Passage of a tracer through frontal zones: A model for the formation of forward-sloping cold fronts

By D. J. PARKER*
University of Leeds, UK

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SUMMARY

It is shown that a forward-sloping cold front may form in a simple two-dimensional model of an atmospheric frontal zone, through differential advection of dry- and wet-bulb potential temperature. As such, two frontal zones may coexist, with opposing slopes. This behaviour is manifest in the neutral Eady edge wave, which exhibits kata-cold front behaviour, while the unstable Eady wave, with an ana-cold front, does not generate a forward-sloping cold front.

These results come about as a consequence of the wave-like behaviour of the fronts in the model, that is, the ability of a front to propagate through the air locally. As such, they highlight the fact that a frontal zone need not be a barrier to airflow and that any air properties, from water vapour to atmospheric pollution, may pass through a front.

KEYWORDS: Cold fronts  Eady wave  Split fronts

1. INTRODUCTION

From the early days of study of atmospheric fronts, the importance of their vertical structure on the associated weather patterns has been recognized. Until recent years, it has been held that the slope of a front with height is such that the warm air overlies the cold. Dynamical models suggest that this is inevitable for intense fronts. Margules’s (1906) formula shows that for the simple two-layer ‘air-mass’ model, the contrary case of a ‘forward-sloping cold front’—a front overlying the warmer air—would imply negative relative vorticity at the surface front, which is not observed. In a more complete frontal model, Hoskins and Bretherton (1972) showed that the limiting frontal slope, which occurs as a singularity is reached at the surface, overlies the cooler air, and that such fronts are inevitably associated with high relative vorticity.

This being said, given that a front is not in fact a mathematical singularity, the basic static stability of the troposphere means that it is perfectly possible to sketch a frontal zone which tilts above the warm air, without being statically unstable, and it is only the dynamical evolution of frontogenesis which precludes such an occurrence in models. From an observational perspective, Browning and Monk (1982) have shown that forward-sloping cold fronts do exist and are common over the British Isles, relating them to ‘split fronts’, where an ‘upper cold front’ runs ahead of the surface cold front, leading to a split in the associated weather patterns. Browning and Monk’s (1982) analysis, based on the wet-bulb potential temperature (\(\theta_w\), homeomorphic to equivalent potential temperature, \(\theta_e\), which will be used here) structure of the frontal zone, showed low \(\theta_e\) protruding into a region of higher \(\theta_e\) ahead of the cold front, leading to convective instability below this region. There have since been various observational descriptions of split fronts, and related phenomena (e.g. Browning and Golding 1995; Browning et al. 1995; Bader et al. 1995; Browning and Roberts 1996), and split fronts have been closely related to the idea of a ‘dry intrusion’ (Young et al. 1987), so named because the low \(\theta_e\) air intruding the warmer air at the cold front is recognizable as a clear, dry slot in the satellite water-vapour imagery. Split fronts and dry intrusions are associated with significant weather patterns, at the position of the upper cold front, notably intense convection, squall lines and tornadogenesis (Browning and Golding 1995).

* Corresponding address: The Environment Centre, University of Leeds, Leeds, Yorkshire LS2 9JT, UK.
The leading edge of a dry intrusion forms the upper cold front: a related concept is the ‘cold front aloft’, a phenomenon of the warm occlusion, whereby the elevated cold front rising up the warm-frontal surface may give rise to significant convergence at the ground (Locatelli et al. 1997). However, the cold front aloft is a feature of the traditional ‘Norwegian’ model, with warmer air overlying cooler air at each frontal surface, or zone, and is not the phenomenon addressed here.

MacVean and Woods (1980) noted that fronts diagnosed in a passive-tracer field (temperature in the ocean) tend to be more intense than those diagnosed by a buoyancy variable, so that observations of the tracer field may not reflect the behaviour of the dynamical front. In the following sections of this paper, it is proposed that forward-sloping cold fronts, of which split fronts are a manifestation, are observed in the atmosphere as a result of assessing the frontal position in terms of $\theta_e$, which may be advected through a conventional (warm-above-cool-air) thermal zone. Thus, there is no discrepancy between the dynamical models of fronts and the observations of split fronts: the two kinds of front may coexist, albeit in different fields, with opposing orientations. The second section reviews models of two-dimensional fronts and their relation to the forward-sloping cold front, section 3 discusses the evolution of $\theta_e$ in such systems, and section 4 demonstrates how a forward-sloping cold front may be generated by tracer advection in a steady shear front. The final section is a summary of these ideas, with some comments on the likely role of moist convection in these fronts.

