Baroclinic instability of semi-geostrophic fronts with uniform potential vorticity. II: Comparison of analytic and numerical solutions

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SUMMARY

The structure of an asymptotic analytic solution for linear baroclinic waves on a frontal basic state is analysed and compared with finite-difference solutions. The accuracy of the geostrophic fields is good for fronts with widths less than about 400 km. There is direct cancellation between along-front advection of the disturbance by the strong frontal jet, and disturbance cross-front advection of the sharp basic-state gradients. This cancellation of small-scale effects leaves the larger scales to control the evolution of the disturbance. It follows that the structure is very similar to that of the classic, meridionally uniform, Eady model. Secondary features, such as the ageostrophic circulation, are more sensitive to the frontal width and hence less accurately reproduced in the asymptotic solution. The analysis illustrates significant differences between the ageostrophic circulation associated with deformation frontogenesis and that which occurs in the linear stage of the instability, forced by along-front curvature. In the latter case, the horizontal ageostrophic wind associated with the rising motion is characterized by along-front convergence and cross-front divergence (i.e. ageostrophic frontolysis).

KEYWORDS: Eady model  Frontal dynamics  Baroclinic instability

1. INTRODUCTION

Part I of this study (Juckes 1998) derived an analytic solution for baroclinic waves on a frontal basic state. The problem is posed as a uniform potential-vorticity quasi-geostrophic flow in the formalism given by Hoskins and Draghici (1977). This paper will analyse the structure of these waves in more detail, and also compare them with the results of finite-difference calculations. For more discussion of the background and the notation the reader is referred to sections 1 and 2, and to appendix D, of part I.

The analytic solution given in part I exploits the fact that the evolution of an arbitrary disturbance in the uniform potential-vorticity model can be expressed in terms of the boundary potential-temperature distribution, and hence in terms of the displacement of potential-temperature contours on the boundaries. It follows that it is possible to determine the growth rates without considering the details of the three-dimensional temperature and vertical-velocity structures which develop within the baroclinic wave. In many applications, however, it is precisely these structures which are of interest and they will, accordingly, be discussed in detail below.

Sections 2 and 3 will describe the basic state and numerical model respectively. The comparison between numerical and analytic results is divided into two parts: section 4 deals with the geostrophic stream function and section 5 looks at the ageostrophic circulation. The latter reveals interesting aspects of the ageostrophic response to the frontal curvature induced by the growing baroclinic wave. These results, as usual in linear-instability studies, depend to a certain extent on the choice of basic state. There are two possible approaches to this choice: taking a realistic basic state from observations or taking a smooth analytic structure. Here the latter route has been taken, and the relevance of the results to more complex flow structures depends, inter alia, on the validity of the assumption that the small-scale details of the basic state are of secondary importance. Some support for this assumption is given in section 6, where it is shown that, for a specific class of fronts, the growth rate can be determined to a good approximation from the temperature contrast across the front and a frontal width defined by an integral.

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2. The basic state

The flow is prescribed by setting the meridional gradient in potential temperature on the boundaries to a Gaussian function, with width \( r \) and amplitude such that the change across the front is \( 2\theta_t \). The precise mathematical formulation and the non-dimensionalization of the variables is described in part I. Figure 1 shows examples of the basic states considered, in both physical and geostrophic (Hoskins 1975) space. The orientation corresponds to that used by Eady (1949), in which the basic-state wind is aligned east–west and the temperature gradients north–south, rather than that generally used in studies of frontal instability; this is because the original motivation was to compare the present flow with the Eady model. There is an easterly jet at the surface and a westerly jet at the tropopause. Although the flow structure is highly idealized it is by no means trivial. Part I showed that the analytic solution gave the growth rate for the flows in Fig. 1(a) and (b) to within 0.1% and 1% respectively. The aim of this paper is to study these solutions in more detail and to show that the method can be generalized to other frontal structures, albeit with the restriction to uniform potential vorticity.

The analysis is carried out in dimensionless variables. In creating dimensionless values from the non-dimensional results this paper will use the Coriolis parameter \( f = 10^{-4} \text{ s}^{-1} \), the Brunt–Väisälä frequency \( N = 10^{-2} \text{ s}^{-1} \), the height of the tropopause (here a solid lid) \( z_{sc} = 8 \text{ km} \), and a time-scale \( t_{sc} = 10^4 \text{ s} \). The corresponding temperature-scale is \( \theta_{sc} = \Gamma z_{sc} = (\theta_{00} N^2/g)z_{sc} = 24 \text{ K} \), where \( \Gamma \) is the lapse rate and \( \theta_{00} = 300 \text{ K} \) is a constant reference value. For any given temperature contrast, \( 2\theta_t^\dagger \) (the \( \dagger \) superscript indicates a dimensional variable), there is a minimum frontal width beneath which the geostrophic coordinate transform becomes singular. For this reason, the narrower front in Fig. 1 is shown with a weaker temperature contrast, \( \theta_t^\dagger = 6 \text{ K} \).

Figure 2 shows the critical value of the temperature contrast (\( \theta_{cv}^\dagger \)), defined as the value at which frontal collapse occurs, as a function of the frontal width in geostrophic coordinates. Empirically, the asymptotic value at small frontal widths is found to be

\[
\frac{2\theta_{cv}^\dagger}{r^\dagger} = 0.125 \text{ K km}^{-1} = 4.17 \frac{\theta_{sc}}{x_{sc}}.
\]

For instance, with a cross-frontal temperature contrast \( 2\theta_t^\dagger = 20 \text{ K} \) frontal collapse occurs at \( r^\dagger \approx 160 \text{ km} \), \( r \approx 0.2 \). The analytic solution gives accurate values (error less than 1%) of the growth rate for frontal widths up to \( r = 0.25 \), and reasonable values (error less than 10%) up to \( r = 0.4 \), and hence includes, for realistic temperature contrasts, a significant range of frontal widths above that at which collapse occurs.

3. Numerical solutions

Part I showed that the problem could be formulated in a single equation for the meridional displacements of material contours on the two boundaries, \( \tilde{\eta}_\mu (k; y, t) \), where \( \mu = 0, 1 \) for the surface \( (z = 0) \) and tropopause \( (z = 1) \) respectively, \( k \) is the zonal wave-number, \( y \) the meridional coordinate and \( t \) time. The tilde is used to denote the Fourier transform in the zonal direction. Each zonal wave number evolves separately; it remains to find the meridional structure, the phase shift between the upper and lower disturbances, and the growth rate or frequency of the disturbance. This can be cast as a one-dimensional eigenvalue problem (by taking equation (28) of part I, assuming exponential time dependence and discretizing the meridional coordinate) which can be solved numerically without great difficulty. This section briefly presents some numerical solutions, obtained with a
Figure 1. Zonal mean wind and potential temperature. Heavy contours show the zonal mean wind, contour interval 5 m s\(^{-1}\), negative contours dashed. Lighter contours show the potential temperature, contour interval 2 K. The axes are marked in non-dimensional units. (a) and (b) are plotted in geostrophic coordinates with \( r = 0.125 \) and 0.25 respectively, and corresponding frontal strengths \( 2\gamma_1 = 12 \) K and 24 K. (c) and (d) are as (a) and (b), but plotted in physical coordinates. See text for further details.
Figure 2. The critical temperature step $\theta_{cr}^c$ (K), at which frontal collapse occurs, as a function of frontal width $r_f$ (km). The shaded area shows parameter pairs which are physically realisable in the sense that the geostrophic coordinate transformation is not singular.

standard finite-difference procedure, which will be compared with the analytic solution obtained in part I. The discretization is as used in Juckes (1995) for a related problem, some details are given in appendix A.

The present analysis differs slightly from the previous works cited in that here there are no meridional boundaries. The disturbance is instead confined by the fact that the boundary potential-temperature gradients decay rapidly away from the front. The structure of the flow in regions without meridional temperature gradients can be described analytically, so grid points are only required where the gradients are significantly non-zero. This allows a cleaner comparison with the analytic results of Part I, but there is no suggestion that it is any more realistic than the earlier work. A realistic treatment of the flow at large distances from the front would have to include the $\beta$-effect at the very least, and spherical geometry.

Some results are given in Table 1, for a linear wave with non-dimensional amplitude unity. The most unstable wave number varies with frontal width, but is always within 1% of $k = 1.6$. The results in Table 1 and in following sections will be presented for $k = 1.6$; the difference from the fastest growing mode is negligible. Here, $\sigma$ is the growth rate non-dimensionalized and normalized by half the temperature change across the front. The parameter values defined above give a dimensional growth rate of $\sigma \approx 1.5 \times 10^{-5}$ s$^{-1}$, corresponding to a doubling time of about 13 hours.

