Baroclinic instability of semi-geostrophic fronts with uniform potential vorticity. I: An analytic solution

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SUMMARY

Baroclinic instability of uniform potential-vorticity flow between solid upper and lower boundaries is analysed. The instability is driven by meridional temperature gradients on the boundaries. The classic Eady model of baroclinic instability uses just this system with the further idealization that the temperature gradients in the basic state are uniform. Here, this last idealization will be replaced with a complementary one in which the basic-state temperature gradients are taken to be concentrated in a front. The analysis then takes advantage of the fact that the boundary temperature anomalies created by the growing baroclinic wave are localized at the front. The dependence of the growth rates and phase structure on wave number are remarkably similar to those of the Eady model. The wave number of maximum instability and the short-wave cut-off differ from those of the Eady model by less than 2% and 10% respectively. The solution is asymptotic in the limit of zero frontal width in geostrophic coordinates. For a physical flow this limit can never be achieved, but comparison with direct numerical solutions shows that the analytic solution is still accurate at physically relevant frontal widths. Part I develops the solution based on an equation for the evolution of the displacement of the surface and upper fronts. Part II will look at the three-dimensional structure of the disturbance in more detail.

KEYWORDS: Baroclinic Instability, Eady model, Frontal dynamics

1. INTRODUCTION

The classic models of baroclinic instability developed by Charney (1947) and Eady (1949) encapsulate the mechanisms through which midlatitude synoptic-scale storms gain energy from a reservoir created by the latitudinal gradient in solar heating. These models consider a flow with horizontally uniform baroclinicity, in contrast to earlier conceptual models developed by the Bergen school which regarded the frontal structure as a precursor to the growth of cyclones. The importance of frontal structures was, of course, well known to Eady and Charney. The instability they considered on a horizontally uniform gradient is important because the relatively simple structure isolates the key mechanisms of baroclinic energy conversions, and also because it explains why the atmosphere does not have the smooth meridional gradients which solar heating alone would generate.

Numerical studies (e.g. Hoskins and Simmons 1975; Davies et al. 1991) have shown how the life cycle of baroclinic waves growing on a broad baroclinic zone can generate near discontinuous frontal structures through frontogenesis (Hoskins and Bretherton 1972). In such life cycle experiments the surface fronts are seen as the consequence, rather than a precursor, of instability.

Recent observational work (Shapiro and Keyser 1990) has directed attention towards the frontal nature of the jets in the upper troposphere. This suggests that, in the upper troposphere at least, the basic state prior to wave development can be regarded as a front rather than a smooth gradient. It is interesting to note that recent work focussing on the Lagrangian motion of sharp boundaries (e.g. Verkley 1994) represents a return to the concepts introduced by the Bergen school (W. Verkley, personal communication).

There are many numerical studies (e.g. Snyder 1995 and references therein) of baroclinic disturbances growing on fronts; the solutions resemble those of the classic Eady

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model in many respects. The present paper develops an analytic solution using semi-geostrophic theory. Apart from the localization of boundary temperature gradient, the problem is that considered by Eady, with constant Coriolis parameter and uniform potential vorticity. The mathematical analysis follows the study of tropopause and surface shear-line instability by Juckes (1995). Despite the different physical situation, there is a degree of similarity with the present work in that both instabilities involve the interaction of boundary potential-temperature anomalies. These anomalies can be considered as generalized potential-vorticity anomalies (Bretherton 1966a).

A conceptual model describing baroclinic instability in terms of interacting Rossby waves has been discussed by Bretherton (1966b) and Hoskins et al. (1985). In this conceptual model the internal structure of the waves is assumed to be of secondary importance. The analysis below shows that this assumption can be made mathematically precise in the limit of a narrow front in which scale-separation leads to a degree of decoupling between the internal structure in the vicinity of the front and the large-scale structure which determines the growth rates. This conclusion is based on an asymptotic analysis in the limit of frontal width tending to zero. The small parameter is the ratio of the frontal width in geostrophic coordinates to the Rossby radius of deformation. For typical atmospheric parameters frontal collapse will occur when the front has a width of 100–200 km in geostrophic coordinates, giving a small parameter of the order 0.1 to 0.25. Comparison with numerical solutions shows that the approximate analytic solution is still accurate in this parameter range.

The problem is formulated in terms of interacting Eady edge waves as defined by Bishop (1993a,b) and Davies and Bishop (1994). The classic Eady edge wave (e.g. Gill 1982, chapter 13) is a normal-mode solution for a uniform potential vorticity semi-geostrophic flow with surface potential-temperature gradients and no upper boundary. The papers cited use the term in a slightly different sense, referring to a flow structure associated with a wave-like potential-temperature anomaly on one boundary, with no potential-temperature anomaly on the opposing boundary. This is no longer a normal mode, but provides a convenient physically based division of the problem. That is, the problem is divided into two stages. The first stage, relating the stream function anomaly to the boundary potential-temperature structure, is non-local but linear. The second, evaluating the tendency of the boundary potential temperature, is local but nonlinear. In contrast to the work cited above, which considered purely sinusoidal boundary temperature anomalies, the present study considers anomalies which are associated with a front and thus localized in the cross-front direction. The analysis is formulated in terms of Green's functions discussed in section 2.

Sections 3 and 4 develop the analytic solution for a front with surface temperature gradients given by a Gaussian function of latitude. This latitudinal structure is convenient for the solution method used. The solution gives the normal-mode growth rates to within 1% for a frontal width of 200 km in geostrophic coordinates. The discrepancy between the semi-geostrophic and primitive equation solutions is significantly greater—of the order of 10% (Snyder 1995) for typical frontal strengths. That is, the present results give accurate solutions of the semi-geostrophic equations but only a qualitative description of the modes that arise in more realistic systems.

The effect of the planetary vorticity gradient, $\beta$, will be neglected here. The introduction of $\beta$ is in a sense a singular perturbation (Bretherton 1966a; Rivest et al. 1992) because it gives non-zero potential-vorticity gradients at a critical line. Lindzen (1994) has noted, however, that $\beta$ can be introduced as a regular perturbation if an appropriate change in static stability is made, such that the meridional isentropic gradient of potential
vorticity remains zero in the troposphere. That is
\[
\frac{\Delta N^2}{N^2} = C \left( \frac{\beta H_p N^2}{f^2 \bar{u}_z} \right) \approx 0.14,
\]
where $N^2$ is the Brunt–Väisälä frequency, $\Delta N^2$ the change required to effect zero meridional potential-vorticity gradients, $H_p$ the density-scale height, $f$ the Coriolis parameter and $\bar{u}_z$ is the vertical shear of the zonal-mean zonal wind. Such a change is by no means insignificant, but if, as suggested by Lindzen’s argument, it can be treated as a regular perturbation, the solution for $\beta = 0$ should provide a reasonable approximation to the solution for $\beta \neq 0$. A glossary of terms and symbols used is given in appendix D.

2. THEORETICAL FORMULATION

(a) Equations of motion and non-dimensionalization.

We consider uniform potential vorticity, Boussinesq, semi-geostrophic flow, on an $f$-plane, with solid boundaries at $z = 0$ and $z = 1$ representing the ground and tropopause. Non-Boussinesq effects are neglected here because they complicate the problem considerably, not least by removing the symmetry between the ground and tropopause, without changing the basic physics of the instability (Eady 1949). The vertical coordinate is $z = z^1 z_{sc}^{-1}$, where $z^1$ is the geometric height and the height-scale $z_{sc}$ is taken to be the height of the tropopause. The theory uses geostrophic horizontal coordinates, $(x, y) = (x^1, y^1) x_{sc}^{-1}$, non-dimensionalized using the Rossby radius given by $x_{sc} = z_{sc} N f^{-1}$. The geostrophic coordinates are related to the geometric coordinates (subscript ‘geom’) by $(x^1, y^1) = (x_{geom} + u^1 f^{-1}, y_{geom} - u^1 f^{-1})$ (Hoskins 1975; Hoskins and Draghi 1977), where $(u^1, v^1)$ is the geostrophic wind (since all horizontal velocities considered in this paper are geostrophic the usual subscript ‘g’ will be omitted to avoid cluttering the equations).

As in Hoskins and West (1979) and Snyder et al. (1991) (and as opposed to Eliassen (1983) and Joly and Thorpe (1990)) a nonlinear term which arises in the semi-geostrophic potential vorticity as defined by Hoskins (1975) has been neglected. The significance of this term is discussed in more detail by Snyder et al. For the uniform potential-vorticity flow considered here it follows that the basic-state variables in geostrophic coordinates are linearly related to each other, so that doubling the surface potential-temperature gradients simply doubles the wind speed and does not change the structure of the flow. The structure of the disturbance is then independent of the strength of the basic flow. Since the emphasis of this paper is on general structure, all the results will be presented in geostrophic space. The solution for various strengths of basic flow, having the same structure in geostrophic coordinates, is then simply obtained by re-scaling the results (see paragraph below equation (5)), provided the transformation from geostrophic to physical coordinates remains valid.

The potential temperature is non-dimensionalized using $\theta_{sc} = \theta_{00} N x_{sc}/(g t_{sc})$ where $\theta_{00}$ is a reference value, $g$ is the acceleration due to gravity and $t_{sc} = f^{-1}$ is the time-scale used in the non-dimensionalization. With these definitions, $\theta_{sc}$ is the change in the static reference potential temperature, $\theta_s(z)$, between the surface and tropopause. Taking $x_{sc} = 800$ km, $g = 10$ m s$^{-2}$, $\theta_{00} = 300$ K and $N = 10^{-2}$ s$^{-1}$ gives $\theta_{sc} = 24$ K (the value of the uniform potential vorticity is then given by $N^2 = 10^{-4}$ s$^{-2}$).

Given the assumption of uniform potential vorticity in $0 < z < 1$, the evolution is determined by advection of the potential temperature on the boundaries $z = 0, 1$. When
linearized the evolution equation is given by:

$$\frac{\partial u_\mu}{\partial t} + \bar{u}_\mu \frac{\partial u_\mu}{\partial x} + v_\mu \frac{\partial \bar{\theta}_\mu}{\partial y} = 0,$$

(1)

where $\bar{u}$ and $\bar{\theta}$ are the basic-state zonal wind and potential temperature respectively (the former non-dimensionalized with $v_c$). The subscript $\mu$ takes values 0 and 1 for the ground and tropopause respectively. $\bar{\theta}$ and $v$ are the disturbance potential temperature and meridional velocity. The geostrophic velocity $(u, v)$ and the potential-temperature anomaly $\theta$ are related to the stream function $\psi$ as follows:

$$(u, v, \theta) = (-\psi_y, \psi_x, \psi_z).$$

(2)

The potential-vorticity anomaly in the interior vanishes, so the stream function satisfies Laplace’s equation with Neumann boundary conditions (e.g. Held et al. 1995):

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad 0 < z < 1.$$  

(3a)

$$\frac{\partial \psi}{\partial z} = \bar{\theta}, \quad z = 0, 1.$$  

(3b)

Equations (1)–(3) can be obtained from Eqs. (24) and (25) of Hoskins and Draghi (1977) by setting $q_y = N^2$ and $\Phi = Z^2 f^2/2 + \psi^\dagger$ and then non-dimensionalizing.