2. MODELS OF FRONTAL DYNAMICS

There are several models of two-dimensional atmospheric fronts, which are applicable in different contexts (see Smith and Reeder 1988). For the purposes of describing forward-sloping cold fronts, two of these may be rejected immediately. The gravity-current model tends to be applied in situations where there is rapid diabatic heating or cooling, principal examples being the sea-breeze front (e.g. Simpson and Britter 1980) and cold-pool fronts driven by evaporative cooling, at squall lines or cold fronts, where moist convection is active (e.g. Thorpe et al. 1980). The gravity current is essentially a mesoscale ‘air-mass’ front, in that the stability of the mixed layer above the following flow inhibits exchange between the air of differing potential temperature behind the head, and this model does not exhibit forward-sloping front behaviour (except for a raised ‘nose’, typically at around 15% of the head height, say 200 m altitude; Simpson and Britter 1979). The two-layer air-mass model (e.g. Margules 1906; Ball 1960; Manton 1981) is somewhat out of favour as a quantitative tool (Smith 1990), but remains in many ways a good model of surface fronts. Margules’s (1906) original frontal-slope relation indicates that in such a model there is no possibility of the frontal discontinuity sloping above the warm air without negative relative vorticity at the front (which is not observed), and that furthermore there can be no transfer of properties across the frontal surface (a process which will be invoked subsequently). Consequently, the gravity-current and air-mass models will not be considered further.

Since the 1940s, synoptic fronts have increasingly been seen as dynamical, rather than physical, phenomena, resulting from the evolution of the larger-scale flows. In particular, frontal zones are seen to evolve in the canonical Charney (1947) and Eady (1949) models of baroclinic instability, as well as in primitive-equation life-cycle experiments (e.g. Simmons and Hoskins 1978). It was the development of the semi-geostrophic (SG) theory of frontogenesis (Hoskins and Bretherton 1972; Hoskins 1975) that showed how singular fronts could form in such systems, by the action of ‘ageostrophic’ advection,
and this has remained the cornerstone of understanding of frontal dynamics. The SG description of fronts describes their formation within synoptic systems, as well as exhibiting observed features of fronts, such as a more diffuse frontal zone, and the possibility of frontal propagation relative to the local flow.

The first front discussed in the SG system is the 'deformation front', formed by the action of a deformation, or strain flow, to intensify a pre-existing thermal gradient. In the SG scheme, the front is a dynamical feature, which may propagate within the flow, but the action of the deformation is always to align advected features along the frontal zone. MacVean and Woods (1980) studied SG deformation models for oceanic frontogenesis, in which the temperature fields are not necessarily coincident with density (as a result of salinity effects). In their solutions, the thermal front was invariably more intense than the density front, particularly at greater depth, but retained the basic slope of the baroclinic zone. Similarly, Banic et al. (1984) simulated aerosol evolution at an SG deformation front and demonstrated that reversed vertical gradients of a tracer could be obtained in a shallow zone beneath the frontal surface. (Such reversals are also apparent in some of MacVean and Woods's (1980) simulations.) Again, in this case, the features were aligned along the thermal cold front. In conclusion, it seems that deformation fronts may not represent forward-sloping cold fronts (without exceptional environmental states mentioned briefly in the final section).

The second, and arguably more interesting form of SG front is that which is formed by horizontal shear of a large-scale thermal gradient. This is typified by the kinds of front which evolve in baroclinic waves (e.g. Charney and Eady waves) growing on the midlatitude planetary temperature gradient, or 'polar front'. Such fronts develop as an intensification of the thermal gradients in a propagating wave, and as such, generally* have speeds differing from (greater than) the local surface wind speed. It is the ability of these fronts to propagate that allows air to pass through the frontal zone (strictly, the frontal zone is a wave which propagates through the air). This behaviour has been discussed by Reeder and Smith (1988) who demonstrated how a shear-front model may exhibit forward-sloping ascent in the air above the surface cold front, in agreement with observations. Here it is shown (section 4) how this cross-frontal flow allows some shear fronts to be related to forward-sloping fronts in $\theta$.

The dynamics of these simple models of baroclinic waves, and their accompanying fronts, is most concisely explained in terms of potential-vorticity dynamics (see discussion by Hoskins et al. 1985), and in a potential-vorticity framework they may be seen as being in many senses equivalent (the Charney and Eady waves were synthesized by Green (1960)). Since the Eady system is analytically and numerically more tractable, it is generally favoured for idealized studies, and it is pursued here.

The Eady basic state (in the SG system, here following the notation of Thorpe and Emanuel (1985)) is one of a meridional (Y-direction) temperature gradient, $\partial \theta / \partial Y < 0$, in thermal-wind balance with a vertical shear of the zonal wind:

$$\frac{\partial u_g}{\partial Z} = -\frac{g}{f \theta_0} \frac{\partial \theta}{\partial Y},$$

where $u_g$ is the cross-front component of the geostrophic wind, $g$ is the acceleration due to gravity, $f$ is the Coriolis parameter and $\theta_0$ is a reference potential temperature. Note that the upper-case coordinates ($X$, $Y$, $Z$) are used to denote the geostrophic coordinates,

* As noted by Smith and Reeder (1988), in the singular limit, strong ageostrophic winds mean that these fronts behave like local air-mass boundaries, which may partly explain why air-mass ideas are still in wide use.
while \((x, y, z)\) are retained for physical space: the coordinate sets are related by,
\[
(X, Y, Z) = (x + v_g / f, y, z),
\]
where \(v_g\) is the along-front component of the geostrophic wind.