The maximum perturbation velocity increases approximately as $r^{-1}$. It will be shown below that this is essentially a displacement of the jet in the basic state. The maximum values of the meridional heat flux, normalized by $(\epsilon^c \theta^c_f)^2$, also increase as $r^{-1}$. The distance over which the heat is transported, however, is of the order $r$. The product (heat flux)\times (transfer distance) approaches a constant. The meridional momentum fluxes increase more rapidly. In the meridionally uniform Eady model these fluxes vanish identically, and Table 1 shows that they are already reduced to an insignificant value at $r = 1$. The total eddy energy also shows an increase as $r$ is decreased, reflecting the increase in eddy velocities.
TABLE 1. RESULTS FOR A LINEAR WAVE WITH NON-DIMENSIONAL AMPLITUDE UNITS

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\sigma$</th>
<th>$u_{\text{max}}$</th>
<th>$\rho_{\text{max}}$</th>
<th>$\vec{u}^2/\rho_{\text{max}}$</th>
<th>$\vec{w}^2/\rho_{\text{max}}$</th>
<th>$E_{\text{tot}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>0.339</td>
<td>3.53</td>
<td>1.42</td>
<td>2.03</td>
<td>1.02</td>
<td>2.33</td>
</tr>
<tr>
<td>0.25</td>
<td>0.303</td>
<td>1.60</td>
<td>1.03</td>
<td>0.94</td>
<td>0.31</td>
<td>1.63</td>
</tr>
<tr>
<td>0.5</td>
<td>0.240</td>
<td>0.65</td>
<td>0.67</td>
<td>0.38</td>
<td>$3.79 \times 10^{-2}$</td>
<td>0.97</td>
</tr>
<tr>
<td>1.0</td>
<td>0.166</td>
<td>0.27</td>
<td>0.39</td>
<td>0.14</td>
<td>$1.08 \times 10^{-2}$</td>
<td>0.44</td>
</tr>
</tbody>
</table>

$\times 10^5$ m  $5 \times 10^{-5}$ s$^{-1}$  10 m s$^{-1}$  8 hPa  30 K m s$^{-1}$  100 m$^2$s$^{-2}$  100 m$^2$s$^{-2}$

The growth rate $\sigma$, maximum eddy zonal velocity $u_{\text{max}}$, maximum pressure anomaly $\rho_{\text{max}}$, maximum meridional eddy fluxes of heat $\vec{u}\rho_{\text{max}}$ and momentum $\vec{w}\rho_{\text{max}}$, and the total eddy energy $E_{\text{tot}}$, for the $k = 1.6$ normal mode at various frontal widths $r$. The quantities in columns 3 to 7 depend on the amplitude of the wave, the values given correspond to non-dimensional amplitude unity. The second from last row gives the factors which must multiply the non-dimensional values to obtain dimensional quantities. The last row gives the values of these factors for the scaling parameters $\zeta_m = 10^4$ s, $\zeta_y = 8 \times 10^3$ m and hence $\delta_0 = 24$ K, frontal strength $\delta_1 = 12$ K and meridional displacement amplitude $\epsilon = 5 \times 10^5$ m. At these parameter values the wave steepness is $\epsilon/\delta_0 = 0.25$, so the linear theory should be reasonably accurate.

The structure of the waves will be discussed in more detail below, with the help of the analytic results from part I.

4. THE STRUCTURE OF THE GEOSTROPHIC STREAM FUNCTION

Part I derived analytic expressions for the growth rate of baroclinic disturbances in the limit of frontal width tending to zero. The solution is expressed in terms of the displacement of material contours. This section looks at the three-dimensional structure of the geostrophic stream function and compares it with the results from the finite-difference model.

The boundary potential-temperature anomalies are given by minus the displacement multiplied by the basic-state gradients:

$$\theta_\mu = \eta_\mu \Lambda, \quad \text{where} \quad \Lambda = -\frac{\partial \overline{\theta}}{\partial y},$$

and $\mu = 0, 1$ for the ground and tropopause respectively. The meridional temperature gradients at the surface are taken here to be equal to those at the tropopause. In the analytic solution the displacement is approximated by the first moment $\eta_\mu(y) \approx \eta_\mu^{(0)}(y) P^{(0)}(y)$, where $P^{(0)}(y)$ is the structure function of the zeroth moment and has a Gaussian form. The width of this Gaussian is of the order of the deformation radius which is assumed to be much greater than the width of the front, so that the displacement can be treated as a constant, $\eta_\mu(y) \approx \eta_\mu^{(0)}$, over the range of $y$ for which there are significant boundary temperature gradients.

The geostrophic stream function is obtained by convolving of the potential temperature anomalies with the Green’s function:

$$\tilde{\psi}(k; y, z) = 2\pi \int_{-\infty}^{\infty} \left[ \tilde{G}_{00}(k; y - y', z; 0) \tilde{\eta}_0^{(0)} + \tilde{G}_{01}(k; y - y', z; 1) \tilde{\eta}_1^{(0)} \right] \Lambda(y') dy',$$

(1)
where $\tilde{G}(k; y - y', z; z')$ is the Green’s function describing the response at height $z$ to a boundary potential-temperature anomaly with structure $\exp(ikx)\delta(y')$ at height $z'$. The structure of $\tilde{G}$ was discussed in part I. The real part of the stream function is symmetric about $z = 1/2$. That is, it is made up of even vertical wave numbers $j$, of which the barotropic component, $j = 0$, dominates. The imaginary part contains the odd vertical wave numbers, antisymmetric about $z = 1/2$. This symmetry is a consequence of the symmetry of the basic state, from which it follows that the Fourier coefficient of the normal-mode stream function at the tropopause is the complex conjugate of that at the surface.

Figure 3 shows the real and imaginary parts of the stream function coefficient at $z = 0$ for $r = 0.25, 0.4$ and 0.5 and for wave numbers $k = 0.4$ and 1.6. The solid and dashed lines are derived from the analytic solution, the circles and stars show the corresponding results from the numerical solution. At $r = 0.25$ the agreement is excellent, with errors of the order 5% at $k = 1.6$ and considerably less at $k = 0.4$. When the frontal width is increased to $r = 0.4$ the errors in the asymptotic solution are significant, and at $r = 0.5$ the asymptotic solution, at least at $k = 1.6$, is no longer close to the true solution.

At $k = 1.6$, which is near the most unstable wave number, the phase shift between upper and lower waves is close to a quarter cycle. This means that, at $y = 0$, the even and odd components have similar amplitudes. As the frontal width is increased the peak amplitude decreases. In the finite-difference solution the relative phase shift between the upper and lower waves remains fairly constant, but the asymptotic solution has a spurious $C(r^4)$ drift which results in an error in the relative amplitudes of the two components plotted.

The structure at $k = 0.4$ is representative of the long-wave limit. In this limit the upper and lower displacements are almost in phase. As a result the stream function anomaly near the front is dominated by the odd (baroclinic) component. The meridional extent of this component varies little between $k = 0.4$ and $k = 1.6$; the even component, on the other hand, is much broader at the longer wavelength. At both zonal wave numbers the even component dominates at distances greater than one deformation radius ($\alpha N/f$) from the front.

A clearer picture of the dependence of the stream function on frontal width $r$ and wave number $k$ can be obtained by looking at the limits of large and small $y$. In these limits the integral in (1) can be evaluated analytically. Part I derived two expressions for the Green’s function, both as infinite series. Both expressions converge to the same result when a sufficient number of terms are taken, but one is asymptotic at large $y$ and the other at small $y$. This means that the Green’s function can be approximated by one or two terms from the appropriate expression at either limit. At order unity values of $y$, (17b) of part I which is asymptotic as $y \to \infty$, can be used. At large $y$ the Green’s function is smooth, so the convolution with the strongly localized potential-temperature anomaly can be approximated in terms of the Green’s function and its second derivative. The leading-order behaviour is given by the first two terms of (17b) of part I, which represent a barotropic component and the first baroclinic mode. Taking

$$\left( \begin{array}{c} \tilde{\eta}_0^{(0)} \\ \tilde{\eta}_1^{(0)} \end{array} \right) = \epsilon \left( \begin{array}{c} e^{i\phi/2} \\ e^{-i\phi/2} \end{array} \right),$$

as in part I, gives:

$$\tilde{\psi}(k; y, z) = -2\epsilon \theta \left\{ 1 + \frac{r^2k^2}{2} \right\} \frac{e^{-k|y|}}{k} \sin \left( \frac{\phi}{2} \right) +$$

(2)
Figure 3. Comparison of the meridional structure of stream function \( \tilde{\psi}_0(k; y) \) in the finite-difference (symbols) and analytic (lines) solutions. The solid lines and stars represent the even component (about \( z = 1/2 \)) and the dashed lines and circles the odd component. (a) \( r = 0.25, k = 1.6; \) (b) \( r = 0.25, k = 0.4; \) (c) \( r = 0.4, k = 1.6; \) (d) \( r = 0.4, k = 0.4; \) (e) \( r = 0.5, k = 1.6; \) (f) \( r = 0.5, k = 0.4. \) The units are dimensionless, the non-dimensional amplitude of the wave is \( \epsilon \theta \). See text for further information.

