Linear evolution of baroclinic waves in saturated air

By MAURIZIO FANTINI*

FISBAT-CNR, Italy

(Received 5 November 1997; revised 22 June 1998)

SUMMARY

A primitive-equation model which includes a moisture parametrization that assumes that ascending air is always saturated, has been used to study the characteristics of moist baroclinic waves, exploring their dependence on the Richardson number, besides zonal and meridional wavelength. Growth rates are shown to increase, in consequence of latent-heat release, in a larger measure for the meridionally structured waves than for the two-dimensional ones. Approximate analytic relationships for the normal-modes growth rates, based on the concept of a weighted average of dry and saturated parameters, compare favourably with the numerical results. Asymmetric horizontal structures, previously obtained in quasi-geostrophic approximation, are shown to be consistent with potential-vorticity generation by diabatic heating in the updraught.

KEYWORDS: Baroclinic waves Moisture Primitive equations

1. INTRODUCTION

The role of latent-heat release has long been recognized in the formation of mesoscale features in extratropical cyclones, but a systematic study of how the presence of moisture in the atmosphere affects baroclinic development, from linear to mature stage, has been delayed by the requirement of large computational resources needed for the representation of moist processes and of the several scales of motion involved.

A simple, but effective, parametrization of the effects of latent-heat release devised by Emanuel et al. (1987) (hereafter referred to as EFT) consists in the use of a uniform, relatively large, value of static stability in the regions of downward motion, and of a different, near zero, value wherever there is ascending motion, accounting in this way for the enhanced buoyancy of an ascending parcel of air in the presence of latent-heat release.

While this representation assumes that ascending air is always saturated, and that a slantwise convective adjustment is taking place in the saturated air, it has been shown to produce the correct features of moist cyclones in simple models. It may also be considered a zero-order approximation when the above assumptions are not satisfied, revealing in essential form the modifications induced by the physical process of condensation of water vapour as an additional energy source present in the updraught of baroclinic cyclones.

This approach has also been used by Joly and Thorpe (1989), who extended the initial two-level EFT results by numerical integration of a two-dimensional (2D) semi-geostrophic (SG) model, and by Whitaker and Davis (1994) in a three-dimensional (3D) primitive-equation (PE) model which allowed for vertical variations of the moist stability parameter and a jet structure in the base wind. Montgomery and Farrell (1991, 1992) examined the growth of a moist baroclinic perturbation up to the finite amplitude stage in an SG 3D model. This author presented a normal-mode study in a quasi-geostrophic (QG) 3D model (Fantini 1995), which gave some peculiar results concerning the spatial structure of the solutions. These will be discussed in relation to the PE results of the present paper.

This paper presents a PE model which was formulated in order to get rid of the possible inconsistencies of the QG, and perhaps also of the SG, approximations. It will mainly be concerned with the normal-mode solutions obtained by long time integrations in the linear regime, comparing growth rates and spatial structures with dry integrations of the same model, which reproduce fairly accurately the analytic results of the Eady (1949)

* Corresponding author: FISBAT-CNR, via Gobetti 101, I-40129 Bologna, Italia.
model. Approximate expressions for the moist growth rates, fitted to the numerical results, will be given. Before this an outline of the model is given in section 2.

2. THE MODEL

The model integrates the primitive equations in an Eady-like set-up, i.e. Boussinesq approximation, f-plane, top rigid lid, uniform vertical shear \( \tilde{u}_z \), and incorporates the effects of release of latent heat, following EFT, by means of a static stability which depends on the sign of the vertical velocity \( w \).

The dimensional prognostic equations are:

\[
\begin{align*}
\dot{u} + (\bar{u} + \gamma u)u_x + \gamma vu_y + w(\bar{u}_z + \gamma u_z) - f_0 v &= -\tilde{\pi}_x + v \nabla^2_H u, \quad (1) \\
\dot{v} + (\bar{u} + \gamma u)v_x + \gamma vv_y + \gamma wv_z + f_0 u &= -\tilde{\pi}_y + v \nabla^2_H v, \quad (2) \\
\dot{\theta} + (\bar{u} + \gamma u)\theta_x + v(\bar{\theta}_y + \gamma \theta_y) + w(r^* + \gamma \theta^*_z) &= v \nabla^2_H \theta, \quad (3)
\end{align*}
\]

where \( \gamma = c_p \theta_0 \), and a Laplacian horizontal diffusion was added for numerical convenience. Symbols are defined in Table 1. Subscripts \( t, x, y \) and \( z \) denote \( \partial / \partial t, \partial / \partial x, \partial / \partial y \) and \( \partial / \partial z \), respectively. The overbar denotes basic-state variables, and all perturbations, except \( \pi \), are divided by \( \gamma \). In this way \( \gamma \) becomes a marker for the nonlinear terms in the equations, i.e. its value is set to zero for the linear runs and to \( c_p \theta_0 \) for the nonlinear runs.