Solution of the SG equations now involves inversion of the potential vorticity, \(q\), a constant field for the Eady basic state, by solution of,
\[
\frac{\partial^2 \Phi}{\partial X^2} + \frac{\theta_0 f^3}{g \rho q} \frac{\partial^2 \Phi}{\partial Z^2} = f^2
\]
for the SG geopotential, \(\Phi\), where \(\rho\) is density. When the basic state, corresponding to
\[
\frac{\partial^2 \Phi}{\partial Z^2} = \frac{g \rho q}{\theta_0 f},
\]
is subtracted, without approximation or linearization, the inversion for the geostrophic geopotential follows (3) with the right-hand side set to zero. The boundary condition for the problem comes from conservation of potential temperature on lower, and possibly upper, boundaries: the SG advection equation for \(\theta\) is
\[
\frac{D \theta}{D t} = \frac{\partial \theta}{\partial t} + \bar{u}_g \frac{\partial \theta}{\partial X} + v_g \frac{\partial \theta}{\partial Y} + w \frac{\partial \theta}{\partial Z} = 0,
\]
where
\[
v_g = \frac{1}{f} \frac{\partial \Phi}{\partial X},
\]
\[
\theta = \frac{\theta_0}{g} \frac{\partial \Phi}{\partial Z},
\]
and \(\bar{u}_g\) is given by the basic shear flow. On the boundaries the vertical velocity, \(w\), is zero so the last term in (5) vanishes. The interior field of \(w\) is found diagnostically, through the SG Sawyer–Eliassen equation for the ageostrophic stream function, \(\psi\):
\[
\frac{\partial^2 \psi}{\partial Z^2} + \frac{\partial}{\partial X} \left( \frac{g \rho q}{\theta_0 f^3} \frac{\partial \psi}{\partial X} \right) = 2 \frac{\partial v_g}{\partial X} \frac{\partial u_g}{\partial Z},
\]
from which we are able to compute
\[
w = -\frac{\zeta}{f} \frac{\partial \psi}{\partial X},
\]
with \(\zeta\), the absolute vorticity, given by
\[
\zeta = \frac{f}{1 - \frac{1}{f} \frac{\partial v_g}{\partial X}}.
\]

At the lower and upper (should it exist) boundaries of the domain, it is possible for edge waves to propagate (see Gill 1982, p. 551; Hoskins et al. 1985): these waves are neutral, and propagate with phase speed
\[
c = \bar{u}_g(0) + \gamma \frac{\partial u_g}{k \partial Z},
\]
where $\bar{u}_g(0)$ is the surface wind speed, $k$ is the horizontal (X-direction) wave number
and
$$y = \left( \frac{\theta_0 f^3}{\rho^2 g} \right)^{1/2},$$
with $\rho$ the density and $q$ the potential vorticity of the basic state: $y$ is equivalent to $N/f$
for the quasi-geostrophic system (where $N$ is the Brunt–Väisälä frequency). Thus the
wave has a steering level, $z_s$, of
$$z_s = \frac{y}{k} = H_R,$$
the Rossby height for the wave. With an upper boundary to the domain, upper and lower
edge waves are able to couple, provided $H_R$ becomes comparable with the domain
height, $H$, that is, for long waves. The coupling leads to an exponentially growing
solution—the classic Eady wave.

(a) Location of the fronts

The choice of diagnostics with which to locate the position of fronts is not a well-
defined problem. Atmospheric fronts exhibit many characteristic features in the thermal,
dynamic and cloud fields, any of which may be chosen to demarcate a front. Since the
current discussion concerns the slope of thermal fronts, the region of maximum thermal
gradient is the frontal zone of interest here. A first estimate of the frontal position is the
point where the thermal gradient is a maximum or minimum, and this is plotted in the
subsequent figures as a bold contour at,
$$\frac{\partial^2 \theta}{\partial x^2} = 0.*$$

(b) The structure of the edge wave

The structure of an edge wave is shown in Fig. 1: this case has wavelength 2000 km,
$q = 3 \times 10^{-7}$ K m$^2$kg$^{-1}$s$^{-1}$, $\bar{\partial}\bar{u}_g/\bar{\partial}Z = 3 \times 10^{-3}$ s$^{-1}$, $f = 10^{-4}$ s$^{-1}$, $\theta_0 = 288$ K,
$\rho = 1.225$ kg m$^{-3}$, and a temperature perturbation amplitude of 6.0 K, giving a steering
level of 2845 m. The frontal zones are marked by the solid line, indicating the positions
of the maximum and minimum in the horizontal gradient (in physical space) of potential
temperature. The principal features of this wave, for the present purposes, are:

(i) The slope of the cold and warm fronts, arising from the ageostrophic advection
terms (these fronts are vertical in the quasi-geostrophic version of this model).