\[
2i \left( 1 + \frac{r^2(k^2 + \pi^2)}{2} \right) \frac{e^{-\sqrt{k^2+\pi^2}|y|}}{\sqrt{k^2 + \pi^2}} \cos(\pi z) \cos \left( \frac{\phi}{2} \right) + \mathcal{O} \left( e^{-2|y|}, r^4 \right) \]

The coefficient of the barotropic term contains a \( k^{-1} \) factor, but in the normal mode it remains regular as \( k \to 0 \) because \( \sin(\phi/2) = \mathcal{O}(k^{3/2}) \) (Eq. (50) of part I). This means that, despite the long wavelength, the structure of the wave remains meridionally localized to some degree.
At small values of $y$ the structure of the stream function is dominated by the contribution from the logarithmic singularity in the Green's function. The evaluation of the convolution of a Gaussian with the logarithmic singularity in the Green's functions is outlined in appendix B. Taking $z = 0$, this gives:

$$
\psi(k; y) = \frac{2e\theta_f}{\pi} \left\{ \left( \ln(\pi r) + I_1 \left( \frac{k}{\pi} \right) - \frac{\pi}{2k} + c_0 r^2 + \frac{y^2}{2r^2} \right) e^{-i\phi/2} + \right.
$$

$$
\left. \left( \frac{\pi}{2k} - S_{-1} \left( \frac{k}{\pi} \right) + c_1 r^2 \right) e^{i\phi/2} + O \left( r^4, y^2, \frac{y^4}{r^2} \right) \right\},
$$

(3)
where

\[
c_0 = \frac{1}{2} \left\{ \frac{\pi k}{2} - \frac{\pi^2}{12} - \frac{k^2}{4} \left( \ln \pi r + I_1 \right) + \pi^2 S_2 \left( \frac{k}{\pi} \right) \right\}
\]

and

\[
c_1 = \frac{1}{2} \left\{ \frac{\pi k}{2} - \frac{\pi^2}{4} - \pi^2 S_{-2} \left( \frac{k}{\pi} \right) \right\}.
\]

The functions \( S_p(k/\pi) \), \( p = -2, -1, 1, 2 \) and the constant \( I_1 \) are defined in part I, appendix C. In deriving this it has been assumed that \( y \ll r \), so the result is only valid within the front itself. The dominant term is \( O(\ln r) \) and out of phase with the displacement of the front.
This gives a cyclone where the displacement is positive (to the north). The corresponding leading-order behaviour of the disturbance geostrophic wind is $u \propto r^{-1}$ at $y = O(r)$ and $v \propto \ln r$ at $y = 0$.

Figure 4 compares (2) and (3) with the finite-difference solutions. Both the large and small $y$ approximations agree very closely with the numerical solution at $r = 0.25$. At $r = 0.4$ there are larger differences. The order $r^2$ corrections are small in the narrower front but become significant at $r = 0.4$. If the frontal width is increased further, the order $r^4$ error term increases rapidly to dominate the solution.

This comparison shows that (2) and (3) provide a reasonable description of fronts
narrower than about 0.4\(\times_{oc}\). At large distances from the front the stream function is dominated by a barotropic mode and the leading baroclinic mode, with decay at rates \(k\) and \(\sqrt{k^2 + \pi^2}\), respectively. Within the front there is a peak whose amplitude varies as \(\ln r\) and width as \(r\). The leading-order behaviour within the front is similar to that of the basic-state wind. Using (B.1) together with (27) and (C.4) of part I gives:

\[
\bar{u}(y, 0) = \frac{2\vartheta_k}{\pi} \left\{ \ln \left( \frac{r \pi}{2} \right) + I_1 - \frac{\pi^2 r^2}{12} + \frac{y^2}{2r^2} \right\} + \mathcal{O} \left( r^4, y^2, \frac{y^4}{r^2} \right).
\]  

(4)
By inspection of (3) and (4), the leading order $\ln r$ and $r^{-2}$ terms satisfy

$$\psi \approx \bar{u} \bar{\eta} = -\eta \bar{\psi}_y.$$  

This anomaly can be regarded as a simple displacement of the basic-state jet which remains, to leading order, parallel to the contours of surface potential temperature.

From (3) it follows that the maximum perturbation vorticity, at $y = 0$, is given by:

$$\zeta_{\text{max}} \frac{2 \epsilon \theta_t}{\pi r^2} \quad \text{or} \quad \zeta_{\text{max}}^* \frac{2 \epsilon \theta_t^* \bar{z}_{ac}}{\pi r^2 \bar{\theta}_{ac}}.$$  

This is not, however, a good estimate of the change in vorticity values; it represents a displacement of the mean jet. The Lagrangian vorticity anomaly is much smaller, only $O(\ln r)$, as will be seen below.

Figure 5 shows various vertical cuts through the three-dimensional structure of the wave. The contours show the potential-temperature anomaly and the arrows show the ageostrophic circulation (discussed in more detail in the next section). Figure 6 shows the corresponding geostrophic stream function. The arrows are as in Fig. 5 and the shading indicates the divergence of the meridional ageostrophic velocity. For these plots the convolution (1) is evaluated numerically to give a field which is valid for all values of $y$ and $z$. The stream function and temperature perturbations at $y = 0$ resemble those of the Eady mode, with a westward slope in the former and eastward in the latter (e.g. James 1994). At $y = 0$ the maximum potential-temperature anomalies occur on the boundaries. The main difference between $r = 0.25$ (Figs. 5(a) and 6(a)) and $r = 0.4$ (Figs. 5(b) and 6(b)) is near the boundaries, where stronger, localized, anomalies are associated with the narrower front. Outside the front (Fig. 5(b)) the boundary anomalies are weak and the maximum is in the mid-troposphere.

Figure 7 shows the horizontal structure of the same waves at $z = 0$. The dark contours show the geostrophic stream function; the arrows and thicker, light contours show the ageostrophic circulation. Also shown are the same fields from the finite-difference solutions. As the frontal width is increased the finite-difference solution develops a central region with a flattened meridional phase structure, made visible by the heavier zero contours. The asymptotic solution is unable to represent this structure.

The phase tilt away from the front is such as to generate a momentum flux decelerating the zonally averaged jet. This can be related directly to the displacement of the front using a result from Plumb (1979). Namely, the effect of growing displacements on a zonally averaged conserved field, such as the surface potential temperature in the present model, can be represented as a diffusion:

$$\frac{\partial \bar{\theta}}{\partial t} = \frac{\partial}{\partial y} \left( \frac{1}{2} \frac{\partial \bar{\eta}^2}{\partial t} \frac{\partial \bar{\theta}}{\partial y} \right).$$  

As the $\bar{\theta}$ profile becomes smoother, so the effective width of the front increases and the peak wind speeds decrease. The diffusive smoothing of the $\bar{\theta}$ profile is a result of the front being advected northwards in one sector of the wave and southwards in another; there are, as will be seen below, some points along the front where the gradients are being enhanced. Nonlinear integrations (e.g. Davies et al. 1991; Thorncroft et al. 1993) show that the subsequent development of an isolated life cycle is decidedly non-diffusive.

5. THE AGEOSTROPHIC CIRCULATION

Changes in the geostrophic vorticity in the vicinity of the front are associated with an ageostrophic circulation. This circulation is not forced by cross-frontal convergence,
Figure 5. x–z sections of the potential-temperature anomaly and ageostrophic circulation of the normal mode at k = 1.6. One wavelength is shown, the x-axis is labelled in units of k/2π. (a) r = 0.25, \( y = 0 \), (b) \( r = 0.25, y = 0.3 \), (c) \( r = 0.4, y = 0 \), (d) \( r = 0.4, y = 0.3 \). The contour interval is 0.4 in (a) and (c), 0.2 in (b) and (d). The scaling of the vectors is enhanced by a factor 4 in (b) and (d) compared with (a) and (c). Maximum values of (\( \theta \), \( u_\alpha \), \( w_\alpha \)) are (a) (3.19, 2.82, 0.67), (b) (0.45, 0.32, 0.15) (c) (2.00, 1.73, 0.68) and (d) (0.66, 0.50, 0.26) respectively. See text for further details.
Figure 6. As Fig. 5, except showing the geostrophic stream function (contours) and divergence of the cross-front ageostrophic velocity (shading). The vectors are as in Fig. 5. The contour interval is 0.2 in (a) and (c), 0.1 in (b) and (d). Maximum values of \((\psi, \partial \psi_{x}/\partial y)\) are (a) (1.05, 1.04), (b) (0.36, 0.32) (c) (0.86, 0.51) and (d) (0.48, 0.15) respectively. See text for further details.
since $\partial v/\partial y$ vanishes at $y = 0$ in the linear wave, but rather by the distortion of the front. In terms of the spatial variation of the large-scale flow this is a higher-order effect; the distortion depends on the second spatial derivatives of the velocity field, whereas the deformation which drives the classic model of frontogenesis (Hoskins and Bretherton 1972) is a first derivative. There is an interesting contrast to the latter model in that the meridional ageostrophic velocity here is frontolytic where the large-scale flow is creating enhanced vorticity and thus forcing ascent. The analysis below aims to clarify the various processes which lead to this result.

