcerting and requires further study.

Evaluation of the simple model using a primitive-equation, nonlinear numerical model has provided justification for the prediction of exponential perturbation growth. Good agreement has been found with the growth-rate dependencies predicted by the simple model. Hence, from a measured growth rate from a given simulation, the growth-rate equation, (36), predicts correctly the growth rates which would be obtained on changing the frictional drag coefficient, the heat and moisture transfer coefficients, the length-scale for the growing vortex, the sea surface temperature and, to a lesser degree, the Coriolis parameter and the magnitude of the maximum azimuthal velocity of the imposed initial vortex. The simulations were observed to deviate from the predicted behaviour when components of the absolute vorticity of the imposed initial vortex were varied. This deviation and to some extent the success of the linear model were explored by considering an indicative solution to an equation set linearized about a nonresting basic state.

The success of the comparison between the predicted and measured growth-rate sensitivities provides support for the minimal physical assumptions and the parametrization for convection (which only permits the operation of the WISHE intensification mechanism) with which the simple model was constructed. The main value of this model is conceptual, where the concepts have been verified by the quantitative experiments performed. It is possible, however, that the expression for the growth rate of tropical-cyclone and polar-low systems could also have some predictive value in the assessment of changes in the growth rate with environmental conditions in nature.

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Appendix A

Transformation of the equation for thermal-wind balance from dry to moist variables

It will be shown that, assuming a saturated atmosphere, reversible thermodynamics and neglecting small terms that depend on the buoyancy effects of moisture, the equation for thermal-wind balance, (14), may be written in moist variables as

\[
\left( \frac{2v}{f} + f \right) \frac{\partial v}{\partial z} = \frac{g}{\theta_e} \frac{\Gamma_m}{\Gamma_d} \frac{\partial \theta_e}{\partial r},
\]

(A.1)

where \( \Gamma_d \) and \( \Gamma_m \) are the dry and moist adiabatic lapse rates respectively.

Now,

\[
\left[ \frac{\partial \theta}{\partial r} \right]_z = \left[ \frac{\partial \theta}{\partial r} \right]_p,
\]

(A.2)

since \( z \) is a function of pressure only (according to (6)) and this can be re-written as

\[
\left[ \frac{\partial \theta}{\partial r} \right]_p = \left[ \frac{\partial \theta}{\partial \theta_e} \right]_p \left[ \frac{\partial \theta_e}{\partial r} \right]_p.
\]

(A.3)

By differentiating the expression for saturated moist entropy (assuming a saturated atmosphere), \( s = c_p \ln \theta_e \), we obtain \( \theta_e \) ds = \( c_p \) d\( \theta_e \), and thus

\[
\left[ \frac{\partial \theta}{\partial \theta_e} \right]_p = \frac{c_p}{\theta_e} \left[ \frac{\partial \theta}{\partial s} \right]_p.
\]

(A.4)

Neglecting small terms associated with the buoyancy effects of moisture, the equation of state can be written as

\[
\alpha p = R_d T,
\]

(A.5)

where \( R_d \) is the gas constant for dry air and \( \alpha \) is the specific volume where \( \alpha = \alpha(p, s) \).

Using (A.5) and the definition of potential temperature, (A.4) becomes

\[
\left[ \frac{\partial \theta}{\partial \theta_e} \right]_p = \frac{c_p}{\theta_e} \left( \frac{p_0}{p} \right)^{\kappa-1} \frac{p}{R_d} \left[ \frac{\partial \alpha}{\partial s} \right]_p.
\]

(A.6)

This can be expressed in terms of \( \Gamma_m \) using the Maxwell relation, \( [\partial \alpha / \partial s]_p = [\partial T / \partial p]_s \) (derived in Emanuel 1986, appendix I), and the hydrostatic equation, \( \alpha \) d\( p = -g \) dh (where \( h \) is height), i.e.,

\[
\left[ \frac{\partial \theta}{\partial \theta_e} \right]_p = \frac{c_p}{\theta_e} \left( \frac{p_0}{p} \right)^{\kappa-1} \frac{p}{R_d} \left[ \frac{\partial T}{\partial h} \right]_s \left[ \frac{\partial h}{\partial p} \right]_s,
\]

(A.7)

\[
\left[ \frac{\partial \theta}{\partial \theta_e} \right]_p = \frac{c_p}{\theta_e} \left( \frac{p_0}{p} \right)^{\kappa-1} \frac{p}{R_d} \Gamma_m \left( -\frac{\alpha}{g} \right),
\]

(A.8)

where \( (\partial T / \partial h)_s = \Gamma_m \). Again using the equation of state, (A.5), the definition of potential temperature, and using the equation for the dry adiabatic lapse rate, \( \Gamma_d = -g/c_p \), gives

\[
\left[ \frac{\partial \theta}{\partial \theta_e} \right]_p = \frac{\theta}{\theta_e} \frac{\Gamma_m}{\Gamma_d}.
\]

(A.9)
Substituting (A.9) into (A.3) and then into the thermal-wind-balance equation in dry variables, (14), we obtain

\[
\left( \frac{2v}{r} + f \right) \frac{\partial v}{\partial z} = \frac{g}{\theta_0} \frac{\Gamma_m}{\rho c_p} \frac{\partial \theta_c}{\partial r},
\]

which from the definitions of \( \theta \) and \( \theta_c \) can be simplified to give the equation for thermal-wind balance in moist variables as given by (15).

**APPENDIX B**

*Calculation of the growth rate of the simulated vortices*

To calculate the growth rates of the simulated vortices the beginning and end points of the calculated regression lines must be defined. The start of convection in a simulation was defined as the first time when the maximum vertical velocity in the domain exceeded 0.1 m s\(^{-1}\) which occurred sharply at the time when convection was first observed in the simulation. For the polar-low simulations, the start of the regression line was set simply to the time of the start of convection plus an additional six hours (to allow for the initial atmospheric overturning, and for intensification to begin). The tropical-cyclone simulations did not begin intensifying until a substantial time after the initiation of convection. Thus, to calculate these growth rates, the start of the regression lines was defined as the time, at least six hours* after the start of convection, at which the smoothed (running averaged) pressure reached its maximum value (i.e. after this time the vortices began intensifying).

From studying the model data the most appropriate end point for the regression lines was determined as a set smoothed pressure depression from the smoothed pressure value at the start of the regression line (of 8 mb and 12 mb for the polar-low and tropical-cyclone simulations, respectively, unless, of course, this value was beyond the end of the data set in which case it was taken as the last data point; e.g. the \(-40\%\) curve in Fig. 1(a)).

The only exceptions to the above calculation procedure were for the set of polar-low simulations in which \( C_D \) was changed from the control value after 30 hours (shown in Fig. 7). For these data sets, the start of the regression line was set at the first time after the change in \( C_D \) (i.e. at 30.25 hours), and the end of the regression line was set at the time when the smoothed pressure depression (below that of the pressure for which the start of the regression line is defined for the control simulation) reached a set value (16 mb).

The regression line was calculated from the 'raw' (15 minute) pressure data for the polar-low simulations. The noise in the data from the simulated tropical cyclone (particularly a relatively strong \( \sim 6\)-hour component), however, made it preferable to use the smoothed data for this purpose and then to acknowledge the associated reduction in the number of independent data values when performing the error analysis. Confidence intervals of 95% for each growth rate were determined from the standard error of the regression coefficient (the gradient of the regression line) using the Student \( t \)-error distribution.

**REFERENCES**


* Except for the simulation in which the heat and moisture transfer coefficients were reduced by 60% from their control values where the six-hour delay was increased to twelve hours.