Entropy production and dynamical complexity in a low-order atmospheric model

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SUMMARY

Lorenz’s classical model of thermal convection is analysed from the standpoint of irreversible thermodynamics. The entropy production, describing how dissipative processes are operating within the system, is expressed in terms of the model variables. It is subsequently evaluated in different parts of the attractor and the values so obtained are compared with various indicators of the complexity of the dynamics, such as the local Lyapunov exponents and a local generalization of the Kolmogorov entropy. As it turns out, dissipative processes are most efficient when the number of locally stable directions increases. In particular, when the local Kolmogorov entropy is close to zero (three stable directions) the system can dissipate over the entire spectrum of values available. Finally, it is shown that the evolution on the attractor does not correspond to any extremal property of the entropy production.

KEYWORDS: Chaotic dynamics  Entropy production  Kolmogorov entropy

1. INTRODUCTION

One of the most important processes underlying atmospheric circulation is the balance of various forms of energy, particularly in the form of kinetic, internal and potential energy (Lorenz 1960). Generally, this balance can be one of two kinds: (i) conversion of a given form into some of the other forms, subject to the condition that the total energy is a conserved quantity; (ii) a net depletion of this form by dissipative processes such as heat conduction and viscosity, whose strength is measured by the entropy production.

On the other hand, it is now established that atmospheric circulation is often manifested through complex dynamics, from baroclinic and thermal convective instability to fully developed chaos (Lorenz 1987). In addition to being at the very origin of the generation of kinetic energy, this dynamical complexity is also largely responsible for the intrinsic limitations to long-term predictions associated with sensitivity to initial conditions and the concomitant growth of small initial errors in the atmosphere.

Starting from the basic laws of fluid mechanics and thermodynamics, one may derive a variety of formal relationships linking energy interconversion and dissipation to the structure of the model equations for the relevant variables involved in the problem at hand (De Groot and Mazur 1962; Van Mieghem 1973). So far, however, few reported attempts have aimed at relating energy, entropy or entropy production to the complexity of the dynamical behaviour generated by the model equations and its various quantifiers such as Lyapunov exponents or Kolmogorov entropy (Ott 1993). In the present paper an attempt in this direction is reported in a case-study involving a simple, low-order atmospheric model.

The particular system considered is Lorenz’s classical model of thermal convection (Lorenz 1963), associated with a three-mode truncation of Saltzman’s representation of Boussinesq’s equations in terms of Fourier variables (Chandrasekhar 1961; Saltzman 1962). There are several reasons for this choice. Firstly, thermal convection is one of the basic processes at the origin of mesoscale variability and, secondly, it contains much of the physics underlying energy interconversion and dissipation. Beyond the Rayleigh–Bénard instability, attained when the temperature gradient across a horizontal fluid layer

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exceeds a critical value, kinetic energy is continuously generated at the expense of internal and potential energy. Part of the kinetic energy is dissipated as viscous heating, internal energy being itself partly subjected to dissipation through heat conduction but, since there is a continuous supply counteracting the losses, the process goes on for ever as long as the thermal constraint is present. Finally, beyond some value of a dimensionless combination of the thermal constraint and other parameters dependant upon the geometry and structure of the system, the flow becomes chaotic and exhibits sensitivity to initial conditions, thereby sharing yet another important characteristic of atmospheric dynamics.

In addition to its conceptual interest, a better understanding of the connection between dissipation and dynamics is also motivated by a number of more practical considerations. To take up an argument developed by Stephens and O'Brien (1993, 1995), the generally accepted approach to the simulation of both atmospheric circulation and climate variability is based on the idea that all processes deemed to be relevant need to be included in the model equations. This introduces a large number of widely separated time- and space-scales as well as a heavy parametrization, often at the expense of a deeper understanding. On the other hand, the principal aim of thermodynamics is to arrive at a unified description of natural systems that is largely independent of the details of the ongoing processes. It is therefore tempting to seek for some ‘organizing principle’ that could underlie some aspects of atmospheric and climate dynamics and give some insight on evolutionary trends that would otherwise be masked by a full-scale analysis.