The ageostrophic circulation can be evaluated diagnostically as residuals in the thermodynamic and momentum equations. From Hoskins and Draghici (1977), Eqs. (10) to (13), these are:

$$\tilde{w}_a \frac{d\tilde{\theta}}{dz} = - (\sigma + ik \tilde{u}) \tilde{\theta} - \tilde{v} \frac{\partial \tilde{\theta}}{\partial y}, \hspace{1cm} (5a)$$

$$\tilde{u}_a f = - (\sigma + ik \tilde{u}) \tilde{v}, \hspace{1cm} (5b)$$

$$\tilde{v}_a f = (\sigma + ik \tilde{u}) \tilde{u} + \tilde{\nu} \frac{\partial \tilde{u}}{\partial y}, \hspace{1cm} (5c)$$

where ($u_a$, $v_a$, $w_a$) are the transformed ageostrophic wind components defined by Hoskins and Draghici (i.e. $u_a^* \tilde{u}$ etc. in their notation). The solution of (5) is not equivalent to solving the omega equation and then deriving the horizontal winds from the continuity equation. That procedure would only give the divergent component of the ageostrophic winds. The divergent winds extend over a meridional-scale comparable to the wavelength (not shown), but the full horizontal ageostrophic winds are concentrated near the front where the forcing in (5) is largest (Fig. 7).

In the true solution $w_a = 0$ is imposed at $z = 0$ and 1, but in the analytic solution this boundary condition is only satisfied to the same degree of approximation as the evolution equations. Errors arise through the severe truncation, to the zeroth moment, and also because the Green's functions are approximated by expansions about $y = 0$. The relative error in the near-surface vertical wind is large because of the normalization by the near-zero true value. The vertical gradient of the vertical wind is not afflicted by this ill conditioning, so it provides a better indicator of the near-surface motion. The thicker contours in Fig. 7 show the difference between the vertical velocity at $z = 0.1$ and at $z = 0$ compared with the vertical velocity at $z = 0.1$ in the finite-difference solution. The differences are minimal.

Table 2 quantifies the magnitude of the spurious boundary vertical wind in terms of its contribution to the potential-temperature advection. The individual terms in the horizontal advection increase rapidly as $r$ is decreased. The sum of the two horizontal advection terms is considerably smaller than the single terms, but still much larger than the spurious vertical

| $r$  | $|ik\tilde{u}\theta|/|w\tilde{\theta}_z|$ | $|ik\tilde{u}\theta + v\tilde{\theta}_y|/|w\tilde{\theta}_z|$ | $|w\tilde{\theta}_z|$ |
|------|--------------------------------|--------------------------------|----------------|
| 0.0625 | 932 | 6510 | 73 | 54 | 432 | 10 | 0.045 | 0.004 | 0.004 |
| 0.125 | 308 | 801 | 18 | 24 | 79 | 4.7 | 0.052 | 0.010 | 0.004 |
| 0.25 | 71 | 63 | 4.7 | 8.2 | 9.6 | 2.1 | 0.083 | 0.045 | 0.004 |
| 0.4 | 11.1 | 9.3 | 1.9 | 1.7 | 1.9 | 1.1 | 0.26 | 0.14 | 0.03 |
| 0.5 | 4.0 | 3.3 | 1.2 | 0.69 | 0.75 | 0.80 | 0.51 | 0.27 | 0.03 |
Figure 7. The geostrophic stream function at $z = 0$ (dark contours, interval 0.1), horizontal ageostrophic wind vectors also at $z = 0$, and the change in vertical wind between $z = 0$ and $z = 0.1$ (light contours, interval 0.1). Negative contours are dashed in both cases. The $x$-axis is labelled in units of $k/2\pi$. (a) and (b) are from the analytic solution with $r = 0.25$ and 0.4 respectively, (c) and (d) are from the numerical solution with respective $r$ values 0.4 and 0.5. See text for further details.
advection at small $r$. At $r \approx 0.5$ the two become comparable. Moving away from $y = 0$ the spurious term first decreases, then changes sign near $y = r$, and increases in relative amplitude as $y$ is increased further. The absolute amplitude of the term is, however, very small in this range. The fourth column (in heavy font) is the most relevant for quantifying the errors, and agrees with other estimates given in the text: the approximation is excellent for $r \leq 0.25$, reasonable for $r$ up to 0.4, and no longer acceptable by $r = 0.5$.

Figure 5 shows, as expected, rising motion in the warm sector. That is, the rising motion is concentrated along the warm front (i.e. the region where the front in the basic state is being advected northwards by the disturbance), whilst the cold front is a region of descent and ageostrophic frontogenesis (see shading in Fig. 6). Away from the front (Fig. 5(b)) there is an interesting banded structure in the ageostrophic circulation, but these velocities are significantly weaker than those occurring near the front. The phase structure can be understood from the relative magnitude of the terms in (5b). At $z = 0$ the advective term dominates, so that $u_x f \approx -\bar{u} \psi_{xx}$. The maximum in $u_x$ therefore coincides with the minimum in $\psi$. At $z = 1/2$ the advective term vanishes, so $u_x f \approx -\sigma \psi_x$. This gives a maximum shifted one-quarter phase towards smaller $x$ relative to $\psi$ and hence in the middle of the cold sector. Consequently the band of maximum $u_x$ slopes sharply in the sense of the basic-state shear. This structure is not apparent at $y = 0$ because the amplitude at $z = 1/2$ is much smaller than that near the boundaries. At $y = 0.3$, on the other hand, $|\bar{k}u|$ is comparable with $\sigma$, so that the amplitude is relatively uniform throughout the troposphere.

The vertical motion at $y = 0$ is fed by convergence of the along-front ageostrophic velocity. The shading in Fig. 6 indicates the divergence of the cross-front ageostrophic velocity. The pattern here has a negative correlation between low-level descent and cross-front divergence. That is, the cross-front divergence is positive where the air is rising and negative where the air is sinking, contrary to what one might expect from two-dimensional models of frontal circulations. This pattern is also visible in the horizontal sections shown in Fig. 7. This point will be discussed further below.

The dominant term forcing $u_x$ is the second term in (5b), the advection of the meridional geostrophic wind by the basic-state wind. In non-dimensional units at the fastest growing wave number we have $\sigma = 0.303$ and $|\bar{k}u| \approx 1.6$ (for $r = 0.25$), so the dominance of the advective term is clear. This fits in with the idea of a conveyor belt, introduced as a synthesis of observations (Harrold 1973; Browning and Monk 1982; Browning 1990), and reproduced in an objective data analysis by Wernli and Davies (1997) and Wernli (1997). In this conceptual model the vertical motion arises, as here, primarily in association with advection along a near-static frontal structure rather than through growth of the wave or frontogenesis (in the present case the frontogenesis function, defined as the scalar product of the $\mathbf{Q}$-vector (discussed below) with the horizontal potential-temperature gradients, vanishes identically at $y = 0$).

For Figs. 5 to 7 the convolution in (1) is numerically evaluated. The ageostrophic circulation near $(y, z) = (0, 0)$ will now be examined in more detail with analytic methods, in order to find out how the structures in these figures are created.

In (5a) to (5c) it is not clear what the leading-order behaviour is, because of cancellation between various terms. In order to gain a clearer picture of the ageostrophic circulation (in particular, the connection between frontogenesis and vertical motion) a more detailed analysis of (5) near $y = 0$ will now be carried out.

The first step is to reformulate (5c) so as to make the cancellation which occurs on the right-hand side more explicit. The definition of $\eta$ implies that:

$$\bar{\nu} = (\sigma + ik\bar{u}) \bar{\eta}. \quad (6)$$
Equation (6) can then be substituted into (5c), leaving:

\[ \tilde{v}_a f = (\sigma + ik\tilde{u}) \left( \tilde{u} + \tilde{\eta} \frac{\partial \tilde{u}}{\partial y} \right). \]  

(7)

As noted above, both \( \tilde{u} \) and \( d\tilde{u}/dy \) have magnitudes of \( \mathcal{O}(r^{-1}) \) within the front. In the combination that occurs in (7), however, it is clear by inspection of (3) and (4) that the \( y/r^2 \) terms cancel leaving a less singular residual. The form of this residual will be clarified below.