The coefficient \( r^* \), which appears in the buoyancy term of the thermodynamic equation (3) includes the effect of latent-heat release. The use of a piecewise constant static stability, function of the sign of \( w \), to represent the effects of latent-heat release on the large-scale flow was adopted by EFT, and in a number of subsequent papers, on the basis of a slantwise convective adjustment taking place in the saturated air, and reducing the lapse rate to moist symmetric neutrality. This lapse rate, much smaller than the dry one, provides the relevant static stability for ascending motion, which is assumed to be always saturated, accounting for the enhanced buoyancy generated by the release of latent heat. The thermodynamic

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity and Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_p )</td>
<td>Specific heat at constant pressure</td>
</tr>
<tr>
<td>( g )</td>
<td>Acceleration due to gravity</td>
</tr>
<tr>
<td>( L )</td>
<td>Latent heat of vaporization</td>
</tr>
<tr>
<td>( R )</td>
<td>Gas constant for dry air = 287 J kg(^{-1})K(^{-1})</td>
</tr>
<tr>
<td>( p )</td>
<td>Pressure perturbation</td>
</tr>
<tr>
<td>( q^*_v )</td>
<td>Saturation mixing ratio of water vapour</td>
</tr>
<tr>
<td>( c_p \theta_0 u )</td>
<td>Perturbation zonal wind</td>
</tr>
<tr>
<td>( c_p \theta_0 w )</td>
<td>Perturbation meridional wind</td>
</tr>
<tr>
<td>( c_p \theta_0 )</td>
<td>Perturbation vertical wind</td>
</tr>
<tr>
<td>( c_p \theta )</td>
<td>Perturbation potential temperature</td>
</tr>
<tr>
<td>( \bar{u} )</td>
<td>Base-state zonal wind = ( \bar{u}_z )</td>
</tr>
<tr>
<td>( \tilde{\pi} )</td>
<td>Base-state non-dimensional pressure = 1 - (( g / c_p \theta_0 ))z - (( f_0 / c_p \theta_0 ))( \bar{u} ) ( y )</td>
</tr>
<tr>
<td>( \tilde{\pi}^*_v )</td>
<td>Non-dimensional pressure perturbation = (( p / p_0 ))^R/cp ( \tilde{\pi} )</td>
</tr>
<tr>
<td>( \theta^*_\gamma )</td>
<td>Buoyancy perturbation—defined in (8)</td>
</tr>
<tr>
<td>( \theta^*_\theta )</td>
<td>Base-state potential temperature = ( \theta_0 + \theta_z ) - (( \theta_0 / g ))f_0( \bar{u} )_z ( y )</td>
</tr>
<tr>
<td>( \theta_{equ} )</td>
<td>Perturbation equivalent potential temperature = ( \theta + (L / c_p \tilde{\pi})q^*_v )</td>
</tr>
<tr>
<td>0 &lt; ( z &lt; H )</td>
<td>Vertical size of the model domain</td>
</tr>
<tr>
<td>0 &lt; ( x &lt; X^* )</td>
<td>Zonal size of the model domain</td>
</tr>
<tr>
<td>0 &lt; ( y &lt; Y^* )</td>
<td>Meridional size of the model domain</td>
</tr>
</tbody>
</table>
equations which include this effect were derived in the above mentioned papers in either the quasi-geostrophic or the geostrophic momentum approximations as:

\[(\partial_t + u \partial_x + v \partial_y) \theta + v \tilde{\theta}_y + w \tilde{\theta}_z = 0\]  \hfill (4)

for descending motion, where \(\partial_t, \partial_x\) and \(\partial_y\) denote \(\partial/\partial t, \partial/\partial x\) and \(\partial/\partial y\) respectively, and