(ii) The symmetry between the warm and the cold front.

* In contrast, Hewson (1998) used the zero contour of $\partial^2 \theta/\partial x^2$ to locate fronts objectively, as the ‘warm-air side
of the frontal zone’. It is worth noting that this contour, at the surface, follows the position of the extremum
of geostrophic vorticity closely throughout the evolution, representing a dynamical motivation for its use by
synopticians. The reason for this may be illuminated by approximating the dynamics to the quasi-geostrophic
form, in which case the lower-boundary condition, (5), for a quasi-steady wave with $\partial \theta/\partial t \approx 0$, implies that
$$u_g \propto \frac{\partial \theta}{\partial x}.$$
From this quasi-geostrophic estimate, it can be seen that an extremum of $\partial^2 \theta/\partial x^2$ approximately located an
extremum of cross-frontal shear, $\bar{\partial}u_g/\bar{\partial}x$, which in this model is equal to the geostrophic relative vorticity.
Figure 1. The structure of an edge wave on the Eady basic state: (a) perturbation potential temperature contoured at 1 K, (b) meridional wind at 5 m s\(^{-1}\), (c) vertical velocity at 0.5 cm s\(^{-1}\), and (d) absolute vorticity contoured at 0.2 \(\times\) \(10^{-4}\) s\(^{-1}\) (maximum value 2.3 \(\times\) \(10^{-4}\) s\(^{-1}\)). Negative contours, and the zero contour, are denoted by dotted lines. The frontal zones are marked by bold solid lines, locating the extrema of the cross-front temperature gradient. Axes are marked in km.

(iii) The location of the maximum of vorticity in the centre of the warm sector.

(iv) Descent in the region of the cold front (‘kata’ behaviour, Bergeron (1937)), with equatorward along-front wind; ascent at the warm front (‘ana’ behaviour), with poleward along-front wind.

(c) The structure of the Eady wave

In the exponentially growing Eady wave, the frontal structure differs somewhat from that of a neutral edge wave: Fig. 2 shows day 3.0 of a case with domain height 10 km, initial potential-temperature perturbation 1 K, \(\theta_0 = 288\) K, wave number \(k = 1.6/L_R\) (where \(L_R\) is the Rossby radius), \(\partial u_g / \partial Z = 3 \times 10^{-3}\) s\(^{-1}\), \(f = 10^{-4}\) s\(^{-1}\) and \(\partial \theta / \partial Z = 4 \times 10^{-3}\) K m\(^{-1}\). The fronts in this wave become singular at day 3.75.

For a small amplitude mode, with weak frontal gradients, the thermal structure, which tilts eastwards with height, is reflected in a cold frontal zone which tilts above
the warm air—this is a feature of the classic quasi-geostrophic Eady wave (not shown here). As such, this represents a candidate for an explanation of forward-sloping fronts. However, the gradients are too weak for this feature to be identified as a true front, and as the gradients intensify, the tendency is for the frontal zone to tilt back, so that it overlies the cooler air. Note that in the quasi-geostrophic form of the Eady wave, with physical structure being constant in time, there would be no change in the frontal slope: it is the intensification of the front, as the vorticity increases, that forces it to tilt back above the cold air in the SG wave.

As in the edge wave, the front is a propagating feature: for the Eady wave the steering level is high, probably unrealistically so (Carlson 1991, p. 234; Gill 1982, p. 560), at altitude $H/2 = 5$ km here. The cold-frontal zone approximately divides the regions of ascent and descent (the cold air descending); this becomes more pronounced as the front intensifies, leading to the behaviour of an ‘ana’ cold front.

(d) Advection terms

For the propagating fronts (neutral or growing) on the Eady basic state, the structure is determined by a balance between the three components of advection, in adiabatic motion. Advection by the basic-state shear tends to tilt surfaces downshear; it is possible to infer, then, that the combination of the meridional and vertical advection tends to tilt features against the zonal shear, in a wave of steady structure. The meridional (along-front) winds give thermal advection of the basic state (positive $v_g$ implies warming) while the vertical motion leads to adiabatic warming and cooling (a smaller, but important term).