The disturbance can be represented in terms of a meridional displacement in geostrophic coordinates, $\eta$, of the isentropes,

$$\eta = \frac{\bar{\theta}}{\Lambda},$$

(4)

where

$$\Lambda = \frac{\partial \bar{\theta}}{\partial y} = \frac{\partial \bar{u}}{\partial z}.$$  

This is convenient in the present study because the meridional displacement is regular in the limit of a frontal discontinuity, whereas the potential-temperature perturbation becomes unbounded in that limit. The displacement of material contours has been widely used in the study of nonlinear vortex dynamics (e.g. Dritschel 1989). There, the full nonlinear evolution of a finite number of discrete material contours is considered, whereas the present study is concerned with the linear evolution of a continuum of material contours describing the structure of a continuous two-dimensional field. This continuous set of contours will later be reduced to a finite set of variables, by taking moments about the centre of the front and truncating the moment expansion.

Equations (1) and (4) can be combined to give the usual linearized kinematic equation for a material contour:

$$\frac{\partial \eta_\mu}{\partial t} + \bar{u}_\mu \frac{\partial \eta_\mu}{\partial x} + v_\mu = 0.$$  

(5)

As noted above, it is possible to generate solutions for a family of basic states by rescaling. That is, given a template consisting of a uniform potential-vorticity basic state with zonal wind $\bar{u}_i(y, z)$, potential temperature $\bar{\theta}_i = z + \bar{\theta}_i(y, z)$, and a corresponding linear disturbance $\eta = \eta_i(x, y, z, t), \theta = \theta_i(x, y, z, t)$, there is a family of solutions, for an arbitrary constant $\alpha$, given by the basic states $\bar{u} = \alpha \bar{u}_i(y, z), \bar{\theta} = z + \alpha \bar{\theta}_i(y, z)$ (which also have
uniform potential vorticity), and the corresponding linear disturbances \( \eta = \eta_0(x, y, z, \alpha t) \),
\( \theta = \theta_0(x, y, z, \alpha t) \). The error terms increase with increasing \( \alpha \), so there is a point beyond which these solutions cease to be relevant to the atmosphere.

The relation between the potential-temperature distribution and the stream function is most easily given in terms of the horizontal Fourier transform, which is used here with the following convention:

\[
\hat{\psi}(k, l; z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, y, z) e^{-i(kx+ly)} dxdy,
\]

\[
\psi(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\psi}(k, l; z) e^{i(kx+ly)} dkdl.
\]

The flow considered will be periodic in \( x \) and unbounded in \( y \). It is convenient to introduce notation for a one-dimensional Fourier transform with respect to the cross frontal coordinate:

\[
\tilde{\psi}(k; y) = F^{-1}[\hat{\psi}] = \int_{-\infty}^{\infty} \hat{\psi}(k, l) e^{ily} dl,
\]

\[
\hat{\psi}(k, l) = F[\tilde{\psi}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(k; y) e^{-ily} dy.
\]

A semicolon is used to separate geometric coordinates from Fourier wave numbers within the argument list of partially transformed functions.

(b) The Green's function

The method which carries his name originated from Green's work on electostatics (reprinted in Green 1970), in which he derived mathematical language to describe potential fields generated by electric charges. Hoskins et al. (1985) and Bishop and Thorpe (1994) have discussed the analogy between the theory of electostatics and the inversion problem posed by (3). Just as an isolated electric charge can be associated with a potential, so an isolated potential-vorticity anomaly or boundary potential-temperature anomaly can be associated with a stream function. Following terminology often used in the context of this method, the right-hand side of (3) will be referred to as the 'source' and the function occurring within the differential operator as the 'response'. This is mathematical terminology and does not imply physical causality. The Green's function is the response in a mathematical idealization in which the source is taken to be concentrated at a single point. A more general response can then be represented as a weighted sum, or integral, of the Green's function relative to all points which have non-zero contributions to the source. This idealization is particularly useful in the present case because the sources (the potential-temperature anomalies) are indeed strongly localized. As a result, much of the structure of the Green's function is retained in the full solution obtained by superposition of Green's functions.

Here, the Green's function \( G(x - x_{src}, y - y_{src}, z; z_{src}) \) is the stream function generated by a boundary potential-temperature anomaly localized at \((x_{src}, y_{src}, z_{src})\), where \( z_{src} = 0, 1 \). Note that the homogeneity of the problem in the \( x \) and \( y \) directions implies that the Green's function depends only on the relative positions \( x - x_{src} \) and \( y - y_{src} \). The boundaries at \( z = 0 \) and \( 1 \) introduce inhomogeneity in the vertical, so the dependence on \( z \) and \( z_{src} \) cannot be simplified in the same way. A semicolon is used here to separate the last
argument, which depends only on the position of the source, from the rest of the argument list.

The solution of (3) in Fourier space is then given by:

$$\tilde{\psi}(k, l; z) = 4\pi^2 \left( \tilde{G}(k, l; z; 0)\tilde{a}_b(k, l) + \tilde{G}(k, l; z; 1)\tilde{a}_1(k, l) \right),$$  \hspace{1cm} (8)

where \( G \) satisfies the following equation, in which, without loss of generality, \((x_{src}, y_{src}, z_{src}) = (0, 0, 0)\) has been assumed:

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = 0, \quad 0 < z < 1,$$ \hspace{1cm} (9a)

$$\frac{\partial G}{\partial z} = \begin{cases} \delta(x)\delta(y), & z = 0, \\ 0, & z = 1. \end{cases}$$ \hspace{1cm} (9b)

The second term on the right-hand side of (8) can be evaluated from the solution of (9) by noting the symmetry under \( z \rightarrow 1 - z, \theta \rightarrow -\theta \), which implies \( G(x, y, z; 1) = -G(x, y, 1 - z; 0) \).

The boundary-value problem (9) can be solved by first reducing it to an unbounded problem with symmetry about \( z = 0 \). The symmetry ensures that (9b) is satisfied for \((x, y) \neq (0, 0)\). To relate the singularity at \((0, 0, 0)\) in the homogeneous problem to that in the boundary-value problem consider a disc shaped volume, \( \mathcal{V} \), with radius \( d_1 \ll 1 \) and thickness \( d_2 \ll d_1 \). Application of the Gauss divergence theorem gives:

$$\int_{\mathcal{V}} \nabla^2 G dV = \int_{\partial \mathcal{V}} (\nabla G) \cdot \mathbf{n} dA \approx 2 \int_{\partial \mathcal{V}} \frac{\partial G}{\partial z} dA = 2,$$ \hspace{1cm} (10)

where \( \partial \mathcal{V} \) is the surface of the volume, \( \partial \mathcal{V}_2 \) is the disc \( x^2 + y^2 < d_2^2, z = 0 \), and \( \mathbf{n} \) is the unit, outward directed, vector normal to the surface. The factor two arises because the integral over \( \partial \mathcal{V} \) is taken over both the upper and lower faces of the disc. The integral over the cylindrical outer surface can be neglected because its area is, by construction, much less than that of the upper and lower discs. It follows that \( \nabla^2 G = 2\delta(x)\delta(y)\delta(z) \) in the vicinity of the singularity. This conclusion can also be obtained by combining a result from Bretherton (1966a), stating that a boundary potential-temperature anomaly can be modelled by a \( \delta \)-function potential-vorticity anomaly just above the boundary \( z = 0 \), with a result discussed in Bishop and Thorpe (1994), showing that the effect of the boundary can be modelled by an image potential-vorticity distribution. In this case the image is a second \( \delta \)-function potential-vorticity anomaly just below \( z = 0 \).

The solution to the homogeneous problem is well known (e.g. Green 1970). In the absence of the upper boundary at \( z = 1 \) the solution to (10) would be:

$$G_{ref}(x, y, z; 0) \overset{\text{def}}{=} -\frac{1}{2\pi(x^2 + y^2 + z^2)^{1/2}} + \text{constant}. \hspace{1cm} (11)$$

The boundary condition at \( z = 1 \) can be satisfied by including an array of image sources, such that the array is symmetric about both \( z = 0 \) and \( z = 1 \). That is:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) G(x, y, z = 2) = \sum_{h=-\infty}^{\infty} \delta(x)\delta(y)\delta(z + 2h). \hspace{1cm} (12)$$

Since all the image sources lie outside \( 0 < z < 1 \) the solution to (12) is also a solution to (9a). The solution to (12) can be written as a sum of solutions of the form (11) provided that
the sum converges. This can be ensured through the appropriate choice of the constants of integration, giving:

\[
G(x, y, z; 0) = -\frac{1}{2\pi} \left\{ \sum_{h=-\infty}^{\infty} \left[ \frac{1}{(x^2 + y^2 + z^2)^{1/2}} + \frac{1}{(x^2 + y^2 + (z + 2h)^2)^{1/2}} - \frac{1}{2|h|} \right] \right\} .
\] (13)

For large \( h_0 \) the terms \( h = h_0 \) and \( h = -h_0 \), which are individually of order \( h_0^{-2} \), combine to be of order \( h_0^{-3} \), so the sum converges uniformly. Figure 1 displays \( G(x, y, z; 0) \). Near the origin the dominant behaviour is given by the first term in (13) and later terms give minor corrections. At large distances from the source, however, a large number of terms need to be included before the series converges. An alternative approach is to express the Green's function as a Fourier sequence with respect to the \( z \) coordinate. This gives:

\[
G = \frac{1}{2\pi} \left\{ \gamma - 2 \log 2 + \log (x^2 + y^2)^{1/2} - 2 \sum_{j=1}^{\infty} \left[ K_0 \left( j\pi (x^2 + y^2)^{1/2} \right) \cos (j\pi z) \right] \right\} ,
\] (14)

where \( K_0 \) is the modified Bessel function. Applying \( \nabla^2 \) to each term gives \( \delta(x)\delta(y) \cos j\pi z \) and summation over \( j \) then gives the \( \delta(z) \) factor. The constant term in (14) has no physical significance, but makes the expression numerically identical to (13). The connection between (13) and (14) can be verified using the 'continuum approximation' (appendix A) for the sum in (13) at large \( x^2 + y^2 \), or for (14) at small \( x^2 + y^2 \).

The horizontal structure of the leading-order baroclinic term, proportional to \( \cos(\pi z) \), is given by the modified Bessel function, which has the following asymptotic form at large
radius:

\[
K_0 \left\{ \pi \left( x^2 + y^2 \right)^{\frac{3}{2}} \right\} = \frac{(x^2 + y^2)^{-\frac{1}{2}}}{\sqrt{2}} \exp \left\{ -\pi \left( x^2 + y^2 \right)^{\frac{1}{2}} \right\} \left\{ 1 + \mathcal{O} \left( (x^2 + y^2)^{-\frac{1}{2}} \right) \right\}, \quad x^2 + y^2 \to \infty.
\]

This gives exponential decay with a length-scale of \( x_{sc}/\pi \), a few hundred kilometres. Consequently, the Green’s function at large radius is dominated by the barotropic term which, as in barotropic vortex dynamics, is logarithmic. Widely spaced temperature anomalies will therefore behave as equivalent barotropic vortices. The associated velocity field, \((-G_y, G_x)\), decays as \((x^2 + y^2)^{-1/2}\), with circulation given by \( \oint u \cdot ds = \int \theta dA \) at large distances from the \( \theta \) anomaly. In dimensional variables this is:

\[
\oint u^d \cdot ds = \frac{f}{\theta_{sc}} \int \theta^d dA.
\]  

(15)

For the parameter values listed above, the dimensional constant is about \( 4 \times 10^{-6} \, (K \, s)^{-1} \). Equation (15) shows that a \( \theta_{sc} \) (here 24 K) boundary potential-temperature anomaly generates the same far field circulation as a barotropic vorticity anomaly with the same area, and amplitude \( 10^{-4} \, s^{-1} \).