A number of approaches based on this philosophy have been reported in the literature. Most familiar are those stipulating the existence of an extremum principle, originally proposed by Paltridge (1975, 1981) and subsequently developed by a variety of authors including the present one (Nicolis and Nicolis 1980; Grassl 1981; Shutts 1981; Mobbs 1982; Wyant et al. 1988; Stephens and O’Brien 1993, 1995). More recently, cloud morphogenesis has been analysed from the standpoint of entropy balance and, as a by-product, an upper bound to the extent of possible change in cloud form has been derived (Duane and Curry 1997). The present study, while sharing the same philosophy and motivation, does not appeal to any variational principle which, in the author’s view, is unlikely to apply to a system as highly nonlinear and far from equilibrium as the atmosphere. Furthermore, while most studies so far reported concern global properties and steady-state behaviour, this work focuses on the behaviour of entropy production during the successive stages of the phenomenon in the course of time, and on its connection with the dynamics of the relevant variables in phase-space, particularly as far as the dynamical instability associated with chaos and loss of predictability is concerned.

In section 2 Lorenz’s model is revisited from the standpoint of irreversible thermodynamics. Starting from the general expression of entropy production, explicit formulas are derived relating viscous and thermal dissipation to the variables involved in the model. Section 3 is devoted to the numerical evaluation of these contributions (or, more precisely, of the excess quantities around the state of rest) for parameter values associated with Lorenz’s classic chaotic attractor. It is found that dissipation takes a small value during the evolutionary stages when the Kolmogorov entropy is a local maximum, while its values present a maximum dispersion (including therefore the highest possible values) when the Kolmogorov entropy is locally near zero. This unexpected trend is confirmed in section 4 by numerical studies in parameter ranges associated with other types of attractors, such as temporal intermittency or even periodic behaviour. In section 5 the connection between entropy production and dynamical instability is further
sharpened by studying the dissipation associated with the transient evolution toward the attractor. The interpretation and repercussions of the results along with suggestions for further studies are discussed in section 6.

2. **Irreversible Thermodynamics of the Lorenz Model**

We consider a horizontal fluid layer of depth $d$ whose upper and lower boundaries are maintained at the temperatures $T_0$ and $T_1$ respectively, and are stress-free. We adopt the Boussinesq approximation treating the fluid as incompressible (div $\mathbf{v} = 0$, $\mathbf{v}$ being the velocity field) and the density $\rho$ as a constant ($\rho = \rho_0$) in the momentum and internal energy balance equations, except for the term expressing the effect of the buoyancy force (Chandrasekhar 1961). It is well known that the state of rest $\mathbf{v} = 0$ of this system becomes unstable beyond a critical value of a dimensionless parameter known as the Rayleigh number,

$$R = \frac{g \alpha \beta d^4}{\nu \kappa}.$$  \hspace{1cm} (1a)

Here $g$ is the acceleration of gravity, $\alpha$ the thermal expansion coefficient, $\beta$ the temperature gradient ($\beta = \frac{T_0 - T_1}{d}$), $\nu$ the kinematic viscosity ($\nu = \frac{\eta}{\rho_0}$, $\eta$ being the shear viscosity coefficient) and $\kappa$ the thermal diffusivity ($\kappa = \frac{\lambda}{\rho_0 c_v}$, $\lambda$ being the thermal conductivity coefficient and $c_v$ the specific heat). Under the above stipulated free-boundary conditions the critical value turns out to be

$$R_c = \frac{27\pi^4}{4}. \quad \hspace{1cm} (1b)$$

The simplest convection pattern arising in the range $R > R_c$ comprises 2-dimensional rolls, extending vertically from the bottom to the top boundary and repeating themselves periodically in the horizontal direction with an inverse wavelength equal to $k = \pi / \sqrt{2}d$. As $R$ increases this pattern tends to be distorted and eventually becomes chaotic in both time and space, but in what follows it is assumed, following Saltzman (1962) and Lorenz (1963), that while the temporal evolution of the variables becomes very complex in this range, the spatial distribution remains essentially unchanged. The validity of this assumption requires, at best, that the aspect ratio $L/d$ (where $L$ is the length in the horizontal direction) is not very large. In what follows we shall take this simplification for granted, thereby limiting ourselves to the regime of purely temporal chaos.