These advective processes may be seen as motion on sloping isentropic surfaces (‘upgliding and downgliding’), combined with the motion of the surfaces themselves (for the growing mode), as outlined by Hoskins et al. (1985). Papers by Snyder and Lindzen (1991) and Parker and Thorpe (1995) have both invoked the correspondence of upward motion with poleward wind in the Eady basic state, particularly for features of short horizontal scale. This rule of thumb accords with the idea of upgliding/downgliding: that poleward motion tends to mean ascent, and vice versa.
(e) Summary of frontal models

In summary, an assessment of the basic models of atmospheric fronts shows that the thermal cold front can not, in these models, overlie the warm air, as a forward-sloping cold front. It is argued here that the observed forward-sloping cold fronts and split fronts are not likely to occur in the potential-temperature field, given simple dynamical frontogenesis mechanisms. Instead, we may look to their formation in a conserved moist variable such as equivalent potential temperature, as a result of differential advection of $\theta$ and $\theta_e$. If this is to be the case, the evolution of $\theta_e$ must differ fundamentally from that of $\theta$ at the front, allowing $\theta_e$ to advect through the front, leaving two thermal gradients of opposite slopes in height, a process which is admitted for non-singular SG fronts. The next section discusses the dynamical implications of taking $\theta_e$ as a frontal parameter, rather than $\theta$.

3. Use of $\theta_e$ as a frontal parameter

Bjerknes and Solberg (1922) noted that a moist variable (they mentioned absolute humidity) is a useful indicator of frontal position, in cases where the temperature contrast is weak. Currently, equivalent potential temperature (see, for example, Emanuel (1994)) is often used to determine frontal location, and usually this gives little discrepancy with $\theta$ fronts at the surface (Hewson 1998). The attraction of appealing to $\theta_e$ is that it is more exactly conserved, in airflows involving cloud processes, than $\theta$: in terms of ‘air-mass thinking’ $\theta_e$ can more clearly demarcate the boundary between air of different origins. Similarly, for fronts affecting western Europe, the warmer air (higher $\theta$) is generally that of higher moisture content (elevated $\theta_e$), so thermal gradients are exaggerated in the moist parameter.

In considering the evolution of $\theta_e$ at fronts, it is first necessary to appreciate the basic structure of this field in comparison with the dry-bulb field, $\theta$. The equivalent potential temperature, $\theta_e$, is more exactly conserved than $\theta$, and, as in the case of split fronts, is generally the more important field in terms of locating cloud features. However, it cannot be forgotten that $\theta$ remains the thermodynamic quantity of primary significance in the dynamics, as it is the field determining the buoyancy. In contrast, $\theta_e$ structure can, in the absence of $\theta$ structure, only influence the atmospheric dynamics indirectly, through cloud evolution resulting from moist instabilities (e.g. conditional symmetric instability; Bennett and Hoskins (1979)). Essentially this is due to the fact that it is possible to produce variations in $\theta_e$ by altering the humidity of the air, without altering the buoyancy.

The differing roles of $\theta$ and $\theta_e$ in atmospheric dynamics may be restated in potential-vorticity terms: a useful, conserved potential vorticity may be defined in terms of any thermodynamic function of pressure and temperature ($p$, $T$), but in general, $\theta_e$ does not satisfy this condition, since it depends also on the air’s moisture content. A definition of ‘SG equivalent potential vorticity’ given by,

$$ q_e = \frac{1}{\rho} \xi e \frac{\partial \theta_e}{\partial Z}, \quad (15) $$

by analogy with dry potential vorticity, $q$, does not give a conserved function. Since $q_e$ is not conserved, the atmosphere may evolve from conditional stability to conditional instability ($q_e < 0$; Bennett and Hoskins (1979)) while still conserving $\theta_e$. Indeed, Cao and Cho (1995) have demonstrated how this may occur in simulated extratropical cyclones and Cho and Cao (1998) have investigated the sensitivity of this process to the moisture distribution of the basic state.
The non-conservation of $q_e$ is an obstacle to its use in SG models: certain studies (e.g. Emanuel et al. 1987) have circumvented this 'problem' by assuming
\[ \theta_e = \theta_e(p, T), \] (16)
as is the case, for example, in saturated regions. From (16), $q_e$ becomes a conserved quantity, seen to be close to zero (e.g. Emanuel 1983), and $q_e \approx 0$ is used in the parametrization of slantwise convection developed by Thorpe and Emanuel (1985). It is important to be aware that there are a number of hurdles in the use of the assumption (16). For example, as $T$ falls in the upper troposphere, $\theta_e \sim \theta$ and an assumption of $q_e \approx 0$ implies also that $q \approx 0$, generally in contradiction of the assumed basic states of simple, balanced cyclogenesis models. Similarly, (16) is not generally a statement of saturated flow (as some authors have incorrectly assumed)—a non-trivial choice of the $\theta$ distribution would be required to ensure this. Often, models using Thorpe and Emanuel's parametrization based on (16) result in negative humidities at upper levels, given no supersaturation in the lower troposphere.*

Returning to Thorpe and Emanuel's (1985) arguments, it remains true that $q_e$ is small in the region of fronts, possibly as a result of conditional symmetric instability, or possibly simply because of the greater capacity of warmer, low-level air to hold water vapour. At higher levels, as remarked, $q_e$ and $q$ become closer in magnitude. These general results are seen in the observations of Thorpe and Clough (1991) and Browning et al. (1995). In considering the meridional gradients, $\partial \theta_e / \partial Y$ (along the front) may be expected to be of the same magnitude as $\partial \theta / \partial Y$ (especially at upper levels, where $\theta \sim \theta_e$), although at low (warmer) levels it has the capacity to differ, and $\partial \theta_e / \partial Y$ is likely to be larger. As a first approximation, then, for low-level features it is reasonable to take
\[ \frac{\partial \theta_e}{\partial Z} \approx 0, \] (17)
\[ \frac{\partial \theta_e}{\partial Y} \approx \frac{\partial \theta}{\partial Y}, \] (18)
with the understanding that (17) becomes less suitable at higher altitudes: if attention is focussed on the lower-level tracer advection, the inconsistency is not important. These basic ideas are used in the choice of a basic state for a passive-tracer field, described in the next section.