\[(\partial_t + u \partial_x + v \partial_y) \theta_z + v \tilde{\theta}_{zy} + w \tilde{\theta}_{zz} = 0\]  \hfill (5)

for ascending, saturated motion. Then (5) is transformed using first the relationship \((\Gamma_w/\theta_0) (d\tilde{\theta}_e)_z \approx (\Gamma_d/\theta_0) (d\theta)_z\) (see Emanuel 1986), where \(\Gamma_d\) and \(\Gamma_m\) are the dry and moist adiabatic lapse rates, respectively, and the differentials of \(\theta\) and \(\theta_e\) are taken at constant \(z\). Secondly the assumption \(\tilde{\theta}_e = 0\) is used to evaluate \(\tilde{\theta}_{ez}\), to

\[(\partial_t + u \partial_x + v \partial_y) \theta + v \tilde{\theta}_y + (w \theta_0/g) \tilde{u}_z^2 = 0\]  \hfill (6)

(see Fantini 1995), so that the piecewise constant coefficient \(r^*\) appearing in (3) is defined as

\[r^* = \begin{cases} \tilde{\theta}_z, & \text{where } w < 0, \\ (\theta_0/g) \tilde{u}_z^2, & \text{where } w > 0. \end{cases} \]  \hfill (7)

In dealing with the PE the extra term \(w \theta_z\) appears in the thermodynamic equation for dry air, and \(w \theta_{ez}\) for moist air in which release of latent heat occurs. The latter can be fully evaluated from the definition of equivalent potential temperature

\[\theta_e = \theta + \frac{L}{c_p \pi} q_v^s,\]

since the saturation mixing ratio of water vapour \(q_v^s(\theta, \pi)\) is a function of the two known variables \(\theta\) and \(\pi\).

We have chosen this explicit formulation of the perturbed buoyancy term, rather than using again the assumption of slantwise convective neutrality. The latter would restrict the study to perturbations having zero equivalent potential vorticity \((q_e' = 0)\), which seems a priori a rather arbitrary assumption, as the slantwise convective adjustment would not have the time to act on the scale of the growth of baroclinic perturbations. Therefore we use

\[\theta^*_z = \begin{cases} \theta'_z, & \text{where } w < 0, \\ \theta'_z + \frac{L}{c_p \pi} q_v^{s'}, & \text{where } w > 0. \end{cases} \]  \hfill (8)

Two diagnostic equations complete the system:

\[u_x + v_y + w_z = 0,\]  \hfill (9)

\[\frac{g}{\theta_0} \theta = \tilde{\pi}_z.\]  \hfill (10)

All linearized experiments were initialized with an analytically defined perturbation, exact solution to the QG Eady problem. Details on the numerics can be found in the appendix.
Figure 1. (a) Dimensional growth rate $\sigma^*$ versus wavelength $X^*$ for two-dimensional waves: dry (lower curves) and moist (upper) modes are shown with three values of Richardson number: $Ri = 5.75$ (curves 1 and 2), $Ri = 11.5$ (curves 3 and 4) and $Ri = 23$ (curves 5 and 6). (b) As (a) except non-dimensional quantities $\sigma$ and $X$. (c) Upper curves: same as (b); small dots: approximation to moist growth rates as from (17); lower curves: dry growth rates transformed as from (20); large dots: Eady growth rates. (d) All curves: same as (c); small squares: approximation given in (18); asterisks: Stone’s approximation in the quasi-geostrophic limit, (21). See text for further explanation.

3. Linear results

(a) Dispersion relation of two-dimensional waves

The results concerning the dispersion relation of two-dimensional waves are summarized in Fig.1. Figure 1(a) displays the dimensional growth rates for three different values of the Richardson number $Ri$, obtained by changing the value of the vertical shear, in the dry case and in the case of saturated upward motion, represented by the ratio of the two terms in (7):

$$ r = \frac{\theta_0 \bar{u}_z^2}{g \frac{\partial \theta}{\partial z}} = \frac{1}{Ri}.$$

(11)
The dimensional values are here shown to emphasize the measure of the dimensional gain in growth rate and destabilization of the short waves induced by latent-heat release, which is partly hidden by a scaling proportional to the Richardson number. We then show, in Fig. 1(b) the non-dimensional values of the same quantities, defined as

\[ \sigma = \sigma^* \frac{NH}{Uf_0}, \quad X = X^* \frac{f_0}{NH}, \]  

(12)

where \( N \) is the Brunt–Väisälä frequency and \( U \) is the zonal wind at the top of the model and scale of horizontal velocity. The non-dimensional curves are qualitatively similar to those obtained by Fantini (1990) from a two-level model. The long-wave side of the curves terminates at about twice the most unstable wavelength. In any domain longer than that, the most unstable mode grows faster than the imposed wavelength and the lower growth rate of the long wave is thus impossible to determine.