This paper is concerned with sinusoidal waves on a zonally symmetric basic state, so the \( x \)-Fourier transform of the Green’s function is required. This describes the stream function for a potential-temperature anomaly which is sinusoidal in the \( x \) direction and localized in \( y \), i.e. a wave on a surface front. The Fourier transform of the differential equation (3) gives:

\[
\left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - k^2 \right) \tilde{G}(k; y, z; 0) = 0,
\]

(16a)

\[
\frac{\partial \tilde{G}}{\partial z} = \begin{cases} 
\delta(y) \over 2\pi, & z = 0, \\
0, & z = 1.
\end{cases}
\]

(16b)

Solutions can be obtained by similar methods to those described above, or by taking the transforms of (13) and (14) respectively, giving for \( k \neq 0 \):

\[
\tilde{G}(k; y, z; 0) = -\frac{1}{2\pi^2} \sum_{h=-\infty}^{\infty} K_0 \left[ k \left\{ y^2 + (z + 2h)^2 \right\}^{1/2} \right] 
\]

(17a)

\[
= -\frac{1}{4\pi} \left[ \frac{e^{-|k|y}}{|k|} + 2 \sum_{j=1}^{\infty} \exp \left\{ -|y|(j^2 \pi^2 + k^2)^{1/2} \right\} \left( j^2 \pi^2 + k^2 \right)^{1/2} \cos(j \pi z) \right].
\]

(17b)

The leading-order baroclinic term in (16b) decays on a length-scale of \( x_{sc}/\pi \) or less, i.e. 250 km or less for a tropopause height of \( z_{sc} = 8 \) km. This means that baroclinic motion and the associated vertical velocity will be localized near the front. The barotropic component of the flow, on the other hand, will extend for a distance of the order of the wavelength away from the front. Near the front the structure is described by the first term in (16a), which has the following asymptotic behaviour at small radius:

\[
K_0 \left( r_x \right) = - \left\{ \gamma + \ln \left( \frac{r_x}{2} \right) \right\} \left( 1 + \frac{r_x^2}{4} \right) + \mathcal{O} \left( r_x^4 \right)
\]

(18)
for \( r_s = k \sqrt{y^2 + z^2} \). That is, the stream function increases logarithmically and the associated velocities as \( 1/y \) as the front is approached. For a front of finite width the stream function and velocity fields will be given by the convolution of the frontal temperature structure, as described in more detail below, with the above Green's function. Instead of increasing without limit, the stream function then reaches a maximum, which is of the order of the logarithm of the frontal width.

The Green's function for the zonally symmetric component cannot be obtained by setting \( k = 0 \) in (17), because both expressions are singular as \( k \to 0 \). In (17b), however, the leading-order term as \( k \to 0 \) is independent of \( y \) and \( z \). Subtracting this term and taking the limit gives:

\[
\widetilde{G}(0; y, z; 0) = -\frac{1}{4\pi} \left\{ -|y| + 2 \sum_{j=1}^{\infty} \frac{1}{j\pi} \cos(j\pi z) \right\}.
\]  

(19)

The process of taking the limit \( k \to 0 \) and subtracting the singular term is valid because the freedom to add a constant, \( 2\pi c_1 \) say, to \( G \) is equivalent to adding a singularity, \( c_1 \delta(k) \), to \( \widetilde{G} \). Alternatively, taking the limit in (17a), the modified Bessel functions are logarithmic at leading order. The sum is divergent in this limit, but can be made convergent by subtracting a constant from each term. The result is:

\[
\widetilde{G}(0; y, z; 0) = \frac{1}{2\pi^2} \left( \ln \pi (y^2 + z^2)^{1/2} + \sum_{j=-\infty \atop j \neq 0}^{\infty} \ln \left[ \frac{(y^2 + (z + 2h)^2)^{1/2}}{2h} \right] \right).
\]  

(20)

These solutions can also be obtained by setting \( k = 0 \) in (14) and solving the resulting two-dimensional Poisson problem. As before the two representations of the Green's function illustrate different aspects of its structure. The leading-order term in (19) gives the large-scale barotropic response, whilst the first term in (20) shows the structure near the singularity. A third, more compact, form of the zonal mean Green's function is given by:

\[
\widetilde{G}(0; y, z; 0) = \frac{1}{4\pi^2} \ln \left[ 2 \left( \cosh \pi y - \cos \pi z \right) \right].
\]  

(21)

This can be derived from (19) by taking the \( z \) derivative, summing the algebraic series that results, and then integrating with respect to \( z \). It can be verified directly by substitution into (14) for \( (y, z) \neq (0, 0) \) and by comparison with (20) in the vicinity of the singularity. There does not appear to be an easy generalization of this compact formula to replace the complex expressions when \( k \neq 0 \). The manipulations of these expressions which are carried out below are, however, facilitated by the identity between (19), (20) and (21). Figure 2 illustrates the structure of \( \widetilde{G} \). As before, there is a transition from near circular isolines around the origin to a barotropic structure at larger distances. The \( k = 0 \) structure (Fig. 2(a)) has a constant horizontal gradient at large distances from the origin, whilst at \( k \neq 0 \) the function decays exponentially to zero. For \( k = 0.4 \) this decay only becomes clear at distances beyond the range plotted (0 < \( y < 2 \)). At larger wave numbers (Figs. 2(c) and 2(d)) the amplitude of the anomaly at the tropopause \( (z = 1) \) is significantly reduced (the dashed contours in these figures have one tenth of the contour interval used elsewhere in Fig. 2).

The strength of the winds at the tropopause associated with a surface anomaly are given by \( k \) times the Green's function. This quantity increases as \( k \to 0 \), dominated by the
Figure 2. \( x \)-Fourier transform of the Green's function, \( \tilde{G}(k; \ y, \ z; \ 0) \). The stream function associated with a surface temperature anomaly \( \theta_0(x, \ y) = \delta(y) (2\pi)^{-1} \cos k x \) is given by \( \psi(x, \ y, \ z) = \tilde{G} \cos k x \): (a) \( k = 0 \), (b) \( k = 0.4 \), (c) \( k = 1.6 \) and (d) \( k = 2.0 \). Contour interval as in Fig. 1. See text for further details.

The first term in (16b). This term is, however, barotropic, so does not contribute to baroclinic energy conversions. The baroclinic component of the wind field tends to zero at large scales. A consequence is that the maximum baroclinic growth rate does not occur at \( k = 0 \) but at a finite wave number.

Taking the \( y \)-Fourier transform of (14) gives:

\[
\left( \frac{\partial^2}{\partial z^2} - k^2 - l^2 \right) \tilde{G}(k, l; z; 0) = 0,
\]

\[
\frac{\partial \tilde{G}}{\partial z} = \begin{cases} 
\frac{1}{4\pi^2}, & z = 0, \\
0, & z = 1,
\end{cases}
\]

which has solution:

\[
\tilde{G}(k, l; z; 0) = -\frac{\cosh \lambda (1 - z)}{4\pi^2 \lambda \sinh \lambda},
\]

where \( \lambda^2 = k^2 + l^2 \) (e.g. Bishop 1993a,b). This is the most compact representation of the Green's function. The barotropic nature of the large-scale flow is reflected in the property:

\[
\tilde{G} \to -\frac{1}{4\pi^2 \lambda^2} \quad \text{as} \quad \lambda \to 0.
\]
(c) The Green’s function formulation of the instability problem

Equations (2)–(4) and (8) determine the evolution of a linear wave. This subsection combines these into a single equation for the evolution of the meridional displacement $\tilde{h}_x(k; y, t)$.

Since the evolution is determined by the boundary potential temperature we can concentrate on the boundary stream function: $\psi_0(x, y) = \psi(x, y, 0)$ and $\psi_1(x, y) = \psi(x, y, 1)$. The inversion relation (8) can be expressed in matrix form as:

$$
\begin{pmatrix}
\hat{\psi}_0 \\
\hat{\psi}_1
\end{pmatrix} = 4\pi^2 \mathcal{G} \begin{pmatrix}
\hat{\theta}_0 \\
\hat{\theta}_1
\end{pmatrix},
$$

(23)

where

$$
\mathcal{G} = \begin{pmatrix}
\hat{G}_{00} & \hat{G}_{01} \\
\hat{G}_{10} & \hat{G}_{11}
\end{pmatrix} = \frac{1}{4\pi^2} \begin{pmatrix}
\frac{-\coth \lambda}{\lambda} & 1 \\
\frac{1}{\lambda \sinh \lambda} & \frac{\coth \lambda}{\lambda}
\end{pmatrix},
$$

(24)

and $G_{\mu\nu}(x, y) = G(x, y, z; z_{\text{src}})$ with $(z; z_{\text{src}}) = (\mu; \nu)$, where $\mu, \nu$ take values 0 and 1 for the ground and tropopause respectively.

The Green’s functions, $G$, are normalized so that the stream function in geostrophic space is the convolution of $G$ with the $\theta$ anomalies; the factor $4\pi^2$ in (23) follows from
the definitions of the Fourier transforms. Combining (4) and (23), using \( \hat{\psi}_\mu = ik\hat{\psi}_\mu \), gives:

\[
\hat{\nu}_\mu(k, l, t) = 4\pi^2ik \left\{ \hat{G}_{\mu 0} \mathcal{F}[\Lambda \tilde{\eta}_0] + \hat{G}_{\mu 1} \mathcal{F}[\Lambda \tilde{\eta}_1] \right\}
= 4\pi^2ik \left\{ \hat{G}_{\mu 0} (\Lambda \circ \tilde{\eta}_0) + \hat{G}_{\mu 1} (\Lambda \circ \tilde{\eta}_1) \right\},
\]

(25)

where \( \circ \) denotes a convolution over \( l \):

\[
(\Lambda \circ \tilde{\eta}_1)(l) = \int_{-\infty}^{\infty} \Lambda(l')\tilde{\eta}_1(l-l')dl'.
\]

The second line in (25) follows from the first via the convolution theorem, which states that the Fourier transform of a product is equal to the convolution of the individual Fourier transforms. In half-transformed space (25) becomes:

\[
\tilde{\nu}_\mu(k, y, t) = 2\pi ik \left\{ \hat{G}_{\mu 0} \circ (\Lambda \tilde{\eta}_0) + \hat{G}_{\mu 1} \circ (\Lambda \tilde{\eta}_1) \right\},
\]

(26)

where \( \circ \) denotes a convolution over \( y \):

\[
(\tilde{G} \circ \tilde{\theta})(y) = \int_{-\infty}^{\infty} \tilde{G}(y-y')\tilde{\theta}(y')dy'.
\]

Similarly, the zonal-mean zonal flow is given by:

\[
\bar{u}_\mu(y) = -\frac{\partial}{\partial y} \sum_{v=0,1} 2\pi \left\{ \tilde{G}_{\mu v}(0; y-y') \circ \tilde{\theta}(y') \right\}
= 2\pi \sum_{v=0,1} \left\{ \tilde{G}_{\mu v}(0; y-y') \circ \Lambda(y') \right\}.
\]

(27)

The second line follows from the first by using the fact that \( \partial/\partial y = -\partial/\partial y' \) when applied to \( \tilde{G}_{\mu v}(0; y-y') \), and then integrating by parts to transfer the derivative from \( \tilde{G} \) to \( \tilde{\theta} \) within the convolution. Alternatively, in Fourier space:

\[
\hat{\tilde{u}}_\mu(l) = 4\pi^2 \sum_{v=0,1} \hat{G}_{\mu v}(0, l)\hat{\Lambda}(l).
\]

Substituting (26) and (27) in the \( x \)-Fourier transform of (5) then gives the evolution equation:

\[
\frac{\partial \tilde{\eta}_\mu(k; y, t)}{\partial t} = -2\pi ik \sum_{v=0,1} \left\{ \tilde{G}_{\mu v}(0; y-y') \circ \Lambda(y') \right\} \tilde{\eta}_\mu(k; y, t)
- \tilde{G}_{\mu v}(k; y-y') \circ \left\{ \Lambda(y')\tilde{\eta}_v(k; y', t) \right\}.
\]

(28)

This determines the evolution of \( \eta \) in a single equation. The basic state is represented by \( \Lambda \) and the dynamical properties of the system are represented by the Green's function. The advection by the zonal-mean zonal wind is described by the first term, which includes the convolution of the \( k = 0 \) Green's function with the basic-state temperature gradient. The second term describes the advection of the basic state by the disturbance transverse velocity. The leading-order result obtained below can be found directly by setting \( \Lambda(y) \propto \delta(y) \) in (28). The corresponding basic state has, however, infinite zonal velocity at \( (y, z) = (0, 0) \). For this reason a somewhat more extended analysis is carried out to calculate the growth rates for a narrow but continuous temperature transition.