Let us set

$$T(x, z, t) = T_0 - \beta z + \delta T(x, z, t), \quad \hspace{1cm} (2)$$

where $x$ and $z$ are the horizontal and vertical coordinates respectively, $t$ is time, and $\delta T$ the deviation of the temperature field from the pure conductive (linear) profile. Since there are only 2 velocity components, hereafter denoted $u$ and $w$, one can define a stream function $\psi(x, z, t)$ such that

$$v_x = u = -\frac{\partial \psi}{\partial z},$$

$$v_z = w = \frac{\partial \psi}{\partial x}. \quad \hspace{1cm} (3)$$
The solution of the Boussinesq equations satisfying the boundary conditions can then be written in the form

\[
\begin{align*}
\delta T(x, z, t) &= \sum_{mn} \theta_{mn}(t) e^{ikx} \sin \frac{n\pi z}{d} \\
\psi(x, t, z) &= \sum_{mn} \psi_{mn}(t) e^{ikx} \sin \frac{n\pi z}{d}.
\end{align*}
\]

When the relations are substituted in the Boussinesq equations one obtains an infinite hierarchy of coupled ordinary differential equations for the modes \(\theta_{mn}(t)\) and \(\psi_{mn}(t)\). Lorenz’s truncation (Lorenz 1963), which we adopt in the sequel, consists of retaining only the modes \(\theta_{11}, \psi_{11}\) and \(\psi_{02}\). Switching to dimensionless amplitudes \(X(t), Y(t), Z(t)\) one then arrives, after some algebra, at the expressions

\[
\psi^* = \kappa^{-1} \psi = 3X(t) \sin \frac{\pi x}{\sqrt{2d}} \sin \frac{\pi z}{d},
\]

\[
\delta T^* = \frac{g\alpha d^2}{\nu \kappa} \delta T = \frac{R_c}{\pi} \left\{ \sqrt{2}Y(t) \cos \frac{\pi x}{\sqrt{2d}} \sin \frac{\pi z}{d} - Z(t) \sin \frac{2\pi z}{d} \right\}.
\]

(4)

Here the stars on the left-hand sides denote the dimensionless stream function and temperature, and \(X(t), Y(t), Z(t)\) satisfy the Lorenz equations (Lorenz 1963)

\[
\begin{align*}
\frac{dX}{dt} &= \sigma (-X + Y), \\
\frac{dY}{dt} &= r X - Y - XZ, \\
\frac{dZ}{dt} &= b Z - XY,
\end{align*}
\]

(5)

where \(r = R/R_c, \sigma = \nu/\kappa, b = 8/3\).

We now proceed to the thermodynamic aspects of the problem. Within the framework of the local formulation of irreversible processes the entropy density, \(s\), depends on exactly the same variables as in equilibrium, namely density \(\rho\) and internal energy \(e\) of the fluid, provided these variables are also locally evaluated (De Groot and Mazur 1962),

\[
s = s\{\rho(x, z, t), e(x, z, t)\}
\]

(6a)

The total entropy per convection cell being given by

\[
S(t) = \int_0^d dz \int_0^{2\pi} dx \int_0^{2\pi} dx \int_0^{2\pi} dx \{\rho(x, z, t), e(x, z, t)\}.
\]

(6b)

Differentiating both sides of (6b) with respect to time and using the balance equations for \(\rho\) and \(e\) in the Boussinesq form one arrives at the entropy balance equation

\[
\frac{dS}{dt} = \frac{d_e S}{dt} + P
\]

(7)

where \(d_e S/dt\) is the entropy flux and \(P\) the entropy production (De Groot and Mazur 1962),

\[
P = \int dx \int dz \phi(x, z, t),
\]