An approximate relationship between the dry and the saturated growth rates can be derived by noticing (e.g. from Fig. 4) that the moist wave is composed of two distinct regions (upward and downward motion, respectively) of well-defined relative width \( \lambda \). Each region appears to scale with its own Richardson number, and the original EFT paper was successful in obtaining the complete spatial structure of the normal modes, in a two-dimensional semi-geostrophic approximation, by matching at the lines of contact the ‘interior’ solutions of separate Eady problems for each one of the two regions. We now extend this concept to the growth rates, assuming that in each region the conversion of available potential energy proceeds according to the results of the Eady model scaled with the environmental parameters of that region, which then contributes to the growth of the entire mode proportionally to its width. Taking for \( \sigma \) a weighted average of the growth rates of the two separate regions, and assuming that they are inversely proportional to the width of the regions themselves, i.e. \( \sigma \sim U/\mathcal{L} \), where \( \mathcal{L} \) is the horizontal scale of motion, we write

\[ \sigma \approx \frac{2}{1 + \lambda} \sigma_{\text{dry}}. \]  

(13)

A dependence of \( \lambda \) on \( r^{1/2} \), for small \( r \), was suggested for the most unstable mode in previous papers. We use it for the whole range of wavelengths explored, with a bit of fitting which gives best results for

\[ \lambda \approx \frac{1}{10} + \frac{4}{3} r^{1/2}. \]  

(14)

The small dots in Fig. 1(c) were obtained from (13) and (14) using the Eady result for \( \sigma_{\text{dry}} \), which, appropriately rescaled, reads

\[ \sigma_{\text{dry}} = \frac{1}{4} \left[ \left( \frac{\pi}{X} - \tanh \frac{\pi}{X} \right) \left( \coth \frac{\pi}{X} - \frac{\pi}{X} \right) \right]^{1/2}. \]  

(15)

Letting

\[ Q = \frac{1 + \lambda}{2}, \]  

(16)

the complete approximation is thus

\[ \sigma \approx \frac{1}{4Q} \left[ \left( \frac{\pi Q}{X} - \tanh \left( \frac{\pi Q}{X} \right) \right) \left( \coth \left( \frac{\pi Q}{X} \right) - \frac{\pi Q}{X} \right) \right]^{1/2}. \]  

(17)
Although reasonably good, the approximation so obtained for the moist growth rates is too low for waves longer than the most unstable. This highlights a feature of the moist-growth-rate curves already remarked upon in previous papers, i.e. that they appear to tend to some finite value as \( X \to \infty \), while the dry curves go to zero in that same limit. The small squares in Fig. 1(d) show a different approximation that tries to take this into account:

\[
\sigma \approx \frac{0.28}{Q} \left\{ 1 - \frac{4 \sin (X/Q)}{\sin (X/Q) + \sinh (X/Q)} \right\}.
\]

(18)

Here the functional dependence on \( X \) was deduced as an approximate solution to the equation (C4) of EFT, while the numeric coefficients were fitted to the curves. Since these appear to exceed the model curves in magnitude, a better fit in the long-wave range could easily be obtained by averaging (17) and (18). However, our main purpose here was to show the robustness of (13) as a starting point for estimates of the growth rate of the moist normal modes. In the next section we will apply similar considerations to the three-dimensional waves.