The horizontal ageostrophic divergence at \( y = 0 \) is given by the sum of

\[ ik\tilde{u}_a = f^{-1}(\sigma + ik\tilde{u}) \left( k^2\tilde{\psi} \right) \]  

(8a)

and

\[ \frac{\partial \tilde{u}_a}{\partial y} = f^{-1}(\sigma + ik\tilde{u}) \left( \frac{\partial \tilde{u}}{\partial y} - \tilde{\eta} \frac{\partial \tilde{\xi}}{\partial y} \right). \]  

(8b)

In (8b) the fact that \( \partial \tilde{u}/\partial y \) vanishes at \( y = 0 \) has been used. It is in general true that the horizontal ageostrophic divergence is related to the absolute vorticity, but (8) shows that in the special case of linear waves on a front a stronger result holds: the divergences of the individual components of the ageostrophic velocity are related to the tendencies of the horizontal shears of individual components of the geostrophic wind. In (8b) the tendency is expressed in a semi-Lagrangian form, \( -\tilde{u}_y + \tilde{\eta}\tilde{\xi}_y \) is the anomalous shear of the zonal wind at \( y \) relative to the undisturbed flow at \( y - \tilde{\eta} \). From (8) it follows that the ageostrophic frontogenesis at \( y = 0 \) (given by \( \tilde{u}_a \partial_y \)) is related to the change in the shear of the along-front flow rather than the change in the vorticity. The following analysis will reveal how these two are related in the linear wave.

Combining (8) with the continuity equation gives an explicit expression for the leading-order behaviour of \( \tilde{u}_a \) at \( y = 0 \) near \( z = 0 \):

\[ \frac{\partial \tilde{w}_a}{\partial z} = -ik\tilde{u}_a - \frac{\partial \tilde{v}_a}{\partial y} \]

\[ = f^{-1}(\sigma + ik\tilde{u}) \left( \tilde{\xi} + \tilde{\eta} \frac{\partial \tilde{\xi}}{\partial y} \right). \]  

(9)

This equation is, in fact, a semi-Lagrangian form of the perturbation vorticity equation, in which the vorticity anomaly relative to the undisturbed position \( y - \eta \) changes by zonal advection and vortex stretching. As in (5a) there is cancellation between terms of order \( r^{-1} \) to give a less singular residual. The advantage here is that the cancellation takes place within the last term, which depends linearly on the basic state. This facilitates the calculation considerably.

The above results show how the ageostrophic circulation is related to the horizontal shears in the growing wave. Using (1) these shears can be related to the displacement of the front. Concentrating on the surface front \( (z = 0) \):

\[ \left( k^2 \tilde{\psi} \right)_{y=0} = -\theta_t \left( C_0^a \tilde{\eta}_0 + C_1^a \tilde{\eta}_1 \right) \]  

(10a)

\[ \left( -\frac{\partial \tilde{u}}{\partial y} + \tilde{\eta} \frac{\partial \tilde{\xi}}{\partial y} \right)_{y=0} = -\theta_t \left( C_0^c \tilde{\eta}_0 + C_1^c \tilde{\eta}_1 \right), \]  

(10b)

where the constants \( C_0^a, C_0^c \) are convolutions of the basic-state temperature gradients with the Green’s functions. The superscripts ‘a’ and ‘c’ refer to the along- and cross-front
shear, and the subscripts refer to the level of the potential-temperature anomaly which is associated with the wind anomalies through the potential-vorticity inversion. This inversion is expressed in the relation between stream function and boundary temperature anomalies given in (1).

Combining the above definitions gives:

$$\left( \tilde{\zeta} + \tilde{\eta} \frac{\partial \tilde{\zeta}}{\partial y} \right)_{y=0} = -\theta_1 (C_0 \tilde{\eta}_0 + C_1 \tilde{\eta}_1),$$

(11)

where $C_0$ and $C_1$ are constants given by:

$$C_\mu = C_\mu^a + C_\mu^c.$$  

Substituting (1) in the left-hand side of (10) and equating coefficients of $\tilde{\eta}_\mu$, $\mu = 0, 1$, gives:

$$C_0^a = 2\pi \int -k^2 \tilde{G}_{00}(k; y) \Lambda(y) dy,$$

(12a)

$$C_0^c = 2\pi \int \left\{ \frac{\partial^2}{\partial y^2} \left( \tilde{G}_{00}(k; y) - \tilde{G}_{00}(0; y) - \tilde{G}_{01}(0; y) \right) \right\} \Lambda(y) dy,$$

(12b)

and

$$C_1^a = 2\pi \int -k^2 \tilde{G}_{01}(k; y) \Lambda(y) dy,$$

(13a)

$$C_1^c = 2\pi \int \frac{\partial^2 \tilde{G}_{01}(k; y)}{\partial y^2} \Lambda(y) dy.$$

(13b)

In (12b) the second half of the integrand, containing the Green’s functions at $k = 0$, represents the vorticity of the zonal mean flow. The rest of (12) and all of (13) represent the vorticity anomaly $\tilde{\zeta}$.

The leading-order behaviour of $C_\mu^a$, $C_\mu^c$ is given by the small $y$ range of the integrands in (12) and (13) (because of the localized structure of $\Lambda(y)$). Appendix B gives the appropriate asymptotic forms of the Green’s functions. The self-interaction terms have a logarithmic singularity at $y = 0$.

The coefficients of $\tilde{\eta}_1$ can be evaluated by approximating the Green’s functions and their second derivatives by their value at $y = 0$ (see (B.3)) and then using $\int_{-\infty}^{\infty} \Lambda(y) dy = 2\theta_1$. This gives:

$$C_1^a = -k + \frac{2k^2}{\pi} S_{-1}(k/\pi),$$

(14a)

$$C_1^c = -\frac{\pi}{2} + k - 2\pi S_{-2}(k/\pi).$$

(14b)

The singular behaviour of the self-interaction terms can be evaluated using the identity $\int_{-\infty}^{\infty} \ln |y| \Lambda(y) dy = 2\theta_1 (\ln r + I_1)$ and using the small $y$ approximations for the Green’s functions (B.2), which include logarithmic terms.

$$C_0^a = k - \frac{2k^2}{\pi} \left( \ln \pi r + I_1 - S_1(k/\pi) \right),$$

(15a)

$$C_0^c = \frac{\pi}{2} - k + \frac{k^2}{\pi} \left( \ln \pi r + I_1 + 0.5 \right) - 2\pi S_2(k/\pi).$$

(15b)
These results give the leading-order contributions, consisting of a logarithmic term and a term independent of \( r \). To get the \( r^2 \) correction it would be necessary to consider the Green’s function expansion out to order \( y^4 \). Given that the errors incurred by the semi-geostratrophic approximation are likely to be significant, the qualitative picture obtained from the leading-order terms is probably the best that can be obtained from the present model.

Comparison of (4) and (15a) shows that the leading order, \( \ln r \), contribution to the along-front shear is given by \(-k^2 \eta_0 \tilde{u} \), which represents the vorticity of the jet flowing along a curved front with no adjustment in structure. The associated convergence, \( u_{a,x} \), is compensated to the tune of 50\% by an opposing term in \( v_{a,3} \), and the rest is taken up in \( w_{a,z} \). Viewed in terms of potential vorticity, this means that the vorticity induced by the along-front curvature is balanced in equal measure by changes in cross-front shear and in static stability so as to maintain uniform potential vorticity. This simple structure is only accurate when \( | \ln(\pi r) | \gg 1 \), which is not satisfied for realistic parameter values \( (r = 0.1 \) to 0.4\). The picture of along-front convergence being split in roughly equal measure between cross-front and vertical divergence is, however, qualitatively correct for \( r = 0.25 \).

Figure 8(a) shows the coefficients \( C_0 \) and \(-C_1 \), which define the relative contributions of the lower and upper waves respectively, to the lower vorticity anomaly. They are equal in amplitude at \( k = 0 \), but \( |C_1| \) decays towards zero whilst \( C_0 \) grows. The surface Lagrangian vorticity anomaly for wave numbers near the most unstable wave number is thus dominated by the contribution which is related to the surface frontal displacement, and which is in phase with that displacement. The ageostrophic stretching term does not have the same phase as the vorticity anomaly, because the zonal advection term dominates over the tendency term, but is displaced a quarter-wavelength upstream. This gives the phase structure seen in Fig. 5, which is the same as seen in the meridionally uniform Eady model. This structure is more easily explained in terms of the energetics (warm air rising to release energy); the present discussion aims to explain how this fits in with the frontolytic and frontogenetic dynamics along the front.