4. TRACER EVOLUTION IN SHEAR-FRONT MODELS

This section describes the use of the wind fields associated with SG models of fronts on the Eady basic state to advect a tracer field, $\chi(X, Y, Z)$. The tracer has been initialized to resemble physically realistic fields of $\theta_e$ (as outlined above). In the case of the edge wave, the analytical solution provides exact wind fields, which have been employed in a centred-difference numerical scheme to advect $\chi$. In the Eady-wave solution, a similar analytical wind field could have been employed, but instead a numerical integration of the transformed SG equations (Heckley 1980; Emanuel et al. 1987) was used, as this is demonstrably very accurate and allows departures from the

* If $\theta_e$ has a fixed value corresponding to saturation at the surface, and constant in height to satisfy $q_e = 0$, a state in which $\theta$ increases steadily with height will mean that there is some altitude above which $\theta_e < \theta$. In practice, with a finite domain height, this kind of problem can generally be eliminated by suitable choice of the reference temperature $\theta_0$ (Parker 1993).
dry normal-mode solution to be investigated. In both cases, the accuracy of the tracer advection was confirmed by initializing with the $\theta$ field: over the relevant integration periods, the departures of this tracer from the model (and analytical) $\theta$ field were marginal.

Given the arguments of the preceding section, fronts are considered with the $\chi$ distributions of:

$$\frac{\partial \chi}{\partial Z} = 0, \quad (19)$$

$$\frac{\partial \chi}{\partial Y} = \frac{\partial \theta}{\partial Y}, \quad (20)$$

$$\chi'(X, Z)_{t=0} = \theta'(X, Z)_{t=0}, \quad (21)$$

where $\chi = \bar{\chi} + \chi'$ and $\theta = \bar{\theta} + \theta'$. It should be recalled that the tracer field merely defines a set of surfaces which are advected in the frontal flows. The values assigned to these surfaces may be altered without altering the results (for instance, $\chi$ could be associated with $\theta_w$ rather than $\theta$). Thus, a state of $\chi < \theta$ need not be 'unphysical', since an additive constant to the tracer field would eliminate the apparent problem. It is, however, useful to consider the local gradients of surfaces. In particular, the fronts in the tracer field are computed as the surfaces of

$$\frac{\partial^2 \chi}{\partial \chi^2} = 0, \quad (22)$$

in line with the definition of the thermal front, (14), and in the figures these tracer fronts are marked by bold dashed lines.

(a) The edge wave

Figure 3 shows the evolution of $\chi$ in the wave of section 2 (Fig. 1). It can be seen that after 0.5 days, $\chi$ surfaces are tilting downshear, and that there is a region of $\partial \chi / \partial z < 0$ (gradient in 'physical $z$') at the cold front. This closely resembles the kinds of features described by Browning and Monk (1982). Note that the $\chi$-front slopes the opposite way with height to the $\theta$-front, and that the symmetry between the warm and cold fronts is broken for the tracer field. In Fig. 1(b), there is a shallow region of rearward slope in the tracer front close to the surface cold front: this effect is independent of the model resolution and is an artifact of the method used to determine the frontal zone.

Given the tracer initialization, the initial advection of $\chi$ differs from that of $\theta$ only in the $u \partial \theta / \partial Z$ term, which is absent for $\chi$. This means that descent leads to increased values of $\theta$, while $\chi$ is unmodified in descent. For the steady edge wave, this implies that descent leads to the advection of low $\chi$ in the region of the cold front: the negative tendencies corresponding to the horizontal advection terms are not balanced, for $\chi$, by adiabatic warming. In more graphical terms, the $\theta$-surfaces are fixed (a steady system), so behind the cold front, air is moving along $\theta$-surfaces downwards and equatorwards (isentropic downgliding). The basic state of $\chi$ has surfaces aligned vertically, and the equatorward (and downward) advection deforms these surfaces so as to bring air of lower $\chi$ from the poleward direction, in the region of the cold front.