To verify the model accuracy as far as the dry modes are concerned we compare the model results with the well-known Eady dispersion relation (15). For this purpose we wish to ‘remove’, so to speak, the PE character of the numerical results, which we do by referring to the approximate solution given by Stone (1966). His equation (4.25) rescaled as from (12) reads

\[
\sigma \approx \frac{1}{\sqrt{3}} \frac{2\pi}{X} \left\{ \frac{1}{2} - \frac{1}{15} \left( 1 + \frac{1}{Ri} \right) \left( \frac{2\pi}{X} \right)^2 \right\}.
\]

(19)

The transformation

\[
\sigma' = \sigma \ D, \quad X' = \frac{X}{D}, \quad D \equiv \left( 1 + \frac{1}{Ri} \right)^{1/2},
\]

(20)

turns this into

\[
\sigma' \approx \frac{1}{\sqrt{3}} \frac{2\pi}{X'} \left\{ \frac{1}{2} - \frac{1}{15} \left( \frac{2\pi}{X'} \right)^2 \right\},
\]

(21)

which has no dependence on the Richardson number and is equal to the quasi-geostrophic limit of (19). We then apply (20) to the dry curves of Fig. 1(b) to obtain the curves shown in Figs. 1(c) and (d) (the lower group of curves) compared with (15) and (21) respectively.

(b) Dispersion relation of three-dimensional waves

Figures 2 and 3 show how the growth rate changes when a meridional structure is introduced, in the form of waves of 6000 and 4000 km meridional length, for various values of the zonal wavelength, and for the three values of the Richardson number considered in Fig. 1.

Figure 2 displays values scaled as per (12). In all cases the growth rate decreases as the meridional wave number increases. This difference is seen to become smaller for large \( X \) and would appear to become negligible if the character of the curves, representing modes with a meridional structure, to be flatter than the ones above them is maintained for zonal wavelengths longer than those shown. As mentioned above, this range cannot be explored with the present method.

The gain in growth rate obtained in going from dry to moist environment is noticeably larger for waves with a meridional structure than for the two-dimensional waves. For
Figure 2. (a) Non-dimensional growth rate $\sigma$ versus zonal wavelength $X$ at Richardson number $Ri = 5.75$ for dry and moist modes with infinite, 6000 and 4000 km meridional wavelength. The curves referring to the two-dimensional cases are labelled as in Fig. 1. The remaining curves are labelled as above with suffixes 6 and 4 referring to the 6000 and 4000 km case, respectively. (b) and (c) As (a) but for $Ri = 11.5$ and 23, respectively.

Figure 3. Same as Fig. 2 except that the transformation (28) was applied to the dry results (lower group of curves), and (28) plus (29) to the moist values (upper group of curves).

Example, for the 4000 km meridional wavelength, the largest moist growth rate appears to be between two and three times larger than the dry one, and the most unstable zonal wavelength about half the dry one. These are near the estimates that were given by EFT for the two-dimensional waves in their SG two-level model, and were successively reduced by consideration of multi-level models (Joly and Thorpe 1989) and primitive-equation dynamics (Fantini 1993).
The applicability of the concept of growth-rate averaging that we introduced in the previous subsection will be tested here on the basis of numerical results for the dry waves, rather than analytic expressions as was done previously. In order to extend the transformation (20) to include non-zero meridional wave numbers, we turn again to the results of Stone (1966), retaining only the terms $k^2$ and $\lambda^2$ in his equation (4.4). This expression is then rescaled as per (12) to give

$$\sigma \approx \frac{1}{\sqrt{3}} \frac{2\pi}{X} \left[ \frac{1}{2} - \frac{1}{15} \left\{ \left( 1 + \frac{1}{Ri} \right) \left( \frac{2\pi}{X} \right)^2 + \left( 1 - \frac{1}{Ri} \right) \left( \frac{2\pi}{Y} \right)^2 \right\} \right].$$  \hspace{1cm} (22)

Applying to the above the transformation (20) for the zonal wave numbers, and

$$Y' = \frac{Y}{E}, \quad E \equiv \left( 1 - \frac{1}{Ri} \right)^{1/2}$$  \hspace{1cm} (23)

for the meridional components, we obtain

$$\sigma' \approx \frac{1}{\sqrt{3}} \frac{2\pi}{X'} \left[ \frac{1}{2} - \frac{1}{15} \left\{ \left( \frac{2\pi}{X'} \right)^2 + \left( \frac{2\pi}{Y'} \right)^2 \right\} \right].$$  \hspace{1cm} (24)

Also, as with any dispersion relation of the kind

$$\sigma = \frac{1}{X} c \left[ \frac{1}{X^2} + \frac{1}{Y^2} \right],$$  \hspace{1cm} (25)

where $c$ is a generic function of $[\cdot]$, we can use

$$\sigma' = \sigma M, \quad X' = \frac{X}{M}, \quad M \equiv \left( 1 + \frac{X^2}{Y^2} \right)^{1/2}$$  \hspace{1cm} (26)

to reduce (25) to

$$\sigma' = \frac{1}{X'} c \left( \frac{1}{X'^2} \right).$$  \hspace{1cm} (27)