Figure 8(b) shows the amplitudes of the contributions to the semi-Lagrangian vorticity anomaly from along- and cross-front shear. The along-front contribution dominates for most wave numbers. The ratio is generally greater than the factor two, which would be obtained in the limit \( r \to 0 \) from the \( \ln r \) terms. The horizontal sections in Fig. 7 show this pattern. The difference between these results and the positive correlation between vertical motion and ageostrophic frontogenesis in the model of Hoskins and Bretherton (1972) can be explained in terms of the different large-scale forcing. The vorticity in the cyclone is increasing, implying positive vertical ageostrophic velocity, but the cross-frontal shear is decreasing, implying ageostrophic frontolysis. The net increase in vorticity occurs because the increase in \( \partial \tilde{u} / \partial x \) more than compensates for the decrease in \(-\partial (\tilde{u} + u) / \partial y \).

From (9) and (11), the modulus of the complex coefficient of the vertical stretching is given by:

\[
\left| \frac{\partial \tilde{u}_a}{\partial z} \right|^2 = \epsilon^2 \theta_t^2 \left( \sigma^2 + k^2 \tilde{u}^2 \right) \left( C_0^2 + C_1^2 + 2 \cos \phi C_0 C_1 \right).
\] (16)

Figure 9 compares (16) (the upper line) with finite-difference results (the upper symbols). The agreement is good. Also shown are the mean gradient between \( z = 0 \) and \( z = 0.1 \) from (1) and (5), using the analytic normal mode solution (dashed line), and the finite-difference solution (solid line). It is clear that the large values of \( \partial w / \partial z \) as \( r \to 0 \) are confined close to the lower boundary and do not extend far into the atmosphere.

Since both \( \tilde{u} \) and \( C_0 \) grow logarithmically as \( r \to 0 \), the growth of the low-level convergence given by (16) is \( \mathcal{O}(\ln r^2) \). This leading-order term is independent of \( \sigma \) and of \( \eta_1 \); it depends only on the amplitude of the lower wave. It represents isentropic up-gliding
of the near-surface jet along the isentropic surfaces perturbed by the surface frontal wave. At a width of \( r = 0.25 \) the dominance of the zonal mean advection, \( k \overline{u} \), over the growth rate \( \sigma \) was noted above. The corresponding values of \( C_0 \) and \( C_1 \) are 2.5 and \(-1.1\) respectively. The discrepancy is not so large, but the first term, representing the effects of the lower wave, still dominates.

Figure 10(a) shows profiles of the zeroth moment of \( w_a \). The range of \( r \) values covered is 1 to \( 1/64 \). At the smallest value there is good agreement between the finite-difference results (stars) and the asymptotic result (solid line through the stars). At \( r = 1/16 \) (circles and nearest solid line) there is already significant error in the asymptotic result. Figure 10(b) shows the variation of the maximum vertical velocity, which occurs at \( z = 1/2 \), with
Figure 9. The vertical gradient of the vertical velocity at \((y, z) = (0, 0)\) against \(\log_2 r\). The solid line is the asymptotic value (Eq. (17)). The circles mark the corresponding numerical values. The change in \(u_v\) between \(z = 0\) and \(z = 0.1\) is given by the dashed line (asymptotic, Eq. (5)) and stars (numeric). See text for further explanation.

the logarithm of the frontal width. As in Fig. 9, it is clear that the \(r = 0\) solution is very different from those at moderate values, \(r \approx 1/4\), which are relevant to atmospheric fronts.

The results of this section can be used to estimate the accuracy of the semi-geostrophic approximation. From (5b) and (7):

\[
\begin{align*}
\left| \frac{\tilde{u}_v}{\tilde{u}} \right| &= \left| \frac{(\sigma + ik\tilde{u})k\tilde{\psi}}{f\tilde{\psi}_y} \right| = \frac{\theta_i}{\theta_{sc}} \mathcal{O}(r|\ln r|^2, 1), \\
\left| \frac{\tilde{v}_v}{\tilde{v}} \right| &= \left| \frac{(\sigma + ik\tilde{u})(\tilde{u} + \tilde{v}\tilde{u}_y)}{f\tilde{\psi}} \right| = \frac{\theta_i}{\theta_{sc}} \mathcal{O}(r|\ln r|, 1).
\end{align*}
\] (17a) (17b)

In each error estimate, the first value applies within the front and the second (i.e. '1') applies outside the front. In (17b), the fact that the Lagrangian vorticity anomaly is \(\mathcal{O}(\ln r)\) within a region of width \(r\) has been used to estimate the Lagrangian zonal velocity anomaly as \(\mathcal{O}(r \ln r)\). The estimated errors within the front decay to zero as the frontal width is decreased, while those outside remain constant. In practice, Snyder (1995) shows that there is a slight increase in error with decreasing \(r\). The discrepancy could be attributed to the fact that the importance of the wind is determined not only by its amplitude but also by its orientation, so that (17) does not give a rigorous bound.

The forcing of the ageostrophic circulation can also be expressed in terms of the Q-vectors (Hoskins et al. 1978). The principal advantage that these diagnostics confer is a convenient means of comparing theory and observations. The wave number \(k\) coefficient of the linearized Q-vector is given by:

\[
\tilde{Q} = (\tilde{v}_y \tilde{\theta}_y, \tilde{u}_x \tilde{u}_y + \tilde{v}_y \tilde{\theta}_x) = (-k^2 \tilde{\psi}_y, -ik(\tilde{u} + \tilde{v}\tilde{u}_y)) \tilde{\theta}_y.
\]

Comparing this expression with (5b) and (7) shows that the Q-vectors for a linear normal
mode are related to the ageostrophic circulation in a surprisingly simple way:

\[ \tilde{\mathcal{Q}} = -\frac{ikf\bar{\partial}_y}{\sigma + ik\bar{u}} (\bar{u}_a, \bar{v}_a). \]

Since the advective term dominates the denominator, the coefficient relating \( \tilde{\mathcal{Q}} \) to the ageostrophic wind is approximately given by \( f\bar{\partial}_y/\bar{u} \), which is real and independent of \( k \). The two vector fields, \( \mathcal{Q} \) and \( (u_a, v_a) \), are therefore approximately parallel, not only in
a single Fourier component, but also in an arbitrary sum of Fourier modes and hence in
an arbitrary linear disturbance. The fact that the potential-temperature gradients have a
narrower meridional structure than the zonal mean flow implies that the \( Q \)-vectors will be
concentrated closer to the front than the ageostrophic circulation. This is also true more
generally, since the vertical velocity is proportional to the inverse horizontal Laplacian of
the divergence of \( Q \), which implies a degree of smoothing.

Keyser et al. (1989) show the ageostrophic circulation in a nonlinear primitive-
equation integration of a frontal baroclinic wave. Their pictures are consistent with the
results of the analytic model discussed here, in so far as the along-front component domi-
nates the ageostrophic circulation, and the convergence of this component is opposed by
that of the cross-front component. These remarks apply to the upper levels of their integra-
tion, where the displacement of the jet remains in the weakly nonlinear regime throughout
the integration.

Surface \( Q \)-vector diagnostics, shown in nonlinear primitive-equation solutions by
Keyser et al. (1992) and in nonlinear semi-geostrophic integrations by Schär and Wernli
(1993), indicate that the pattern of along-front alignment remains valid in the bent-back
warm front which forms to the north of the cyclonic anomaly. Keyser et al. (1992) suggest
that the associated vertical motions are responsible for comma-cloud formation. The cold
front is much closer to the Hoskins and Bretherton paradigm with \( Q \)-vectors perpendicular
to the front.

Keyser et al. (1988) and Keyser et al. (1992) show that the component of the \( Q \-
vector parallel to the potential-temperature contours is related to the rotation of \( \nabla \theta \) along
a trajectory. In a recent case-study of a mature midlatitude cyclone Martin (1998) shows
that the along-front component of the \( Q \)-vector is the dominant forcing term for vertical
motion in the occlusion.

6. OTHER BASIC STATES

(a) Strongly localized fronts

The solution has been developed for a specific class of basic states for which the
meridional gradient of potential temperature on the boundary has a Gaussian form. This
section will show that the results can be generalized to cover other basic states. The easiest
case, dealt with first, is for fronts in which the meridional temperature gradients are strongly
localized in the vicinity of the maximum. In this case the results that have been derived
earlier can be applied, once the appropriate value of \( r \) has been determined.