In general both the mean flow, \( \bar{u} \), and potential-temperature gradient, \( \Lambda \), are functions of \( y \), so a simple analytic solution is not possible. The Eady solution (e.g. Gill 1982) is obtained when \( \bar{u} = \Lambda_e(z-1/2) \) and \( \tilde{\theta} = -\Lambda_e y \) for a constant \( \Lambda_e \).
The aim of this paper is to look at the structure of baroclinic instability over a range of basic states with varying widths of the baroclinic zone. For simplicity we will concentrate on the case in which the tropopause and surface potential-temperature gradients are equal. It follows that the surface wind is minus the tropopause wind. More realistic flows, with a strong tropopause jet and relatively weak surface winds, can be constructed using unequal potential-temperature gradients. This, however, makes the calculation more complicated and would obscure the main focus of this paper. Such flows will be considered separately, as will solutions using more realistic Green’s functions which can represent the effect of finite static stability in the stratosphere and finite density-scale height.

The meridional temperature gradient in the fronts is taken to have a Gaussian form:

\[
\Lambda(y) = \sqrt{\frac{2}{\pi}} \frac{\theta_t}{r} \exp \left( -\frac{y^2}{2r^2} \right),
\]  

(29)

The normalization in (29) is chosen so that the change in \(\bar{\theta}\) between \(y = \pm \infty\) is equal to \(2\theta_t\), for some constant \(\theta_t\). Figure 3 shows \(\bar{\theta}\) and \(\bar{u}\) for a range of \(r\) values between 1/16 and 1. This paper looks at the limit \(r \to 0\). The larger values of \(r\), for which the flow approaches that of the classic Eady problem, will be considered separately.
A value of $\theta_l = 0.5$ corresponds to a front in which the highest surface potential temperature is equal to the lowest tropopause potential temperature. Values in the range 0.25 to 0.5 are then of practical interest. For such values frontal collapse occurs at around $r = 0.1$ (more details in part II), so frontal widths in the range 0.1 to 0.2 are of interest (i.e. fronts which are slightly broader than the point of collapse).

3. Solution by Moment Expansion

The asymptotic analysis below makes use of a moment expansion; this section lays out the general structure of that expansion. The $n$th moment of a variable is taken as an integral of that variable multiplied by a polynomial of order $n$ and an envelope function as defined below. The original structure of the variable can then be recovered as the sum of the moments multiplied by the appropriate basis functions. The asymptotic result is achieved by tuning the parameters which define the meridional scales of these functions. The zeroth moment will describe the meridional displacement of the jet. The next order describes the modification of this structure by the meridional shear.

The overall form of the moment expansion depends on the choice of envelope function, the purpose of which is to ensure that the integrals are well behaved and dominated by the structure near the front. A Gaussian envelope achieves this, and has the additional feature that its Fourier transform is a Gaussian. The latter property is useful since the Green’s functions are conveniently expressed in terms of their Fourier transforms. It is then natural to use Hermite polynomials to define the moments, since these satisfy an orthogonality relation when integrated with Gaussian weighting.

This form of expansion has been used by Simmons (1974) to look at baroclinic instability on broad jets, and also by Juckes (1995) to look at the instability of surface and tropopause shear lines bounded by sharp fronts. The fact that the same mathematical representation is well suited to both broad and narrow fronts, leads to a solution which is asymptotic in both limits $r \to 0$ (below) and $r \to \infty$; it also provides a good approximation over the entire range of $r$ values.

The Hermite polynomials, $\mathcal{H}^{(m)}$, satisfy the orthogonality relation:

$$
\int_{-\infty}^{\infty} e^{-s^2/2} \mathcal{H}^{(m)}(s) \mathcal{H}^{(n)}(s) ds = \delta_{mn} n! \sqrt{2\pi},
$$

(30)

and can be evaluated from the recurrence relation:

$$
\mathcal{H}^{(n+1)}(s) = s \mathcal{H}^{(n)}(s) - n \mathcal{H}^{(n-1)}(s), \quad n \geq 1
$$

(31a)

with

$$
\mathcal{H}^{(0)}(s) = 1, \quad \mathcal{H}^{(1)}(s) = s
$$

(31b)

(e.g. Hochstrasser 1965). The definition is scaled differently here, so that the exponent in (30) is $-s^2/2$ instead of $-s^2$; this makes the algebra below slightly clearer.

The meridional displacement, $\tilde{n}_\mu(k; y, t)$, is represented as a sum of coefficients $\tilde{n}_\mu^{(m)}(k, t)$ multiplying basis functions:

$$
\tilde{n}_\mu(k; y, t) = \sum_{m=1}^{\infty} \tilde{n}_\mu^{(m)}(k, t) P_\mu^{(m)}(y),
$$

(32a)

$$
P_\mu^{(m)}(y) = \frac{1}{m!} \mathcal{H}^{(m)} \left( \frac{y}{R_\mu} \right) \exp \left( -\frac{y^2}{2b_\mu^2} \right),
$$

(32b)
where \( R_\mu \) and \( b_\mu \) are constants defining the scale of the expansion polynomials. These constants could, in general, take different values at the tropopause and the ground, but because of the symmetry of the basic state considered here they take on the same value for both \( \mu = 0 \) and \( \mu = 1 \). The scales \( a, b, R \) and functions \( P^{(m)} \) will henceforth be given without subscripts. It follows from (30) that the functions

\[
P^{(m)}(y) = \frac{1}{R \sqrt{2\pi}} \tilde{g}^{(m)} \left( \frac{y}{R} \right) \exp \left( -\frac{y^2}{2a^2} \right) \quad \text{with} \quad \frac{1}{R^2} = \frac{1}{a^2} + \frac{1}{b^2},
\]

are the adjoints to the basis functions, in that they satisfy:

\[
\int_{-\infty}^{\infty} P^{(m)}(y) P^{(n)}(y) dy = \delta_{mn}.
\]

Consequently, the coefficients in (32) are given by:

\[
\tilde{\eta}^{(m)}_{\mu}(k, t) = \int_{-\infty}^{\infty} P^{(m)}(y) \tilde{\eta}^{(n)}_{\mu}(y, t) dy.
\]

There are thus two independent constants determining the scales of the polynomials: an external scale \( b \), giving the meridional scale of the Gaussian envelope of the basis functions, and an internal scale \( R \). The first plays a secondary role in the small \( r \) limit because the external scale of the disturbance is much greater than that of the basic state. This is true provided the baroclinicity in the front is sufficiently localized, as will be made precise in Part II.

Taking moments of the evolution equation (28) gives:

\[
\frac{\partial \tilde{\eta}^{(m)}_{\mu}}{\partial t} = ik_\ell \sum_{n=0,\infty}^{\infty} M^{(mn)}_{\mu \nu} \tilde{\eta}^{(n)}_{\nu},
\]

where the interaction matrix \( M^{(mn)}_{\mu \nu} \) describes the influence of moment \( n \), level \( \nu \) on the evolution of moment \( m \), level \( \mu \). Note that \( M \) is a function of \( k \) but the tilde is omitted in this case to simplify the notation, given that the geostrophic space form of \( M \) will not be needed. Using Fourier and geometric representations in the meridional direction, the elements of this matrix are given, respectively, by:

\[
M^{(mn)}_{\mu \nu}(k) = 4\pi^2 \int_{-\infty}^{\infty} \left\{ \tilde{G}_{\mu \nu}(k, l) \tilde{A}^{(mn)}(k, l) - \delta_{\mu \nu} \sum_{\nu'} \tilde{G}_{\mu \nu'}(0, l) \tilde{B}^{(mn)}(k, l) \right\} dl
\]

\[
= 2\pi \int_{-\infty}^{\infty} \left\{ \tilde{G}_{\mu \nu}(k, y) \tilde{A}^{(mn)}(k, y) - \delta_{\mu \nu} \sum_{\nu'} \tilde{G}_{\mu \nu'}(0, y) \tilde{B}^{(mn)}(k, y) \right\} dy,
\]

where

\[
\tilde{A}^{(mn)} = \frac{2\pi}{\theta_\ell} \tilde{P}^{(m)} \left( \tilde{P}^{(n)} \circ \tilde{A} \right)
\]

represents the eddy meridional velocity, and

\[
\tilde{B}^{(mn)} = \frac{2\pi}{\theta_\ell} \left( \tilde{P}^{(m)} \circ \tilde{P}^{(n)} \right) \tilde{A}
\]
represents the advection by the zonal-mean zonal flow. An algorithm for evaluating \( \hat{A} \) and \( \hat{B} \) is given in appendix B. \( \delta_{\mu\nu} \) is the Kronecker delta—unity for \( \mu = \nu \) and zero otherwise; this term represents the advection of the disturbance by the basic-state wind.

The representation (36a) for the interaction matrix is convenient numerically, because of the relatively simple form of the Green’s functions in Fourier space, and is also advantageous in the broad-jet limit because the integrals are then dominated by the contribution from \( l \ll 1 \), making it relatively easy to obtain an analytic estimate. The representation (36b) is convenient in the narrow-jet limit when the integrals are dominated by contributions from small \( y \).

The infinite sum over basis functions in (35) can be truncated to give a finite representation of the system which can be solved numerically. This system is no simpler than a standard finite-element discretization but does, however, have the advantage that a very small number of terms are sufficient to give a good approximation to the full solution. In fact, truncating to a single coefficient, the zeroth moment, still retains the structure of the instability. However, it is important to choose the length-scales \( a, b \) and \( R \) correctly; this is discussed below.

The main disadvantage of the moment expansion as compared with finite-element methods is that it is not convergent. If the number of terms is increased the error decreases at first, but will eventually increase and overwhelm the solution. For this reason results from a standard finite-difference solution, described in more detail in part II, will be used to evaluate the accuracy of solutions obtained with the zeroth-moment approximation.