The combination of (20), (23) and (26) gives

$$\sigma' = \sigma P, \quad X' = \frac{X}{P}, \quad P \equiv \left\{ \left( 1 + \frac{1}{Ri} \right) + \frac{X^2}{Y^2} \left( 1 - \frac{1}{Ri} \right) \right\}^{1/2}.$$  \hspace{1cm} (28)

Application of (28) to the dry-model results makes them collapse to a single curve—the QG 2D result, as seen in Fig. (3) (lower group of curves). By the same principle adopted above, we also apply the transformation (28) to the moist curves, but everywhere use the weighted average of the Richardson numbers $Ri_{eff}$

$$\frac{1}{Ri_{eff}} = \frac{1}{Ri} \frac{1 + Ri \lambda}{1 + \lambda}.$$  \hspace{1cm} (29)

The collapse of the moist curves shown in the same Fig. (3) is imperfect, and reasonably good only for the short waves, up to the most unstable one. Whether this is due to the roughness of the above considerations or to numerical errors in the simulation of the longer waves, or in other words if it is our numerical results or the above transformations that are closer to reality (not to mention how close), we are unable to say at this point.
Figure 4. (a)–(f) Horizontal view of (a) and (d) perturbation pressure, (b) and (e) potential temperature and (c) and (f) vertical velocity, at (a)–(c) midlevel (5 km) and (d)–(f) lowest internal level (455 m) for the saturated wave at \( X^* = 2000 \) km and \( Y^* = 6000 \) km and for \( Ri = 11.5 \). (g)–(j) Vertical \((x-z)\) cross-section at \( y = Y^*/4 \) of (g) meridional wind, (h) perturbation potential temperature, (i) vertical velocity and (j) perturbation pressure for the same wave. Dashed contours represent negative values. See text for further explanation.
Figure 5. Same as Fig. 4 except for wave of zonal length $X^* = 3200$ km (most unstable). Horizontal size of figure adjusted proportionally to Fig. 4.
Figure 6. Same as Fig. 4 except for wave of zonal length $X^* = 4400$ km. Horizontal size of figure adjusted proportionally to Fig. 4.
(c) **Spatial structure of three-dimensional waves**

Figures 4, 5 and 6 show the spatial structure for waves of zonal length 2000 km (shortest unstable wave), 3200 km (most unstable) and 4400 km (longest) and meridional length 6000 km for $Ri = 11.5$ (the intermediate case).

The surface wave appears deformed in the sense described also in the QG model (Fantini 1995), i.e. the shortest wave extending in the zonal direction and the longest one extending in the meridional direction. The latter leads, when waves longer than twice the most unstable one are considered, to the appearance of a double wave, as the updraught regions keep extending and eventually join each other (not shown). This behaviour may lead to the suspicion that this mechanism is at work even in the case of Fig. 6. Although we cannot rule this out with certainty, model runs of 600 simulated hours have failed to show any change of spatial structure or growth rate of the mode of Fig. 6 or shorter. On the other hand Fig. 2 shows how close to each other the numerically obtained curves of growth rate for different meridional structures become, in the saturated case, for long zonal wavelengths, so that even longer runs might be required to discriminate the desired ones.

The vertical cross-sections of the normal modes (shown in Figs. 4(g)–(j), 5(g)–(j) and 6(g)–(j)) seem to imply that the updraught region keeps more or less the same width, while the rest of the perturbation is redistributed on what is left of the domain. Moreover the updraught width seems dependent only on the environmental Richardson number, as shown in Fig. 7 for all the experiments discussed so far. In most cases the differences are within the model (zonal) resolution, which is shown by the bars at the lowest edge—the model is run with a fixed number of grid points (64) so the resolution changes with the width of the domain. The result of an updraught width independent of the wavelength would imply

$$\lambda \approx \mathcal{L}^*/X^*, \quad (30)$$

where $\mathcal{L}^*$ is a constant with dimensions of length representing a typical updraught width. This is not inconsistent with the constant $\lambda$ used in the considerations of the previous subsections, as $\mathcal{L}^*/X^*$ is small enough to be nearly constant in the range of $X$ considered. It is possible, however, that (30) could be used to improve the estimates (17) and/or (18) of the growth rate of moist modes.