Consider a general temperature gradient \( \Lambda (y; L) \), where \( L \) is a parameter defining
the frontal width. To apply the previous results this structure must be represented as a sum
of Hermite polynomials:

\[
\Lambda (y; L) = \left\{ \Lambda^{(0)} + \Lambda^{(2)} h^{(2)} \left( \frac{y}{R} \right) + \Lambda^{(4)} h^{(4)} \left( \frac{y}{R} \right) + \ldots \right\} \frac{1}{r} \sqrt{\frac{2}{\pi}} \exp \left( \frac{-y^2}{2r^2} \right). \quad (18)
\]

The width of the Gaussian, \( r \), should be related to \( L \). As in part I, the scale \( R \) of the Hermite
polynomials is given by:

\[
\frac{1}{R^2} = \frac{1}{b^2} + \frac{1}{r^2},
\]

where \( b \) is the meridional-scale of the disturbance. The leading-order results derived in the
previous section will give a solution for the more general problem to the same asymptotic
accuracy if the coefficient of the second moment, \( \Lambda^{(2)} \), in (18) vanishes. This is because
the fourth and higher moments give contributions or order \( r^4 \) and less as \( r \to 0 \). That is, we seek to approximate the frontal structure \( \Lambda(y; L) \) with a Gaussian, and the form of the approximation is objectively determined by the formalism of the moment expansion introduced in part I. The relation between \( r \) and \( L \) is determined by using the orthogonality of the Hermite polynomials to express \( \Lambda^{(2)} \) in terms of a weighted integral of \( \Lambda(y; L) \):

\[
\Lambda^{(2)} = \int_{-\infty}^{\infty} \Lambda(y; L) \exp \left( \frac{-y^2}{2b^2} \right) g^{(2)} \left( \frac{y}{R} \right) dy = 0. \quad (19)
\]

For \( L \ll b \) the Gaussian term in (19) is approximately unity and \( R \approx r \), which leads to:

\[
r^2 = \frac{\int_{-\infty}^{\infty} \Lambda(y; L) y^2 dy}{\int_{-\infty}^{\infty} \Lambda(y; L) dy}. \quad (20)
\]

As an example, consider the step function:

\[
\Lambda_{s}(y; L) = \begin{cases} 
0, & |y| > L, \\
\frac{\theta}{L}, & |y| \leq L.
\end{cases} \quad (21)
\]

From (20) it follows that:

\[
r = \frac{L}{\sqrt{3}}.
\]

The growth rate is then given by (46) of Part I. Table 3 gives some results at \( k = 1.6 \). The error is of the order of 4% at \( L = 0.5 \) (corresponding to 400 km for a tropopause height of 8 km) and decreases as \( L \) is decreased. The step function, however, has unbounded vorticity at \( y = \pm L \), so the transformation from geostrophic to physical space is not well defined. For the next example, consider the ‘witch’s hat’ function (this is the simplest piecewise polynomial function which gives finite surface vorticity):

\[
\Lambda_{\text{wh}}(y; L) = \begin{cases} 
0, & |y| > 2L, \\
\frac{\theta}{2L} \left( 2L - |y| \right), & |y| \leq 2L.
\end{cases} \quad (22)
\]

Substitution into (20) gives \( r = L \sqrt{2/3} \). The resulting growth rates are also shown in Table 3.

Equation (20) extends the results of this study to cover a wide range of frontal profiles. The next subsection looks at an example that does not fall within this range.
(b) Weakly localized gradients

As noted earlier, the method given above requires that the front be sufficiently localized. Equation (20) makes this precise: the method will work provided the integral in the numerator converges and is determined by \( \Lambda(y; L) \) in \( |y| \ll b \).

The next example does not fit into the methodology so cleanly. The structure analysed by Hoskins and Bretherton (1972) and many subsequent authors has meridional gradients given by:

\[
\Lambda_{HB}(y) = \frac{\theta_t}{\pi} \frac{L}{L^2 + y^2}.
\]  

(23)

This decays only as \( y^{-2} \) at large \( y \), so the integral in (20) is not convergent. Because of the relatively slow decay of the gradients away from the front, the Gaussian factor in (19) cannot be neglected. Going back to (19), the limit \( L \to 0 \) gives:

\[
r^2 = \frac{Lb}{\sqrt{2\pi}}.
\]  

(24)

Until now the meridional-scale of the disturbance, \( b \), has played no active role, but it is now required in order to relate \( r \) to \( L \).

The meridional-scale of the disturbance can be determined by requiring that \( \eta^{(2)} \) vanish in the limit \( L \to 0 \). Putting this condition into the integral which determines \( \eta^{(2)} \) ((34) of part I with \( m = 2 \)), after some manipulation, gives:

\[
r^2 = \frac{\int y^2 \tilde{\eta} e^{-\frac{x^2}{2\sigma}} dy}{\int \tilde{\eta} e^{-\frac{x^2}{2\sigma}} dy}.
\]

Integration by parts then yields:

\[
\frac{1}{b^2} = \frac{\int \tilde{\eta}_{yy} e^{-\frac{x^2}{2\sigma}} dy}{\int \tilde{\eta} e^{-\frac{x^2}{2\sigma}} dy} \approx \frac{\tilde{\eta}_{yy}(k; 0)}{\tilde{\eta}(k; 0)}.
\]  

(25)

This ratio can be evaluated from the tendency equation for \( \tilde{\eta} \) ((28) of part I). Taking the second meridional derivative of that equation gives:

\[
\sigma \tilde{\eta}_{yy} = -2\pi ik \left[ \tilde{u} \tilde{\eta}_{yy} + \left[ \tilde{u} - \tilde{G}(k; y - y') \Lambda(y') \right]_{yy} \tilde{\eta} \right] + \mathcal{O}(1), \quad \text{at} \quad y = 0,
\]

using \( \tilde{u}_y(y = 0) = 0 \). The leading-order terms are of order \( \ln r \). Up to this point the logarithmic terms have been treated as being of order unity for the purposes of deciding which terms to retain. In the present case, however, retaining the order unity terms along with the logarithmic terms leads to a huge increase in complexity. In this case (25) contains a non-trivial \( r \) dependency, so that (24) would have to be solved by an iterative numerical scheme. Neglecting the order unity terms gives the much simpler result

\[
\tilde{\eta}_{yy} - \frac{k^2}{2} \tilde{\eta} = 0.
\]  

(26)

As might have been expected, the meridional-scale is proportional to \( k^{-1} \). Substituting (25) and (26) into (24) gives:

\[
r = \sqrt{\frac{L}{k\sqrt{\pi}}}.
\]  

(27)
TABLE 4. Comparison of finite difference and asymptotic results for the Hoskins and Bretherton front (Eq. (23)).

<table>
<thead>
<tr>
<th>L</th>
<th>r</th>
<th>σ_{fd}</th>
<th>σ_{as}</th>
<th>r</th>
<th>σ_{fd}</th>
<th>σ_{as}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.420</td>
<td>0.256</td>
<td>0.194</td>
<td>0.840</td>
<td>0.065</td>
<td>0.099</td>
</tr>
<tr>
<td>0.25</td>
<td>0.297</td>
<td>0.279</td>
<td>0.259</td>
<td>0.594</td>
<td>0.068</td>
<td>0.081</td>
</tr>
<tr>
<td>0.125</td>
<td>0.210</td>
<td>0.307</td>
<td>0.299</td>
<td>0.420</td>
<td>0.072</td>
<td>0.072</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.148</td>
<td>0.327</td>
<td>0.324</td>
<td>0.297</td>
<td>0.076</td>
<td>0.071</td>
</tr>
</tbody>
</table>

This level of approximation turns out to be sufficiently accurate for present purposes. Errors do become large, however, for $k \approx 0.4$.

In the previous examples the amplitude of the zeroth moment of the frontal profile was given by the temperature change across the front; in the present case, however, there is an $O(L/b)$ correction:

$$
\Lambda^{(0)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Lambda_{HB}(y) \exp\left(\frac{-y^2}{2b^2}\right) \, dy \approx 2\theta_f \left(1 - \frac{L}{\sqrt{2\pi} b}\right).
$$

Some results are given in Table 4. The errors are larger than those found at comparable frontal widths with the frontal profiles considered earlier, as might be expected from the extra approximations involved. At $k = 1.6$ the asymptotic solution converges towards the true solution as $L$ is decreased. At $k = 0.4$, however, the relative error introduced by the approximation in (26) is larger. There is agreement at $L = 0.125$, presumably fortuitous, and then the asymptotic result moves away from the true result as $L$ is further reduced.

This basic state considered in this subsection corresponds to that used by Snyder (1995), but he used the full form of the semi-geostrophic potential vorticity as opposed to the linearized form here. His flow I corresponds to $L = 0.5$ in Table 4 with $\theta_f = 0.5$ (for which value the warmest surface isentrope is also the coldest tropopause isentrope). The present results then give $\sigma^* = 1.28 \times 10^{-5} \text{ s}^{-1}$. This is larger than that found by Snyder, but the difference may plausibly be attributed to differences in the equations, model (his model includes a diffusive term), and side-boundary conditions.

7. Discussion

The analytic asymptotic solution presented in part I has been found to accurately describe the three-dimensional structure of baroclinic waves growing on a range of fronts. As far as the geostrophic stream function and the potential-temperature anomalies are concerned, the analysis does not bring any surprises. The phase structure at $y = 0$ is similar to that of the classic Eady model. The horizontal structure has the phase lines in the stream function anomaly angled into the surface jet, indicating a down-gradient momentum transport.