4. Asymptotic analysis for \( r \to 0 \).

In narrow fronts the surface potential-temperature anomaly will be concentrated near the front. The stream function, and also the displacement fields, will then have a meridional-scale of the order of the Rossby radius, determined by the Green’s function. This means that the meridional-scale \( b \) will take an \( O(1) \) value. This represents an external-scale, the scale over which anomaly fields decay away from the front. There is also an internal-scale \( R \), which describes the scale of oscillations of the basis functions within the front. This scale does not appear explicitly in the equations when they are truncated at the zeroth moment, but choosing the correct value is essential if the truncation is to give a good approximation. The projection-scale \( a \) (see 33), which does appear in the zeroth-moment equations, is related to the internal scale \( R \) by the orthogonality condition on the basis functions. The asymptotic solution is obtained by choosing these scales such that the interaction between the zeroth moment and higher moments vanishes in the limit \( r \to 0 \) (e.g. Hinch 1991). The leading-order solution can then be obtained from the zeroth moment alone.

As the frontal width tends to zero the integrals which determine the coefficients of the interaction matrix are dominated by small \( y \) or large \( l \), in the geostrophic and Fourier space representations, respectively. At large values of \( l \), corresponding to the small scales, the Green’s function is given by \( \overline{G}_{00}(k, l) = -\frac{1}{4\pi^2 l^3} + \mathcal{O}(l^{-4}) \). This slow decay at large wave number represents the logarithmic singularity which occurs at \( y = 0 \) in the half-transformed Green’s function \( \overline{G}_{00}(k, y) \) (and a \( |x|^{-1} \) singularity in the geostrophic space Green’s function \( G_{00}(x, y) \)). It is significant that the singularity is independent of \( k \); this allows a cancellation to take place between the singularity in the basic state \( (k = 0) \) and that in the disturbance, such that the evolution is determined by regular components of the
flow. At large $l$, substituting the results of appendix B into (37) and assuming $r \ll b$:

$$
\hat{A}^{(02)} \approx \frac{-r^4l^2(a^2 + b^2)}{4\pi^2a^2b^2} \exp \left\{ -\frac{l^2(a^2 + r^2)}{2} \right\},$

$$
\hat{B}^{(02)} \approx \frac{-r^2l^2}{4\pi^2} \exp \left\{ -\frac{l^2(R^2 + r^2)}{2} \right\}.
$$

If $R$ is given a finite value as $r \to 0$, then the matrix elements are of order unity due to the contribution arising from $\hat{B}^{(02)}$ at order unity wave numbers. This represents the tendency of the mean flow to distort the disturbance so that it projects onto the second moment. Letting $R \to 0$ makes $\hat{B}$ smaller, but also makes the Gaussian function broader. If $r \ll R$ then the dominant term is of the form:

$$
\int \frac{R^2}{2} \exp \left( -\frac{l^2R^2}{2} \right) dl,
$$

which is independent of $R$. At small $R$ the integrand attains its maximum at wave numbers of order $R^{-1}$. Similarly, if $R \ll r$ an order unity contribution is given by the large $l$ range of the ‘$A$’ integral. The latter integral represents the tendency of the meridional structure in $v$ to generate higher moments in the displacement field. In other words, if $r \ll R$ the basic-state shear couples the zeroth moment to higher moments, and if $r \gg R$ the eddy meridional velocity leads to similar coupling. If $R$ and $r$ are of the same order of magnitude both terms are significant. Taking

$$
a = r,
$$

and hence

$$
\frac{1}{R^2} = \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{r^2} + O(1),
$$

(39) gives:

$$
\hat{A}^{(02)} = \hat{B}^{(02)} + O \left\{ r^2 \exp(-r^2l^2) \right\}, \quad \text{as } r \to 0 \text{ with } rl = O(1).
$$

This identity, together with the fact that $\hat{G}(k, l) \approx \hat{G}(0, l)$ at large $l$, implies that the leading-order large $l$ terms of the integrand in (36) cancel. The integral is then uniformly convergent and the integrand has amplitude $O(r^2)$; hence

$$
M^{(02)}_{\mu
u} = O(r^2).
$$

(40)

The scale of the Hermite polynomials is given by $R$, so (39) implies that the scale of non-zero moments is given by that of the front. The zeroth moment, however, has a meridional-scale of the order of the Rossby radius. The decoupling between the various moments in the limit $r \to 0$ expressed by (40) can be seen as a consequence of the scale-separation between the zeroth and higher moments.

From (39) and the results of appendix B it follows that:

$$
\hat{A}^{(00)} = \hat{B}^{(00)} = \frac{1}{\pi} \exp \left( -l^2r^2 \right) + O(1), \quad \text{as } r \to 0.
$$

(41)

The equality of $A^{(00)}$ and $B^{(00)}$ implies that the self-interaction coefficients $M^{(00)}_{00}$ and $M^{(00)}_{11}$ are determined, in (36), by a weighted integral of the difference between the Green’s function at wave number $k$ and at wave number $0$. This quantity is $O(l^{-3})$ as $l \to \infty$. It follows that the integrals in (36) are uniformly convergent as $r \to 0$, and are determined by
values of the Green's function at order unity wave numbers. This means that the dynamics of the flow are dominated by scales of the order of the Rossby radius. That is, the mean displacement remains a large-scale flow structure despite the appearance of a singularity at the front. The higher moments, however, are influenced by the scale of the front.

As in Juckes (1995) the growth rate can be evaluated by truncating the moment expansion. Inspection of the interaction matrices shows that the coefficients of the zeroth-order matrix have a correction of $O(r^2)$ relative to the $r = 0$ solution, and that the $n$th moment brings a correction of $r^{2n}$. The first moment vanishes identically because of the symmetry of the problem, and the second moment gives an $O(r^4)$ correction which can be neglected, so the leading-order result is obtained simply from the zeroth-order interaction matrix. The coefficients of this matrix are given by integrals of the Green's functions weighted by the Gaussian functions $A$ and $B$ (see (36)). The evolution can then be written:

$$\frac{\partial}{\partial t} \left( \begin{array}{c}
\eta_0^{(0)} \\
\eta_1^{(0)}
\end{array} \right) = -i\theta_t \left( \begin{array}{cc}
D & E \\
-E & -D
\end{array} \right) \left( \begin{array}{c}
\eta_0^{(0)} \\
\eta_1^{(0)}
\end{array} \right) + \mathcal{O}(r^4), \quad (42)$$

where

$$D = kM_{00}^{(0)} \quad \text{and} \quad E = kM_{01}^{(0)}.$$

Using the method set out in appendix C, section (a), to evaluate (36) in the limit $r \to 0$ we find:

$$D(k, r) = -\frac{2k}{\pi} \left[ \mathcal{K}_0(k; 0) - \gamma - \ln(k/\pi) - r^2 \left\{ \frac{k^2}{2} \left( \ln \frac{kr}{\sqrt{2}} + \gamma + I_1 \right) - \mathcal{K}_0''(k; 0) + \frac{\pi^2}{6} \right\} \right] + \mathcal{O}(r^4), \quad (43a)$$

$$E(k, r) = \frac{2k}{\pi} \left[ \mathcal{K}_1(k; 0) + r^2 \mathcal{K}_1''(k; 0) \right] + \mathcal{O}(r^4), \quad (43b)$$

where $''$ denotes the second derivative with respect to $y$, $\gamma = 0.5772156649$ is Euler's constant:

$$I_1 = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{-y^2/2} \ln |y| dy = -0.6352,$$

and $\mathcal{K}_i$ are sums of modified Bessel functions.

As in the discussion of Green's functions, an alternative representation of these functions is possible. The sums in (43) are well conditioned at large $k$, but at small $k$ the convergence is slow. An alternative is derived from the vertical-mode form of the Green's functions. The calculation is outlined in appendix C, section (b). The results are:

$$D(k, r) = -1 + \frac{2k}{\pi} \left\{ \ln 2 - S_1 \left( \frac{k}{\pi} \right) \right\} + r^2 k \left\{ \frac{\pi}{2} - k + \frac{k^2}{\pi} \left( \ln \sqrt{2\pi} r + \frac{1}{2} + I_1 \right) - 2\pi S_2 \left( \frac{k}{\pi} \right) \right\} + \mathcal{O}(r^4), \quad (44a)$$

$$E(k, r) = 1 - \frac{2k}{\pi} S_{-1} \left( \frac{k}{\pi} \right) - r^2 k \left\{ \frac{\pi}{2} - k + 2\pi S_{-2} \left( \frac{k}{\pi} \right) \right\} + \mathcal{O}(r^4). \quad (44b)$$

The functions $S_p$, $p = -2, -1, 1, 2$ are infinite sums of algebraic functions, defined in appendix C. The expressions in (43) are, despite appearances, numerically identical to
those in (44), but the sums in the latter are well conditioned for $k/\pi \leq 1$, which includes the entire range of wave numbers of interest here.

The $r = 0$ terms can be traced back to various terms in the vertical expansion of the Green’s function (16b). The ‘1’ in both $D$ and $E$ comes from the barotropic term. In $D$ the contribution to the tendency of a boundary disturbance, due to the difference between the zonal and eddy advection associated with temperature gradients and anomalies on the same boundary, is represented by $S_1$; the contribution due to the advection by the zonal advection, associated with the temperature gradients on the opposing boundary, is represented by the term linear in $k$. In $E$ the sum $S_{-1}$ represents the contribution of eddy advection associated with temperature anomalies on the opposing boundary. In both sums the first term comes from the first baroclinic mode, and successive terms are associated with higher-order vertical structures.

Figure 4 shows $D$ and $E$ for $r = 0, 0.125$ and 0.25. There is not much sensitivity to $r$, but the effect is amplified when the eigenvalues are calculated. At larger values of $k$ the interaction between the two levels, represented by $E$, decays to zero, while the self-interaction term $D$ increases as $k \ln k$. This result is valid for $kr \ll 1$; the speed of disturbances localized on the front is represented by $D/k$, and this speed is in reality bounded by the maximum wind speed in the basic state.

The results up to this point are valid for general time-dependence of the two frontal waves. Attention will now be restricted to normal modes with exponential time-dependence $\exp(\sigma \theta f t)$, where $\sigma$ is the growth rate non-dimensionalized using $t_{sc}$, and normalized by $\theta f$. When this time-dependence is substituted into (42) one obtains the following eigenvalue problem:

$$
\begin{pmatrix}
D + i \sigma & E \\
-E & -D + i \sigma
\end{pmatrix}
\begin{pmatrix}
\eta_{0}^{(0)} \\
\eta_{1}^{(0)}
\end{pmatrix} = 0.
$$

The eigenvalue is given by:

$$
\sigma(k, r) = \sqrt{E^2 - D^2} = \sqrt{(E + D)(E - D)}.
$$

The dimensional growth rates are given simply by:

$$
\sigma^+ = \frac{\theta f}{\theta_{sc}} \sigma.
$$
That is, the growth rate depends on the ratio of meridional to vertical potential-temperature differences, the Coriolis parameter, and the factor $\sigma$.