An understanding of the peculiar shape of the shortest unstable modes may come from the consideration that, in the linearized version of the equations of motion, the potential vorticity (PV) generated in the updraught part of the wave decays away from there on a horizontal scale $\tilde{u}/\sigma$. With growth rates from 2 to $5 \times 10^{-6}$ s$^{-1}$ (see, for example, Fig. 2) and $\tilde{u} \approx 10$ m s$^{-1}$ this gives a horizontal scale between 2000 and 5000 km, i.e. longer than the wave itself, so that the effect of the positive (negative) PV anomaly generated in the updraught at low (high) levels is felt horizontally throughout the wave.

Figure 8 shows cross-sections of linearized PV

$$q = f_0 \theta_z + (u_z - u_y) \tilde{\theta}_z + \tilde{u}_z \theta_y + \tilde{\theta}_y u_z \quad (31)$$

for the modes of Figs. (4), (5) and (6), i.e. at $Ri = 11.5$ and $Y = 6000$ km for $X = 2000$, 3200 and 4400 km, respectively, at a meridional location chosen in the middle of an updraught. The marks on the bottom indicate the position of the updraught at midlevel. Maximum positive (negative) PV generation is seen to coincide with the rear (forward) edge of the updraught itself, following its slight upshear tilt. Decay scale goes like $\tilde{u}(z)$, i.e. linearly with height. The two longer modes appear to be rather similar, because of their similar growth rates (see Fig. 3(a)), while the shortest mode exhibits a longer decay scale of the PV anomaly, due to its smaller $\sigma$. 
Referring to the most unstable of the above modes ($X^* = 3200$ km—Fig. 5) we notice the differences from the most unstable dry mode at the same $Ri$ and $Y$ ($X^* = 4400$ km—Fig. 9) in the narrow updraught, namely stronger vorticity in the cold-frontal area, thermal perturbation centred at midlevel where diabatic heating is a maximum instead of at the boundaries as in the dry solution, all of which also belong to the 2D case. A comparison of dry and moist north–south cross-sections through the centre of the updraught (Fig. 10) shows that no contraction of the updraught occurs in the meridional direction. Both the dry and the moist modes exhibit the northward tilt with height that is already included in SG dynamics—the geostrophic coordinate transformation (Hoskins 1975) from an upright QG solution would give a tilt of $\tilde{u}_z H/f_0 = 300$ km over the height of the model, against the 270 km estimated graphically from Fig. 10.

4. Conclusion

Although in recent years observational studies have revealed an average tropospheric state of nearly uniform PV, despite the existence of a planetary vorticity gradient, thus
enhancing the status of Eady's model among the classical models of baroclinic instability, the gap between such an idealized study as the present one and the observed behaviour of midlatitude cyclones is wide. The reason for its continued use is its ability to describe a complex physical process in a most elementary form, providing a bridge to more refined representations. On this basis we have designed the present model, which, by integrating the primitive equations in a three-dimensional domain, avoids the possible inconsistency problems of previous studies by this and other authors mentioned in section 1, which were either dynamically approximate (QG, SG) or two-dimensional. Whitaker and Davis (1994) have already used a 3D PE model very similar to the present one, to study a jet-like base state. We were more interested, as a first step, in obtaining results with the uniform wind shear and uniform stratification of Eady's model for an easier comparison with earlier numerical and analytical results, although the future extension of this work to life-cycle studies will of necessity take into consideration a more structured base state.

The current emphasis in idealized studies is on transient growth events (Farrell 1984; Davies and Bishop 1994) rather than individual exponentially growing modes which would
not have the time, nor presumably find the uniformity of environmental conditions, to appear out of the background flow; but it has also been shown (Rotunno and Fantini 1989; Rotunno and Bao 1996) that the same kind of behaviour can be obtained by superposition of discrete spectrum modes. The presence of water vapour near saturation, besides baroclinicity, in the troposphere, might be thought of as inducing a mixture of the two families of solutions just mentioned, in that the release of latent heat produces a PV anomaly advected by the mean wind, therefore projecting on the continuous spectrum of Eady modes while at the same time modifying the discrete modes. In actuality the presence of a phase transition in the process makes it intrinsically nonlinear—at least one of the terms of the equations of motion is discontinuous, and at no amplitude is there a superposition principle to provide a physical argument in favour of exponentially growing solutions, which is the sole meaning we can give the term 'normal mode' in this context. On the other hand there is a correspondence between each dry normal mode, in the proper sense, and the moist solution into which it evolves in a time integration like those we have performed, and this is what we have explored in the present paper.