The ageostrophic circulation in the vertical plane also shows a familiar structure, with rising motion in the warm sector and sinking in the cold sector. A new feature is revealed in the horizontal plane. In two-dimensional models of frontal circulation, upwards motion is associated with frontogenesis. Here the reverse is true, the rising motion in the warm sector coincides with a region of ageostrophic frontolysis. The difference arises because the geostrophic frontogenesis forcing vanishes identically in the linear wave. This allows a higher-order effect, the distortion of the front, to come to the fore. The distortion of
the front generates cyclones and anticyclones with cyclonic and anticyclonic along-front shear. The contribution of the anomalous cross-front shear to the vorticity, however, is in the opposite sense. This can be seen as a negative feedback; the front adjusts its structure to reduce the vorticity anomaly associated with the curvature. It is the cross-front shear which determines the sharpness of the front. Hence this tendency for the cross-front shear to oppose the vorticity anomaly generated by the frontal distortion implies ageostrophic frontogenesis where a cyclone is being created, and ageostrophic frontogenesis in the anticyclone.

Despite these differences, the ageostrophic circulation can lead to frontal collapse in the growing wave. Viewed in terms of the cross-front shear alone, the ageostrophic response is a positive feedback. That is, an increase in cross-front shear implies a change in the geostrophic coordinate transformation which gives a further increase in the physical space cross-front shear. The dashed lines in Fig. 8(a) show the amplitude of the change in cross-front vorticity. At $r = 0.25$ the maximum is about 0.5. If we take a wave amplitude at which linear theory might still be qualitatively accurate, $\epsilon = .3$ (giving a wave slope $ke \approx 0.5$), and $\tau^\dagger = 12$ K, this implies an anomaly of $0.5 \epsilon \tau^\dagger f = 0.075 f$. Larger changes can no doubt be generated by persistent deformation, but the size of the change here is by no means negligible. Comparison with observed structures in developing waves is difficult, because of the unrealistic vertical symmetry in the basic state used here. It is hoped that further work will provide insight into the effect of relaxing the symmetry about $y = 0$ and anti-symmetry about $z = 0.5$.

The vertical motion is dominated by isentropic up-gliding, with time evolution of the flow playing a relatively minor role. This is because the strong along-front jet implies an advective time-scale which is considerably shorter than the inverse growth rate.

The present results are of course limited by the validity of the semi-geostrophic equations. In particular, it has been found that the distortion of the front generates an ageostrophic circulation, but this result is only accurate so long as the curvature of the front is small (e.g. Craig 1993). Similarities with the nonlinear primitive-equation solutions of Keyser et al. (1992) suggest, however, that the results are of at least qualitative relevance to the later development.

The results can also be generalized to arbitrary frontal profiles, provided they are sufficiently localized (section 6). The growth rate then depends on the temperature contrast and to a lesser extent on $\int_{-\infty}^{\infty} y^2 \partial_y \theta, dy$. This underlines the point that it is primarily the temperature contrast which determines the growth rate, and not the structure within the front. These comments only apply to the baroclinic mode of instability associated with the interaction between the ground and tropopause. Internal structure in a surface front can induce instabilities of a more barotropic nature (e.g. Schär and Davies 1990; Joly and Thorpe 1990).

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APPENDIX A

Numerical methods

The Green's function representation is discretized, as in Juckes (1995). There are no meridional boundaries, but the numerical grid needs only to cover those points where the potential-temperature anomaly is non-zero. That is, in the discretization of the integral in
(28) of part I, points where the integrand has negligible amplitude can be omitted. For frontal structures with strongly localized potential-temperature gradients this means that all the grid points can be concentrated in a region no more than several frontal widths wide, ensuring that accuracy is maintained as the frontal width is reduced.

As in Juckes (1995), terms of the order:
\[ d_y^2 \frac{d^2 K_0}{dy^2}, \quad d_y^2 \frac{d^2 \Lambda(y)}{dy^2} \]
are retained, whilst those of order:
\[ d_r^2 \frac{d^2 \tilde{\eta}_l}{dy^2} \]
are neglected, where \( d_y \) is the meridional grid spacing. The retention of the former reduces the error by approximately a factor of four for the case \( r = 1, k = 1.5 \). With these parameters, the growth rate varies with the number of grid points as 0.17274 (\( M = 21 \)), 0.17287 (\( M = 41 \)) and 0.17289 (\( M = 81 \)). The error decays smoothly as \( M^{-2} \), and increasing the resolution further does not change the eigenvalue to the accuracy quoted.

For the Gaussian profile the domain size is taken as \( |y| \leq 4.5r \), which guarantees that the temperature perturbations are \( \leq \exp(-4.5^2/2) \approx 5 \times 10^{-5} \) at the boundaries. For the Hoskins and Bretherton (1972) front \( |y| \leq 18r \) is used. In this case the gradients of the basic state decay only as \( y^{-2} \), and the decay of the disturbance stream function as \( \exp(-k|y|) \) is also significant in determining the amplitude of the error induced by the finite domain size. This means that maintaining accuracy becomes increasingly expensive as the frontal width is reduced, since smaller scales must be resolved but the domain size does not shrink at a corresponding rate. At \( r = 0.0625 \), for instance, convergence to 3 significant figures is obtained with \( M = 160 \), compared with convergence to 5 significant figures at \( M = 80 \) with the Gaussian profile.

APPENDIX B

Algebraic methods

(a) Convolution of a Gaussian and a logarithm.

The Green's functions have a logarithmic singularity at the origin. This subsection shows how this singularity is smoothed out when it is convolved with a Gaussian weighting function. In order to evaluate fields within the frontal zone we consider:
\[
\mathcal{G} \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi r}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{(y - y')^2}{2r^2} \right\} \ln |y'| dy'
\]
for \( y \leq r \). The Gaussian can be approximated with a Taylor series expansion about \( y = 0 \), giving:
\[
\mathcal{G} = \frac{1}{\sqrt{2\pi r}} \int_{-y_r}^{y_r} \exp \left( -\frac{y'^2}{2r^2} \right) \left\{ 1 + \frac{yy'}{r^2} - \frac{y^2}{2r^2} + \frac{y^2y'^2}{2r^4} + \mathcal{O} \left( \frac{y^3y'^3}{r^6} \right) \right\} \ln |y'| dy'
\]
\[ + \mathcal{O} \left( \exp \left( -\frac{y^2}{2r^2} \right) \right), \]
where \( r \leq y_r \leq r^2/y \) is chosen to guarantee that both the error terms are small, one arising from truncating the integral and a second from the Taylor expansion within the integral.
These two conditions can only be fulfilled simultaneously if \( y \ll r \). After expansion the range of the integrand can be extended to infinity, again without changing the order of the error. The term linear in \( y \) vanishes identically on integration. For the remaining terms substituting \( \xi = y'/r \) gives:

\[
\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{\xi^2}{2} \right) \left\{ 1 - \frac{y^2}{2r^2} \left( 1 - \xi^2 \right) \right\} \ln r |\xi| d\xi.
\]

Evaluating the integrals gives:

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{(y - y')^2}{2r^2} \right) \ln |y'| dy' = \ln r + I_1 + \frac{y^2}{2r^2}, \tag{B.1}
\]

where \( I_1 = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \exp \left( -\xi^2/2 \right) \ln |\xi| d\xi \).

(b) Asymptotic forms of the Green’s functions.

The leading-order behaviour is given by the small \( y \) range of the integrand (because of the localized structure of \( \Lambda (y) \)). The results of appendix C in part I give the form of the Green’s function at small \( y \). The self-interaction terms (subscript 00) have a logarithmic singularity. The leading-order behaviour is:

\[
\tilde{G}_{00}(k; y) = -\frac{1}{4\pi} \left[ k - \frac{2}{\pi} \left\{ \ln \pi |y| - S_1 \left( \frac{k}{\pi} \right) \right\} \right] + \mathcal{O}(y^2)
\]

\[
\left( \tilde{G}_{00}(k; y) - \tilde{G}_{00}(0; y) - \tilde{G}_{01}(0; y) \right)^n
\]

\[
= -\frac{1}{4\pi} \left\{ k - \frac{\pi}{2} - \frac{k^2}{\pi} \left( \ln \pi |y| + \frac{1}{2} \right) + 2\pi S_2 \left( \frac{k}{\pi} \right) \right\} + \mathcal{O}(y^2). \tag{B.2}
\]

The cross interaction terms are regular. Evaluating at \( y = 0 \) gives:

\[
\tilde{G}_{01}(k; 0) = \frac{1}{4\pi} \left\{ \frac{1}{k} - \frac{2}{\pi} S_{-1} \left( \frac{k}{\pi} \right) \right\},
\]

\[
\tilde{G}_{01}^{\prime}(k; 0) = \frac{1}{4\pi} \left\{ k - \frac{\pi}{2} - 2\pi S_2 \left( \frac{k}{\pi} \right) \right\}. \tag{B.3}
\]

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