The $r = 0$ solution is shown in Fig. 5(a), along with some approximations. The first is obtained by truncating the expansions for $D + E$ and $D - E$ in $k$ after two and three terms, respectively:

$$D + E = \left(\frac{k}{\pi}\right)^3 \left\{ \frac{7Z(3)}{4} - \frac{3Z(5)}{64} \left(\frac{k}{\pi}\right)^2 + \mathcal{O}\left(\frac{k}{\pi}\right)^7 \right\}$$  \hspace{2cm} (47a)$$

$$E - D = 2 - \frac{4k \ln 2}{\pi} - \frac{Z(3)}{4} \left(\frac{k}{\pi}\right)^3 + \mathcal{O}\left(\frac{k}{\pi}\right)^5,$$  \hspace{2cm} (47b)$$

where $Z(m) = \sum_{j=1}^{\infty} j^{-m}$, $m = 3, 5$, are constants ($Z$ is the Riemann zeta function). Equation (47) is not only accurate at small $k$ but also gives a good estimate of the cut-off wave
number, which is determined by the condition $E - D = 0$. The leading-order behaviour for small $k$ is given by:

$$
\sigma(k, r = 0) \sim \left(\frac{7Z(3)}{2\pi^3}\right)^{1/2} \left(1 - \frac{2\ln 2}{\pi} k\right)^{1/2} k^{3/2}.
$$

(48)

The growth rate decays as $k^{3/2}$ as $k \to 0$, compared with the linear approach to zero in the Eady model.

The eigenvectors are determined from (45) and have the form:

$$
\tilde{\eta}_0^{(0)} = \epsilon e^{i\phi/2}, \quad \tilde{\eta}_1^{(0)} = \epsilon e^{-i\phi/2},
$$

where

$$
\cos \phi = -\frac{D}{E}, \quad \sin \phi = \frac{\sigma}{E},
$$

(49)

and $\epsilon$ is a real constant. The approximation corresponding to (48) is

$$
\phi \rightarrow \left(\frac{7Z(3)}{2\pi^3}\right)^{1/2} \left(1 - \frac{2\ln 2}{\pi} k\right)^{-1/2} k^{3/2}.
$$

(50)

The dependence of $\phi$ on $k$ is illustrated in Fig. 5(b). For the approximation corresponding to (47) $D \approx -(1 - 2k \ln 2/\pi + k^3Z(3)/\pi^3)$ is used. At small $k$ the displacements are almost in phase, which means that the stream function anomalies oppose each other. At the cut-off wave number the stream function anomalies reinforce each other. The approximated forms of the interaction coefficients clearly reproduce the phase structure of the disturbance very well. The structure of the stream function will be discussed in more detail in part II.

Figure 6(a) shows the variation of the growth rates as functions of $r$ for a selection of wavelengths, verified against the finite-difference solutions described in part II. The resolution, domain size and precision with which the Green’s functions are calculated have all been varied, to ensure that the finite-difference results are accurate to three significant figures. The comparison shown in Fig. 6(a) verifies the accuracy of the asymptotic result to the order quoted, and shows that the amplitude of the error is small at frontal widths relevant to atmospheric flows (the asymptotic analysis guarantees that the error decreases in proportion to $r^4$, but does not predict the constant of proportionality). The largest errors occur for $k = 2$, which is close to the cut-off wave number. Consequently, small errors in the phase structure can generate large errors in the growth rate. Apart from $k = 2$ the results are good at $r = 0.25$ and the error is negligible at $r = 0.125$. At larger $r$ the asymptotic result fails catastrophically, predicting that the growth rates tend to zero when they actually remain relatively constant. Such sudden failure is by no means unusual in asymptotic analysis, since the method essentially uses a polynomial to extrapolate results from $r \ll 1$ to finite values of $r$, and extrapolation using polynomials is well known to be ill-conditioned. As noted earlier, the expansion method used here is also suitable for the analysis of the broad-jet limit, $r \to \infty$. Work in progress shows that the two asymptotic analyses can be combined to give a result which interpolates between the two limits, and is thus well conditioned.

Figure 6(b) shows the phase shifts corresponding to the growth rates in Fig. 6(a). Again the accuracy is good for $r \ll 0.25$, except near the cut-off wave number. The approximation breaks down abruptly at higher values of $r$.

The wave number of maximal instability ($k_m = 1.590$ and 1.630 at $r = 0$ and 0.25 respectively) and the short-wave cut-off ($k_c = 2.163$ and 2.266) do not differ greatly from
those of the Eady model \((k_m = 1.605 \text{ and } k_c = 2.399)\). The present results do not offer a clear explanation for the surprising insensitivity of these wave numbers to the meridional scale of the baroclinic zone. The dominance of the scales of the order of the Rossby radius in the integrals which determine the interaction coefficients is certainly a factor, but does not adequately explain why \(k_m\) is invariant to within 2%. An analysis of the broad-jet limit may help. The Eady growth rate is proportional to a gradient, whereas the growth rate here is proportional to a temperature difference. A comparison between the two must rely on a choice of a meridional length-scale which is to a certain extent arbitrary, but which can reasonably be chosen as the Rossby radius. With this choice the maximum growth rate for the Eady model (with unit temperature change over one Rossby radius) is \(\sigma_m = 0.310\) compared with the present results of \(\sigma_m = 0.360\) and 0.304 at \(r = 0\) and 0.25 respectively. Again, the differences are small.

Figure 6. (a) Normalized growth rates versus \(\log_2 r\). The lines show results of the asymptotic analysis and the markers show results from a finite-difference calculation, at wave number \(k = 0.5\) (stars and solid lines), 1.0 (circles and long dashes), 1.5 (\(\times\) and medium dashes) and 2.0 (+ and short dashes). (b) Corresponding phase shifts, \(\phi/\pi\). See text for further details.
5. DISCUSSION

The flow studied here is qualitatively different from the classic Eady model, in that the meridional temperature gradients, which provide the energy to drive the instability, are concentrated within a meridional-scale much smaller than the extent of the wave. Previous numerical studies (e.g. Davies et al. 1991) have shown, and this study confirms, that the essential dynamics are nevertheless the same as that for the classic Eady wave. The asymptotic analysis carried out here shows how the evolution of the wave becomes independent of the frontal width when scale-separation leads to decoupling between the two. The wave number of the most unstable mode is surprisingly insensitive to the structure of the front (see the last paragraph of section 4).

Comparison with finite-difference solutions shows that the analytic solution derived here is accurate for \( r \leq 0.25 \), corresponding to a frontal width in geostrophic coordinates of 200 km (taking the tropopause height to be 8 km). Here the equations of motion have been reduced to a set for the frontal displacement, so the solution is obtained without explicit reference to the full three-dimensional structure of the disturbance. This structure will be discussed in more detail in Part II.

The fact that the difference between the semi-geostrophic solution and the primitive-equation solution (Snyder 1995) does not diverge as frontal collapse is approached, but rather remains of the order 10\%, could be related to the fact that the evolution is determined by the larger scales having Rossby number of the order 0.1 (Snyder used the full semi-geostrophic system of Hoskins (1975) which is slightly more accurate than that used here). Further analysis of the higher-order terms in the Rossby number expansion would be required to make this hypothesis more specific.

The numerical and analytic solutions taken together show convincingly that reducing the jet width does not lead to a stable state of the form suggested by Lindzen (1993). That suggestion was based on asymptotic results for a baroclinic zone with width tending to infinity, such that the problem approaches the Eady model (Ioannou and Lindzen 1986). The results showed that finite width reduced the growth rates relative to the Eady model, a result which could be attributed to disruption of the normal mode by meridional shear. It is shown here that this effect does not increase monotonically as the width is decreased. Ioannou and Lindzen (1990) showed that their asymptotic results give reasonably accurate growth rates for widths down to one Rossby radius. The present results apply for widths up to one quarter of this value, so there is no contradiction. It should be noted that the expansion parameter used by Ioannou and Lindzen is the width of the basic-state jet. In the broad-jet limit this is close to the width of the surface temperature gradients. In the narrow-jet limit, however, the two are distinct. This, together with the fact that the broad-jet growth rates scale with the baroclinicity whereas the frontal growth rates scale with the temperature change, needs to be borne in mind when comparing the two sets of results. Work in progress, extending the Hermite polynomial expansion method to look at the broad-jet limit, following Simmons (1974), will discuss the transition between the two limits in more detail.

APPENDIX A

Continuum approximation for asymptotic evaluation of infinite sums

In a number of instances the analysis produces infinite sums in which, for a particular parameter range, the terms vary slowly as the index is increased. In this case the sum can be approximated by a continuous integral which can, in turn, be evaluated analytically to give the leading order asymptotic behaviour.
Consider the sum given by $S_j(\kappa)$ defined in Table C.1 (appendix C). For large $\kappa$ this can be approximated by substituting $\tau = j/\kappa$ and then approximating the sum over $j$ as an integral over $\tau$. The first term in $s_j^{(j)}(\kappa)$ varies slowly with $\tau$ over each interval of width $\Delta \tau = 1/\kappa$, but the second term has large variations when $j = O(1)$. To deal with this we first replace this term using the identity $\lim_{N \to \infty} \left( \sum_{j=1}^N j^{-1} - \ln N \right) = \gamma$, giving:

$$S_1(\kappa) = \lim_{N \to \infty} \left( \frac{1}{\kappa} \sum_{j=1}^N \frac{1}{\sqrt{1 + j^2 \kappa^{-2}}} - \ln N \right) - \gamma.$$ 

The sum can now be approximated by an integral, with an error of order $\Delta \tau^2 = \kappa^{-2}$. Using the identity:

$$\int_{1/(2\kappa)}^{N/\kappa} \frac{1}{1 + \tau} d\tau = \ln \frac{N}{\kappa} + O(\kappa^{-1}), \quad \text{for} \quad N \gg \kappa$$

gives:

$$S_1(\kappa) = \lim_{N \to \infty} \int_{1/(2\kappa)}^{N/\kappa} \left( \frac{1}{\sqrt{1 + \tau^2}} - \frac{1}{1 + \tau} \right) d\tau - \ln \kappa - \gamma + O(\kappa^{-1}).$$

Since the integral is regular at both upper and lower limits, we can extend the limits to 0 and $\infty$ without increasing the order of the error term. The integral can then be evaluated (using the substitution $\tau = \sinh \theta$) as $\ln 2$. The final result is:

$$S_1(\kappa) = -\ln \frac{\kappa}{2} - \gamma + O(\kappa^{-1}), \quad \kappa \to \infty.$$ 

**APPENDIX B**

**Evaluation of $A$ and $B$**

The required transforms of polynomials multiplying Gaussians can be evaluated using the following two identities:

$$\frac{1}{(2\pi)^{1/2} r} \int_{-\infty}^{\infty} e^{ily} e^{-y^2/(2r^2)} dy = e^{-l^2 r^2/2}$$

(B.1)

and

$$\frac{1}{(2\pi)^{1/2} r} \int_{-\infty}^{\infty} e^{ily} y^2 e^{-y^2/(2r^2)} dy = r^2 (1 - l^2 r^2) e^{-l^2 r^2/2}.$$ 

(B.2)

The first can be evaluated by changing the integration variable to $y - ilr^2$, and the second by using integration by parts to reduce it to a multiple of the first.

It follows that the transform of the meridional temperature gradient is given by:

$$\tilde{\nabla}(\theta) = \frac{\theta_l}{\pi} e^{-l^2 r^2/2}.$$ 

Specific values for the zeroth moment and its interaction with the second moment can be evaluated using (B.1) and (B.2), replacing $r$ with the different Gaussian widths as appropriate. The results are displayed in Table B.1.