ACKNOWLEDGEMENTS

This work was supported by the Programma Nazionale di Ricerche in Antartide. Additional computer resources were provided by a SuperComputing Grant of the CINECA Computer Centre.

APPENDIX

Equations (1), (2), and (3) are integrated in time with a leap-frog scheme on a B-grid in a doubly periodic domain. Only the vertical velocity \( w \) is known on the top and bottom boundaries, and it is vertically aligned with the \( \theta \) points. All other variables are at intermediate levels between two \( u \) levels.

Since we are working with a top rigid lid the hydrostatic relation (10) is not sufficient to determine \( \bar{\pi} \) everywhere. A boundary value for \( \bar{\pi} \) is needed either at the top or the
bottom. We then derive an extra diagnostic equation for the column-integrated value of non-dimensional pressure

\[ \tilde{\Pi} \equiv \int_0^h \tilde{\pi} \, dz \]

(as was done e.g. by Williams (1967) in a two-dimensional context) as follows.

From the momentum equations, which we rewrite as:

\[ u_i = -\tilde{\pi}_x + F_{[u]} \]

\[ v_i = -\tilde{\pi}_y + F_{[v]} \]

(A.1)

(A.2)
we form the horizontal divergence, and use the continuity equation (9), to get:

\[ -w_x = -\nabla_H^2 \tilde{\pi} + (F_{(u)})_x + (F_{(v)})_y. \]  

(A.3)

Then, integration on a column leads to:

\[ \nabla_H^2 \tilde{\Pi} = \int_0^H \text{d}z \left\{ (F_{(u)})_x + (F_{(v)})_y \right\} \equiv \mathcal{F}. \]  

(A.4)

Or, by making explicit the integral on the right-hand side, and performing integration by
parts where appropriate:
\[
\nabla_H^2 \tilde{\Pi} = \int_0^H dz \left\{ f_0(v_x - u_y) - 2(v_x \tilde{u}_y + w_x \tilde{u}_z) - \gamma (u_x^2 + v_y^2 + w_z^2) - 2\gamma (v_x u_y + w_x u_z + w_y v_z) \right\}.
\]

(A.5)

However, (A.5) is never needed, as the model computes \( F_{[s]} \) and \( F_{[v]} \) to advance (1) and (2), and those quantities are then differentiated numerically for use in (A.4). Therefore, after time-stepping (1) to (3), we invert the horizontal Laplacian in (A.4) to obtain the column-average value of \( \tilde{\Pi} \). This provides the needed condition on \( \tilde{\Pi} \) for (10).

Finally, \( w \) is obtained from (downward) vertical integration of (9), using the condition \( w = 0 \) at the top boundary. In doing this, the numerical errors of the whole procedure accumulate on the value of \( w \) at the bottom boundary, which becomes non-zero. Taking this fact into account, the column integral of (A.3) actually reads
\[
\nabla_H^2 \tilde{\Pi} = \mathcal{F} - w_l(z = 0),
\]

(A.6)
or, with the leap-frog time discretization,
\[
\nabla_H^2 \tilde{\Pi}^{(0)} = \mathcal{F}^{(0)} - \frac{w^{(+)}(z = 0) - w^{(-)}(z = 0)}{2\Delta t},
\]

(A.7)
with the superscripts now indicating time level. Forcing \( w^{(+)}(0) \) to vanish, we obtain
\[
\nabla_H^2 \tilde{\Pi} = \mathcal{F} + \frac{w^{(-)}(z = 0)}{2\Delta t}.
\]

(A.8)
The extra term on the right-hand side performs a damping of numerical errors, and would vanish in the ideal situation of infinite precision. Practically, if we use (A.4) without the damping term the accumulated errors at the far boundary in the vertical integration of the continuity equation appear to grow catastrophically in the course of the time integration.

REFERENCES

Eady, E. T. 1949 Long waves and cyclone waves. Tellus, 1, 33–52
1995 Moist Eady waves in a quasi geostrophic three dimensional model. J. Atmos. Sci., 52, 2473–2485