The net 'cooling', in $\chi$, around the cold front, and the equivalent 'warming' over the warm-frontal zone, breaks the symmetry enjoyed by the $\theta$ field. This leads to the cross-frontal advection of the $\chi$ contours, and the development of the forward-sloping 'cold' front in $\chi$. 
Figure 3. The structure of the tracer field in the edge wave of Fig. 1, at times (a) 0.5 days and (b) 1.0 days. The initial tracer field was that of the potential-temperature perturbation, in Fig. 1(a). Negative contours, and the zero contour, are denoted by dotted lines. The tracer fronts are marked as bold dashed lines, while the thermal fronts remain as bold solid lines. Axes are marked in km.

Figure 4. The structure of (a) the perturbation potential temperature, and (b) the (total) tracer field, at time 3 days, in the Eady wave of Fig. 2. Both fields are contoured at 4 K. Negative contours, and the zero contour, are denoted by dotted lines. The tracer fronts are marked as bold dashed lines, while the thermal fronts remain as bold solid lines. Axes are marked in km.

(b) The Eady wave

To study the evolution of the tracer in the growing baroclinic wave, \( \chi \) has been initialized in a mode for which the surface cold front was sufficiently intense to tilt above the cold air (day 2.0 for the model described above). As the wave grows, the slope of the \( \chi \)-frontal zone evolves with height (as does the \( \theta \)-front). However, the tilt is similar to that of the \( \theta \)-front near the surface; indeed, the \( \theta \) and \( \chi \) fronts remain similar up to mid-levels (see Fig. 4, at time 3 days).

The steering level of the wave, at 5 km, is high, and any low-\( \chi \) air appearing behind the cold front (by equatorial cold advection, in the absence of compensating subsidence warming, as for the edge wave above) is advected to the west in the cross-frontal shear. Given the relative flow from the east, in the lowest 5 km of this model, ‘instability’ in the
\( \chi \) field, \( \partial \chi / \partial z < 0 \), could only be attained by destabilizing ahead of the cold front, in the warm air. The relatively strong mid-level advection, at the steering level of the wave ahead of the cold front, precludes this by maintaining a warm mid-troposphere, and a stable environment ahead of the front. Notably, in this wave, the region of ‘instability’ in the tracer field, where \( \partial \chi / \partial z < 0 \), is located well behind the cold front, in the cold air: here the mid-level cold advection dominates.

The generation of convective instability in the cold sector demonstrates the way in which baroclinic waves may induce ‘air-mass’ properties (in this case those of polar maritime air) ‘locally’, by differential advection, without the low-level air being of polar origin. In fact, in this model it is the \textit{failure} of the air to behave as a coherent air mass that leads to the formation of structure resembling polar maritime air in the cold sector.

In terms of a synoptic interpretation of the evolution, the cold front in the mature Eady wave is an ‘ana’ front, with the warm air rising and the cold air sinking. More significantly, the front is also a division between equatorward-moving air, behind, and poleward-moving air ahead. In contrast with the edge wave, where low-\( \chi \) air was advected into the frontal zone, and then ahead of the front, above the steering level, the Eady-wave cold front experiences positive meridional advection of \( \chi \) just ahead, and this is advected back through the frontal zone by the zonal wind.

\( (c) \) \textit{Sensitivity to initialization}

Use of other initial states for the tracer field perturbation, \( \chi'(X, Z) \) (not shown) seems to have little influence on the qualitative evolution of the tracer field. This may be explained by splitting the tracer perturbation field in the \( (X, Z) \) plane into a part \( \chi_1(X, Z, t) \), corresponding to advection of the initial perturbation, and a part \( \chi_2(X, Z, t) \), corresponding to advection of the basic-state field, \( \chi(Y, Z) \). Then, the condition,

\[
\frac{D \chi}{Dt} = \frac{\partial \chi'}{\partial t} + u \frac{\partial \chi'}{\partial X} + v \frac{\partial \chi'}{\partial Y} + w \left( \frac{\partial \chi'}{\partial Z} + \frac{\partial \chi}{\partial Z} \right) = 0,
\]

where \((u, v, w)\) is the imposed (SG) advecting velocity, leads to the set:

\[
\frac{\partial \chi_1}{\partial t} + u \frac{\partial \chi_1}{\partial X} + w \frac{\partial \chi_1}{\partial Z} = 0,
\]

\[
\frac{\partial \chi_2}{\partial t} + u \frac{\partial \chi_2}{\partial X} + w \frac{\partial \chi_2}{\partial Z} = - \left( v \frac{\partial \chi}{\partial Y} + w \frac{\partial \chi}{\partial Z} \right),
\]

with initial conditions

\[
\chi_1(X, Z, 0) = \chi'(X, Z)_{t=0},
\]

\[
\chi_2(X, Z, 0) = 0.
\]

From (24) and (26), it can be seen that \( \chi_1 \) evolves so as to rearrange the contours in the \( (X, Z) \) plane: the values of \( \chi_1 \) can not exceed the extrema of the initial state. In contrast, \( \chi_2 \) is ‘forced’ by a function derived from the known advecting winds and the basic-state tracer field: \( \chi_2 \) is not limited in magnitude, particularly in the case of the Eady wave, where the amplitude of the advecting winds is exponentially growing. For example, in the Eady wave, the dominance of the large-scale advection over the advection of the initial perturbation is significant ahead of the warm front, at mid-levels (around the wave steering level) where persistent and growing positive advection of \( \chi \) leads to a ‘stabilization’ of the low-level tracer field.
5. Discussion and Summary