(8a)

\[
\phi = -\frac{1}{T^2} I_q \nabla T - \frac{1}{T} \Pi : \nabla v.
\]

(8b)
Here $J_q$ and $\Pi$ stand, respectively, for the heat flux and the dissipative part of the pressure tensor. Within the framework of local thermodynamics they are given by the constitutive relations (De Groot and Mazur 1962)

$$J_q = -\lambda \nabla T,$$

$$\Pi_{ij} = -\eta \left( \frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} \right) \quad i, j = 1, 2,$$

(9)

where $v_i, v_j$ stand for the velocity components $u$ and $w$, and $r_i, r_j$ for the space coordinates $x$ and $z$, respectively for $i, j = 1, 2$.

Equation (8b) becomes

$$\varphi = \frac{\lambda}{T^2} (\nabla T)^2 + \frac{\eta}{2T} \sum_{ij} \left( \frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} \right)^2.$$

(10)

The right-hand side of this expression is to be evaluated from Eqs. (2) to (4). Using the fact that for 2-dimensional rolls the relevant variables depend only on $x$ and $z$, we first write (10) in the intermediate form

$$\varphi = \frac{\lambda}{\{T_0 - \beta z + \delta T(x, z, t)\}^2} \left\{ \left( \frac{\partial \delta T}{\partial x} \right)^2 + \left( -\beta + \frac{\partial \delta T}{\partial z} \right)^2 \right\}$$

$$+ \frac{2\eta}{T_0 - \beta z + \delta T(x, z, t)} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 + \frac{\partial u}{\partial z} \frac{\partial w}{\partial x} + \frac{1}{2} \left\{ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right\} \right].$$

(11)

In what follows we shall be interested in the excess dissipation created by the convective instability. We shall also introduce the simplification, supported by observational evidence, that the variation of the temperature across the layer is small compared to the reference value $T_0$. Finally, upon integrating further over $z$ and $x$, one is led to identify the following set of quantities describing the dissipation produced by a convection cell:

$$P_{\text{excess}} = P_q + P_v,$$

(12)

where the thermal and viscous parts $P_q$ and $P_v$, respectively, are given by

$$P_q(t) = \left( \frac{\nu k}{g d^3 \pi} \right)^2 \frac{\lambda \pi^2}{T_0^2} \left( \frac{3}{\sqrt{2}} Y^2(t) + 4\sqrt{2} Z^2(t) \right)$$

(13a)

$$P_v(t) = 81 \kappa^2 \frac{\pi^4}{4\sqrt{2} d^2} \frac{\eta}{T_0} X^2(t)$$

(13b)

As seen from (13a) and (13b) the concrete value of $P_{\text{excess}}$, (12), depends upon the characteristics of the fluid and the geometry. Since we search for universal trends linking dissipation to dynamics, in the following analysis we shall treat separately the thermal and viscous contributions. Furthermore we shall omit multiplicative factors, and focus on the variable-dependent parts of $P_q(t), P_v(t)$ which also happen to be the dimensionless values of the thermal and viscous dissipation.

3. DISSIPATION ON THE CLASSIC LORENZ ATTRACTOR

It is well known (Lorenz 1963; Sparrow 1982) that for parameter values $r = 28$, $\sigma = 10$ and $b = 8/3$, Eqs. (5) generate a chaotic attractor of fractal dimension close to
2.06, possessing one positive Lyapunov exponent $\bar{\sigma}_1$ and hence exhibiting sensitivity to initial conditions. As always, when dealing with a continuous time dynamical system (a flow), there also exists a vanishing Lyapunov exponent $\bar{\sigma}_2 = 0$, the third one being necessarily negative, $\bar{\sigma}_3 < 0$. The sum of these three exponents is equal to the mean rate of phase-space-volume contraction, which in turn, is given by the divergence of the 'velocity field' (Ott 1993) defined by the right-hand sides of (5). In the Lorenz model this happens to be a constant depending only on the parameters:

$$\bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_3 = -(\sigma + b + 1) \approx -13.66.$$  