Recurrence relations can also be derived to calculate higher-order interaction coefficients. The eigenvalues obtained using a number of moments, instead of the zeroth-moment truncation described in the text, converge rapidly, giving an independent check on the finite-difference solutions, but the meridional structure is less robust. For this reason only the leading-order terms in the asymptotic limit are discussed in the text.
TABLE B.1. Specific values for the zeroth moment \((n = 0)\) and its interaction with the second moment \((n = 2)\)

<table>
<thead>
<tr>
<th></th>
<th>(n = 0)</th>
<th>((n = 2)/(n = 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{F}[\rho^{(n)} A])</td>
<td>(\frac{\theta_f}{\pi} \frac{b}{\sqrt{b^2 + r^2}} \exp\left(-\frac{b^2 r^2}{2(b^2 + r^2)}\right))</td>
<td>(\frac{\theta_f}{\pi} \frac{b^2}{2b^2} \left(\frac{b^2}{2b^2} - \frac{b^2}{2(b^2 + r^2)}\right))</td>
</tr>
<tr>
<td>(\mathcal{F}[\rho^{(n)}])</td>
<td>(\frac{\theta_f}{\pi} \frac{b}{\sqrt{b^2 + r^2}} \exp\left(-\frac{b^2 r^2}{2b^2}\right))</td>
<td>(\frac{\theta_f}{\pi} \frac{b^2}{2b^2} \left(1 - (a^2 + b^2)t^2\right))</td>
</tr>
<tr>
<td>(\mathcal{F}[\rho^{(n)}] F^{(n)})</td>
<td>(\frac{\theta_f}{\pi} \frac{b}{\sqrt{b^2 + r^2}} \exp\left(-\frac{b^2 r^2}{2b^2}\right))</td>
<td>(-\frac{\theta_f}{\pi} \frac{b^2}{2b^2} t^2)</td>
</tr>
</tbody>
</table>

The first column lists the quantities evaluated, the second column gives the values for \(n = 0\) and the last column gives the ratio of the \(n = 2\) value to the \(n = 0\) value.

APPENDIX C

Interaction matrix elements as \(r \to 0\)

(a) From the modified Bessel function representation. As with the Green's functions discussed in section 2, there are two possible representations of the interaction coefficients. The method given in this subsection is the most straightforward, but the result is a series with slow convergence, especially at small wave numbers. A result which is harder to derive but easier to evaluate is given in the following subsection.

From (41), the weighting functions \(A\) and \(B\) are given by:

\[
\tilde{A}^{(00)} = \tilde{B}^{(00)} = \frac{\theta_f}{\sqrt{\pi} r} \exp\left(-\frac{y^2}{4r^2}\right) + \mathcal{O}(r^4).
\]  

(C.1)

Now consider the expansion of the Green's functions about the singularity at \(y = 0\): the general form of the function for small \(y\) is given by:

\[
F(y) = F^{[0]} + F^{*[0]} \ln |y| + \frac{1}{2} y^2 \left(F^{[2]} + F^{*[2]} \ln |y|\right) + \mathcal{O}(y^4 \ln |y|),
\]  

(C.2)

for constants \(F^{[n]}\) and \(F^{*[n]}\). The interaction coefficients are given by the convolution of the weighting functions in (C.1) with expansions of the form (C.2), which can be evaluated as follows:

\[
\frac{\theta_f}{\sqrt{\pi} r} \int_{-\infty}^{\infty} F(y') \exp\left(-\frac{y'^2}{4r^2}\right) dy' = 2\theta_f \left\{ F_0 + \left(\ln \sqrt{2}r + I_1\right) F^{*[0]} + r^2 F^{[2]} + r^2 \left(\ln \sqrt{2}r + 1 + I_1\right) F^{*[2]} + \mathcal{O}(r^4 \ln |r|)\right\}.
\]  

(C.3)

The \(\ln |y|\) terms in (C.2) generate \(\ln r\) terms in (C.3). \(I_1\) is a constant (see appendix D).

The zonal-mean components of the Green's functions occur only in a combination representing the total mean wind:

\[
\tilde{G}_{00}(0; y) + \tilde{G}_{01}(0; y) = \frac{1}{4\pi^2} \ln \left(\frac{\cosh \pi y - 1}{\cosh \pi y + 1}\right) = \frac{1}{2\pi^2} \ln \left(\frac{\pi |y|}{2}\right) - \frac{y^2}{24} + \mathcal{O}(y^4).
\]  

(C.4)
The Green’s functions at wave number \( k \) can be written as a sum of the singular component arising from the boundary singularity and a regular component arising from the image sources:

\[
\begin{align*}
\tilde{G}_{00}(k; y) &= \tilde{G}_{11}(k; y) = -\frac{1}{2\pi^2} \left\{ K_0(k|y|) + \mathcal{F} K_0 \right\}, \\
\tilde{G}_{10}(k; y) &= \tilde{G}_{01}(k; y) = -\frac{1}{2\pi^2} \mathcal{F} K_1,
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{F} K_0(k; y) &= 2 \sum_{h=1}^{\infty} K_0 \left\{ k \left( y^2 + 4h^2 \right)^{1/2} \right\}, \\
\mathcal{F} K_1(k; y) &= -2 \sum_{h=1}^{\infty} K_0 \left\{ k \left( y^2 + (2h - 1)^2 \right)^{1/2} \right\}.
\end{align*}
\]

Combining (C.4) and (C.5a), using the expansion for the modified Bessel function given in (18), gives:

\[
2\pi^2 [\tilde{G}_{00}(k; y) - (\tilde{G}_{00}(0; y) + \tilde{G}_{01}(0; y))] \approx \\
\gamma + \ln \frac{k}{\pi} - \mathcal{F} K_0(k; 0) - \frac{y^2}{2} \left\{ \frac{k^2}{2} \left( 1 - \gamma - \ln \frac{k|y|}{2} \right) + \mathcal{F} K_0''(k; 0) - \frac{\pi^2}{6} \right\},
\]

where the prime denotes a derivative with respect to \( y \). Similarly, the Green’s function for the interaction between the two levels is:

\[
2\pi^2 \tilde{G}_{10}(k; y) \approx \mathcal{F} K_1(k; 0) + \frac{y^2}{2} \mathcal{F} K_1''(k; 0).
\]

The asymptotic forms, (43a) and (43b), for the components of the zeroth-moment interaction matrix are obtained by identifying the expansions in (C.7) and (C.8), respectively, with (C.2) and using (C.3).

(b) From the vertical mode expansion. The calculations of the previous subsection give the correct result, but the sums are in some ways inconvenient. In particular, at small \( k \) the modified Bessel functions have a logarithmic singularity and it is far from clear what the result of adding a large number of such terms will be. The continuum approximation of appendix A can be used, with the help of the identity \( \int_0^\infty K_0(x)dx = 2 \), to show that \( \mathcal{F} K_0 \to 1/k \). This subsection takes a different route, producing a representation which is convergent at all values of \( k \) and asymptotic for \( k \to 0 \).

The derivation starts from equation (17b) and (19). The main difficulty arises from the fact that the second derivatives of the expression given in (17b) are unbounded at \( y = 0 \). This is related to the logarithmic singularity at \( y = 0 \). We can use the identity between (20) and (19) to isolate this singularity. The combination:

\[
\tilde{G}_{\text{reg}} \overset{\text{def}}{=} \tilde{G}_{00}(k; y) - \tilde{G}_{00}(0; y) \left( 1 + \frac{k^2y^2}{4} \right),
\]

has no logarithmic singularities in the first and second order. Obtaining a small \( y \) expansion is still non-trivial, because the sums which define the Green’s functions become divergent.
TABLE C.1. Definition of functions $S_p(\kappa) = \sum_{j=1}^{\infty} s_p^{(j)}(\kappa), \quad p = -2, -1, 1, 2$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$s_p^{(j)}(\kappa)$</th>
<th>$S_p(\kappa)$ for $\kappa \to 0, \infty$.</th>
</tr>
</thead>
</table>
| 1   | $\frac{1}{\sqrt{j^2 + \kappa^2}} - \frac{1}{j}$ | $\left\{ -\frac{\gamma}{\kappa} Z(3) + \frac{3 \gamma^4}{\kappa^4} Z(5) + \mathcal{O}(\kappa^6),
- \ln \frac{\kappa}{\gamma} - \frac{1}{\kappa} + \mathcal{O}(\exp(-2\pi\kappa)). \right\}$ |
| -1  | $\frac{(-1)^{j+1}}{\sqrt{j^2 + \kappa^2}}$ | $\left\{ \ln 2 - \frac{3 \gamma^2}{\kappa} Z(3) + \frac{4 \gamma^5}{\kappa^4} Z(5) + \mathcal{O}(\kappa^6),
\frac{1}{\kappa} + \mathcal{O}(\exp(-\pi\kappa)) \right\}$. |
| 2   | $\sqrt{j^2 + \kappa^2} - j - \frac{\kappa^2}{2j}$ | $\left\{ -\frac{\gamma^4}{\kappa^2} Z(3) + \mathcal{O}(\kappa^6),
\frac{\gamma^2}{4} \left( -\ln \frac{\kappa}{\gamma} + \frac{1}{2} - \gamma \right) - \frac{\gamma}{2} + \frac{\gamma^2}{12} + \mathcal{O}(\exp(-2\pi\kappa)) \right\}$. |
| -2  | $(-1)^{j+1} \left[ \sqrt{j^2 + \kappa^2} - j \right]$ | $\left\{ \frac{\kappa^2 \ln 2}{\pi} - \frac{\gamma^4}{\kappa^2} Z(3) + \mathcal{O}(\kappa^6),
\frac{\kappa}{2} - \frac{\gamma}{\kappa} + \mathcal{O}(\exp(-\pi\kappa)) \right\}$. |

The last column gives asymptotic expansions for small (above) and large (below) arguments. The first is obtained by a Taylor expansion of each term, the second by comparison with the result obtained from the modified Bessel function representation of the Green's functions. The latter is easily evaluated at large $k$ because the contributions from the image sources are then exponentially small. The expansion about $k = 0$ converges for $|\kappa| < 1$. Since $\kappa = k/\pi$ and the range of unstable wave numbers is $(0, 2.5)$, it is the small $\kappa$ expansion which is of most interest. $Z$ is the Riemann zeta function (appendix D).