This paper has discussed the principal models of frontal evolution, and indicated that one of these, the Eady edge wave, can give rise to a forward-sloping cold front, through differential advection of equivalent and dry-bulb potential temperature. Given this understanding, it is informative to return, briefly, to the other frontal models:

- The gravity-current model is clearly unable to support a difference between the $\theta$ and $\theta_e$ fields within its own flow, but if a gravity current exists in a background vertical shear, advecting a passive $\theta_e$ field with downshear variation, the $\theta_e$ contours will be tilted downshear, in the ambient flow.

- The deformation front tends to align tracer contours along its frontal axis. However, if an along-front gradient of the tracer field is imposed in such a model, so that the low-level jet advects 'warmer' tracer values into the low levels of the cold front, this destabilizes the tracer field. Note that this configuration, for typical UK cold fronts, is unphysical, implying air of higher $\theta_e$ in the poleward direction.

The fact that the SG Eady wave does not support a forward-sloping cold front has been seen to be a result of the high steering level for this wave, and the fact that the cold front is an ana-front. Static 'instability' of the tracer, $\partial \chi / \partial z < 0$, is seen in the model ahead of the cold front, above the steering level, where cooler air brought from the poleward side, behind the cold front, is advected ahead of the front in the zonal shear. At these higher levels, the tracer initialization is not representative of a realistic $\theta_e$ field, nor is the model a faithful representation of the upper troposphere. It may be said that the mechanisms of formation of forward-sloping fronts persist in this model, but are only favourable above the steering level: a lowered steering level could create forward-sloping behaviour even for an ana-cold front. However, it is difficult, and probably risky, to attempt to generalize these processes further, without reference to the behaviour of analysed systems in the atmosphere.

In observed cases, forward-sloping cold fronts are associated with split front behaviour: an upper cold front runs ahead of the surface cold front, and the low-level frontal surface overlies the warm air (Browning and Monk 1982). The upper cold front is a distinct 'nose' of intruding cold air, at which moist convection is observed. Such a feature does occur in the edge-wave tracer-advection model, if the basic-state vertical gradient of $\chi$ is taken to increase with height to that of $\theta$: in such a case (not shown) the upper-level tracer behaviour is similar to that of $\theta$ and the low-level behaviour is forward-sloping, as would be expected.

No latent-heat release has been included in the simple frontal models described here, but the qualitative effect of a band of convection may be discussed (see the sketch in Fig. 5). Convection will lead to very little change in the Lagrangian conservation of the equivalent potential temperature, but the convectively induced circulation will lead to a modification of the tracer advection. The tendency of the convectively induced circulation will be to cause horizontal convergence of the tracer field at the bottom of the convective band, and horizontal divergence above. Such a flow may be expected to retard the cross-front advection at higher levels and enhance it below: in this way, the occurrence of a 'nose' of colder air, at the upper cold front, may be expected to be maintained, at the lower levels of the convective band. The effect of such a band on the field of dry-bulb potential temperature would not be expected to change the basic cold-front characteristics, since the tendency of the latent heating will be to warm at the level of the convection and to cool (through adiabatic ascent) below the convection, thereby stabilizing the atmosphere at low levels.
Figure 5. A sketch of the flow modification which may be induced by a band of convection ahead of the cold front. The horizontal convergence at the base of the convection, and the horizontal divergence above, is expected to lead to the maintenance of a more pronounced 'nose' in the tracer field which is being advected ahead of the surface cold front: this nose may be identified with an upper cold front.

A more complete discussion of the influence of convection on the generation of negative values of $q_e$ is given by Cho and Cao (1998), who make the point that although moist convection can not directly modify $q_e$, it can lead to disparity of the $\theta$ and $\theta_e$ fields, which may in turn yield advective $q_e$ tendencies.

The simple waves on the Eady state do not give perfect representations of atmospheric fronts: they lack diffusion, cloud processes, deformation flow and potential-vorticity gradients. However, the elements of these waves can be applied directly to real cases, and thereby increase the understanding of frontal evolution. In this case, it has been seen how the interplay between $\theta$ and $\theta_e$ advective terms can give rise to coexisting 'dry' and 'moist' fronts which have opposing slopes. This allows the split front to be understood simply in terms of known frontal dynamics, and highlights the behaviour of the front as that of a dynamical feature (or wave), as opposed to an air-mass boundary: the atmospheric frontal zone need not be a barrier to the transport of air properties such as moisture, trace gases or particulates. The strong implication of these results is that we may look to passive advection to explain such frontal features, and thus the important weather signatures which accompany them.

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