In fact there is a high variability of the Lyapunov exponents around the means, $\bar{\sigma}_1 = 0.9$, $\bar{\sigma}_2 \sim 0$, $\bar{\sigma}_3 = -14.6$, as one moves along a typical chaotic attractor (Abarbanel et al. 1991). The local (instantaneous) values of $\sigma_1$, $\sigma_2$ and $\sigma_3$ are still bound to satisfy the above expression, viz.

$$\sigma_1(t) + \sigma_2(t) + \sigma_3(t) = -(\sigma + b + 1) \approx -13.66,$$  \hspace{1cm} (14)

but, taken individually, they may vary considerably and even change their sign. There exist for a low-order model, such as (5), reliable numerical algorithms for evaluating these quantities very accurately (Eckmann and Ruelle 1985). Figure 1 depicts the probability density of $\sigma_1$, $\sigma_2$ and $\sigma_3$ obtained using these algorithms, confirming entirely the above anticipated high variability for the system at hand. Notice, however, that $\sigma_3 < 0$ applies throughout.

An important indicator of dynamical instability is the Kolmogorov entropy, which in a closed, confined system may be identified to the sum of the mean positive Lyapunov exponents (Ott 1993)

$$K_0 = \sum_{\sigma_i > 0} \sigma_i.$$  \hspace{1cm} (15a)

Since $\bar{\sigma}_1$ is the only mean, positive Lyapunov exponent in the Lorenz model, one has $K_0 = \bar{\sigma}_1$. On the other hand, in view of the considerable variability of the local Lyapunov exponents stressed above, one is naturally led to also introduce a local (instantaneous) Kolmogorov entropy, as suggested already in Grassberger et al. (1988). This quantity, hereafter denoted by $K$, will be defined by

$$K = \sum_{\sigma_i > 0} \sigma_i.$$  \hspace{1cm} (15b)

This is expected to fluctuate considerably across the attractor, while remaining non-negative throughout. Figure 2 describes the probability density of $K$, confirming this point.

Since, according to Fig. 1, $\sigma_1$ and $\sigma_2$ may be locally positive with an appreciable probability, one may ask whether they could both take, locally, a positive value entailing the presence of two unstable directions in this part of the attractor. Detailed analysis shows that this is indeed so, in the range of values of $\sigma_1$ and $\sigma_2$ between 0 and 5. An instructive representation illustrating this property is provided by Figs. 3(a) and 3(b), showing a 'phase-space' plot of $K$ versus $\sigma_1$ and $\sigma_2$. While there is a wide dispersion of values of $K$ for $\sigma_1$ and $\sigma_2$ less than 5, beyond this threshold $K$ varies linearly with either $\sigma_1$ or $\sigma_2$, indicating the presence of only one unstable direction. On the other hand Fig. 3(c) reveals that very negative values of $\sigma_3$ correspond to large values of $K$, while for values of $\sigma_3$ close to and above $-(\sigma + b + 1)$, $K$ drops to zero. In view of this and
Figure 1. Probability density of the local Lyapunov exponents: (a) $\sigma_1$; (b) $\sigma_2$; (c) $\sigma_3$ as obtained from model Eqs. (5) after an integration of 1000 time units with a sampling time $\Delta t = 0.1$ units. Parameter values $b = 8/3$, $\sigma = 10$, $r = 28$. 
the constraint imposed by (14) such values correspond, by necessity, to attractor regions with *three stable directions*.

A final indicator of the dynamics, introduced for later use, is the linear velocity along the trajectory:

\[ v_1 = \sqrt{(dX/dr)^2 + (dY/dr)^2 + (dZ/dr)^2}. \] (16)

In Fig. 4(a) the probability density of the values of \( v_1 \) is given, showing once again a wide dispersion around the mean. Figure 4(b) depicts the 'phase-space' plot of \( K \) versus \( v_1 \). We see that for high speeds the range of values of \( K \) is limited and close to the minimum (zero value) while, on the contrary, for low speeds the range of variation of \( K \) is a maximum, comprising the maximum value of \( K \) available. This might seem surprising at first sight, since one would tend to associate high velocities with unstable behaviour. The explanation proposed is that during the slow stage the system is more vulnerable to perturbations, since there is an eigenvalue of the local Jacobian close to zero. A small difference in the initial conditions is, then, more likely to affect this value which is, precisely, the principal signature of the instability of motion.