In this limit, from (16b):

$$\tilde{G}_{reg}(k; y) = -\frac{1}{4\pi} \frac{\exp(-k|y|)}{k} \left( 1 - \frac{\kappa^2 y^2}{4} \right) \exp(-|y|j\pi)$$

(C.10)

The leading order can be obtained by simply setting $y = 0$, giving:

$$\tilde{G}_{reg}(k; 0) = -\frac{1}{4\pi} \left\{ \frac{1}{k} + \frac{2}{\pi} S_1 \left( \frac{k}{\pi} \right) \right\},$$

for $S_1$ as defined in Table C.1. Since the series (C.10) is uniformly convergent for $y \neq 0$, we can take the derivative term by term to find:

$$\tilde{G}_{reg}'(k; y) = -\frac{k}{4\pi} \exp(-k|y|)$$

(C.11a)

$$-\frac{1}{2\pi} \sum_{j=1}^{\infty} \left\{ \left( j^2 \pi^2 + \frac{k^2}{2j\pi} \right) \exp(-|y|j\pi) \right\}$$

(C.11b)

$$+ \sqrt{j^2 \pi^2 + k^2} \left\{ \exp(-|y|\sqrt{j^2 \pi^2 + k^2}) - \exp(-|y|j\pi) \right\}$$

(C.11c)

$$- \frac{j\pi k^2 y^2}{4} \exp(-|y|j\pi) + k^2 y \exp(-|y|j\pi)$$

(C.11d)

The first term in the summation is uniformly convergent in the limit $y \to 0$; the others require closer attention. Each term individually tends to zero in this limit, but in such a way that the sum of an infinite number of terms remains finite. The sum can be split into
a finite series \( 1 \leq j < N \), which is regular as \( y \to 0 \), and a remainder, from \( N \to \infty \), with \( 1 \leq N \ll y^{-1} \). In the remainder the first term (C.11b) contributes \( \mathcal{O}(N^{-2}) \), the second term (C.11c) can be approximated, using \( j \pi \gg k \), as:

\[
-\frac{k^2y}{2} \exp (-|y|j\pi) + \mathcal{O}(j^{-2}).
\]

This contribution to the remainder can now be summed using:

\[
\sum_{j=N}^{\infty} y \exp (-|y|j\pi) = y \frac{\exp (-|y|N\pi)}{1 - \exp (-|y|\pi)} = \frac{1}{\pi} + \mathcal{O}(y). \tag{C.12}
\]

The contribution to the remainder from the third term can be summed similarly, as follows:

\[
\sum_{j=N}^{\infty} y^2 j\pi \exp (-|y|j\pi) = -y^2 \frac{\partial}{\partial y} \sum_{j=N}^{\infty} \exp (-|y|j\pi) = -y^2 \frac{\partial}{\partial y} \frac{\exp (-|y|N\pi)}{1 - \exp (-|y|\pi)}
= \frac{1}{\pi} + \mathcal{O}(y).
\]

The contribution from the last term can also be summed using (C.12). Combining these results, taking the limit \( y \to 0 \) followed by the limit \( N \to \infty \), gives:

\[
\tilde{\mathcal{G}}_{\text{reg}}(k; 0) = -\frac{k}{4\pi} - \frac{1}{2} \left\{ S_2 \left( \frac{k}{\pi} \right) + \frac{k^2}{4\pi^2} \right\},
\]

where \( S_2 \), which arises from the regular component of the sum, (C.12b), is as defined in Table C.1.

Combining the above results into (C.9) gives

\[
\tilde{\mathcal{G}}_{\text{reg}} = -\frac{1}{4\pi} \left\{ \frac{1}{k} + \frac{ky^2}{2} + \frac{k^2y^2}{4\pi} + \frac{2}{\pi} S_1 \left( \frac{k}{\pi} \right) + \pi y^2 S_2 \left( \frac{k}{\pi} \right) \right\} + \mathcal{O}(y^4). \tag{C.13}
\]

The first two terms come from the barotropic component of the disturbance Green's function; the third term is the contribution from the remainder term. This non-uniform convergence is a consequence of the singularity at the front, so this term is related to the frontal singularity. The last two terms then represent regular contributions from the series of baroclinic terms. Using (20), we also have, to \( \mathcal{O}(y^4) \):

\[
\frac{k^2y^2}{4} \tilde{\mathcal{G}}_{00}(0; y) = \frac{k^2y^2}{8\pi^2} \ln \pi |y| \quad \text{and} \quad \tilde{\mathcal{G}}_{01}(0; y) = -\frac{1}{4\pi} \left( \frac{2 \ln 2}{\pi} + \frac{\pi y^2}{4} \right). \tag{C.14}
\]

Combining (C.13) and (C.14) to obtain an expansion for

\[
\tilde{\mathcal{G}}_{00}(k; y) - \{ \tilde{\mathcal{G}}_{00}(0; y) + \tilde{\mathcal{G}}_{01}(0; y) \} = \tilde{\mathcal{G}}_{\text{reg}}(k; 0) + \frac{1}{4} k^2y^2 \{ \tilde{\mathcal{G}}_{00}(0; y) - \tilde{\mathcal{G}}_{01}(0; y) \}
\]

and substituting into (C.3) gives (44a).

Similarly, the cross-interaction elements can be evaluated by first considering:

\[
\tilde{\mathcal{G}}_{01}(k; y) = \tilde{\mathcal{G}}_{01}(k; y) + \tilde{\mathcal{G}}_{01}(0; y) - \tilde{\mathcal{G}}_{01}(0; y)
= \frac{1}{4\pi} \left[ \frac{1}{k} + \frac{ky^2}{2} - \frac{2 \ln 2}{\pi} - \frac{\pi y^2}{4} - 2 \sum_{j=1}^{\infty} (-1)^{j+1} \left\{ \frac{1}{\sqrt{j^2\pi^2 + k^2}} - \frac{1}{j\pi} + \frac{y^2}{2} \left( \sqrt{j^2\pi^2 + k^2} - j\pi \right) \right\} \right] + \mathcal{O}(y^4). \tag{C.15}
\]
The first two terms come from the barotropic term of the eddy Green's function, the next two from the expansion of (20) about \( y = 0, z = 1 \). The sum contains the difference between eddy and zonally symmetric Green's functions. The latter is the sum of all the baroclinic terms evaluated at \( k = 0 \). In this case there is no contribution from the remainder term in the sum, because the alternation in the sign of the term, summed makes convergence faster than was the case in (C.12). Substituting (C.15) into (C.3) gives (44b).

The results of this subsection can also be obtained from the Fourier expression for the Green's function, (22), by using the method of contour integration in the complex \( l \)-plane. The marginally convergent sums encountered here are replaced by marginally convergent integrals.

**APPENDIX D**

**Glossary of terms**

**Definition of variables**

- \( A \): Surface of control volume \( V \).
- \( A_2 \): Surface formed by collapsing \( A \) onto the \( x, y \) plane.
- \( \mathcal{F} \): Fourier transform with respect to \( y \), Eq. (7).
- \( \mathcal{G} \): Matrix of Green's functions, Eq. (23).
- \( \mathcal{K}^{(m)} \): Hermite polynomial of order \( m \).
- \( \mathcal{O} \): Indicates an error term of magnitude bounded by a constant times the argument in the limit under consideration.
- \( \mathcal{S} \): Sums of modified Bessel functions, Eq. (C.6).
- \( \mathcal{V} \): Control volume used to analyze the Green's function in the vicinity of the singularity.
- \( A, B \): Weighting functions which enter into the calculation of the interaction matrix.
- \( D, E \): Elements of the interaction matrix for the zeroth moment.
- \( F(y) \): Arbitrary function used in appendix C.
- \( G \): Green's function.
- \( G_{reg} \): Regularized combination of Green's functions, Eq. (C.10).
- \( H_0 \): Density-scale height.
- \( I_1 \): Constant, \( I_1 = (1/\sqrt{2\pi}) \int_0^\infty e^{-y^2/2} \ln |y| dy = -0.6352 \).
- \( K_0 \): Modified Bessel function of zeroth order.
- \( M \): Interaction matrix relating displacements to their tendencies.
- \( N \): Brunt–Väisälä frequency, \( N^2 = (g/\theta_0)\Gamma \).
- \( p^{(m)}, p^{+(m)} \): Basis function and its adjoint.
- \( R \): Internal meridional-scale of moment expansion.
- \( S_p \): Sums of algebraic functions (Table 2).
- \( Z(m) \): Riemann Zeta function \( Z(m) = \sum_{j=1}^{\infty} j^{-m} \); \( Z(3) \approx 1.202, Z(5) \approx 1.037 \).
- \( a, b \): Projection-scale and external meridional-scale of moment expansion.
- \( f \): Coriolis parameter.
- \( g \): Acceleration due to gravity.
- \( h \): Index over images of boundary temperature anomaly.
- \( i \): \( i = \sqrt{-1} \).
- \( j \): Index of vertical modes.
- \( k, l \): Zonal and meridional wave number respectively.
- \( m, n \): Index of meridional moment expansion (\( m \) is also used in the definition of \( Z \)).
- \( n \): Unit outward normal to \( V \).
- \( p \): Subscript labelling various \( S \) functions (Table C.1).
- \( r \): Parameter determining the width of the front, Eq. (29).
- \( r_e \): Radius in \( y, z \) plane times wave number.
- \( s_p^{(j)} \): \( j \)th term in the sum defining \( S_p \), see Table C.1.
- \( t \): Time.
- \( \mathbf{u} = (u, v, w) \): Three-dimensional wind vector, see section 2 for definitions of various subscripts and superscripts.
- \( (x, y, z) \): Geostrophic coordinates.
- \( \Gamma \): Lapse rate \( \partial\theta/\partial z \).
- \( \Lambda \): Minus the zonal mean meridional gradient of boundary potential temperature.
\( \alpha \) Real constant used to label a family of basic states.

\( \beta \) Gradient of planetary vorticity.

\( \gamma \) Euler's constant, \( \gamma \approx 0.5772 \).

\( \delta(x) \) Dirac delta function: \( \delta(x) = 0, x \neq 0 \) and \( \int_{-\infty}^{\infty} \delta(x)dx = 1 \).

\( \delta_{\mu\nu} \) Kronecker delta: \( \delta_{\mu\nu} = 1 \) when \( \mu = \nu \) and zero otherwise.

\( \epsilon \) Amplitude of meridional displacements.

\( \eta \) Meridional displacements.

\( \theta \) Potential temperature.

\( \theta_l \) Half amplitude of the frontal-temperature transition.

\( \theta_0 \) Constant reference potential temperature.

\( k \) \( k/\pi \) occurs as argument of \( S \).

\( \lambda \) Horizontal wave number, \( \lambda^2 = k^2 + l^2 \).

\( \mu, \nu \) Indices for the two levels, 0 for surface and 1 for tropopause.

\( \rho \) Density.

\( \sigma \) Normal-mode growth rate.

\( \tau \) Dummy variable in appendix A.

\( \phi \) Phase shift between tropopause and surface wave.

\( \psi \) Geostrophic stream function.

Subscripts (the variables for which the definitions apply are given in brackets).

0, 1 Surface and tropopause values respectively (\( \eta, u, v, \theta, G, M \)). See also \( l_1, K_0 \) and \( S_p \).

geom Geometric (as opposed to geostrophic) coordinates (\( x, y \)).

c Cut-off wave number (\( k \)) or critical temperature gradient (\( A \)).

m Pertaining to maximum growth rate (\( k, \sigma \)).

s Static reference atmosphere (\( \theta \)).

sc Dimensional constant used to non-dimensionalize a variable (\( x, z, t, \theta \)).

t 'Template', i.e. a basic state which can be used to construct a family of such states (\( \overline{u}, \overline{\theta}, \eta, \theta \)).

src Position of source (\( x, y, z \)).

\( x, y, z \) Derivatives (\( \psi, \theta, u \)).

Superscripts (the variables for which the definitions apply are given in brackets).

\( m, m \) Of order \( m \) in the expansion given by Eq. (32) (\( \eta, P, \mathcal{H} \)) or pertaining to the interaction between order \( m \) and \( n \) (\( A, B, M \)).

\( [n], [\eta n] \) Coefficients of \( y^n/n! \) and \( (y^n/n!) \ln |y| \), respectively, in expansion of \( F(y) \).

\( \dagger \) Adjoint (\( P \)) or dimensional version of variables (\( x, y, z, u, v, \theta, \sigma \)).

\( / \) Derivative with respect to \( y \).

Mathematical notation

\( \nabla \) three-dimensional gradient operator.

\( \overline{O} \) zonal mean.

\( \mathcal{O} \) Horizontal Fourier transform, Eq. (6).

\( \mathcal{O} \) Along-front Fourier transform, Eq. (7).

\( \alpha, \overline{\alpha} \) Convolution with respect to \( y \) and \( l \) respectively.

REFERENCES


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