We are now in a position to assess the relation between dynamics and entropy production. Figures 5(a) and 5(b) depict our main result, in the form of a 'phase-space' plot of thermal and viscous dissipation versus \( K \). We observe a pattern similar to Fig. 4(b): minimum dissipation when \( K \) is locally maximal, and a broader range of dissipation values when \( K \) decreases, encompassing the maximum value of dissipation available when \( K = 0 \). We recall (Fig. 3) that the smaller the local \( K \), the larger the number of stable directions in this part of the attractor.

Figures 6(a)–6(c) provide an alternative view of this phenomenon through a comparison of the time series of \( K \), \( P_0 \), and \( P_v \) and linear velocity \( v_1 \). We see that \( P_0 \) and \( P_v \) are in phase opposition with \( K \), and essentially in phase with each other as well as with \( v_1 \).

Summarizing, it seems that dissipative processes are most efficient when the number of locally stable directions increases. An extreme situation arises in the neighbourhood of \( K = 0 \) (3 stable directions), where the system can dissipate over the entire spectrum of values available. Inverting this statement, one would thus be led to observe that dissipation plays a stabilizing role, and that a highly unstable situation is indicative of poor dissipation.
Figure 3. Phase-space plot of the local Kolmogorov entropy, $K$, versus the local Lyapunov exponents: (a) $\sigma_1$; (b) $\sigma_2$; and (c) $\sigma_3$. Parameter values as in Fig. 1.
The above interpretation is further supported by the observation, summarized in Figs. 7(a)–7(c), that dissipation is favoured when the sum of negative Lyapunov exponents is maximum (its absolute value being a minimum). Remarkably, in this limit there is locally no positive Lyapunov exponent. Because of this, the sum of the (three) negative exponents is then bound to be minus the rate of phase-space-volume contraction, (Eq. (14)). As a corollary $\sigma_3$ will be smaller in absolute value than this rate, see Fig. 3(c).

4. DISSIPATION VERSUS DYNAMICS FOR OTHER PARAMETER VALUES

The Lorenz equations admit a wide variety of solutions, both chaotic and periodic, as the parameter $r$ is varied beyond the value $r = 28$ (Sparrow 1982). Of special interest is the regime of intermittency (Manneville and Pomeau 1979), arising at about $r_c \approx 166.07$. In this section we explore the behaviour of dissipation in the vicinity of this critical point, looking at the regime of intermittent chaos ($r > r_c$) and the regime of complex symmetric periodic orbits ($r < r_c$).

We shall consider more specifically the chaotic attractor arising at $r = 169$ and the periodic attractor arising at $r = 159$. Figures 8(a) and 8(b) depict the corresponding
projections of the attractors on the \((Z, X)\) plane. As in section 3, all three Lyapunov exponents can locally be positive or negative, their sum being still given by \((14)\).

The main result on the behaviour of the dissipation in these dynamical regimes is summarized in Figs. 9(a) and 9(b), and also 10(a) and 10(b) where thermal and viscous dissipation, respectively, are plotted against the local Kolmogorov entropy. We observe a pattern very similar to that of Figs. 5(a) and 5(b) corresponding to \(r = 28\) : a small entropy production if \(K\) is maximal, and a broad spectrum of values of entropy production (including the maximal ones available) as \(K\) decreases to zero. There exist, however, two minor differences:

- at \(K_{\text{max}}\) there is a second 'lobe' in the phase-space plot corresponding to intermediate values of the dissipation, \(P_\theta(t)\);
- the maximum dispersion of dissipation values is not occurring strictly at \(K = 0\) as in section 3, but at some small positive value.

These complications are probably due to the intermittent character of the process, as a result of which the time series of heat and viscosity are almost, but not quite, in antiphase with the time series of the Kolmogorov entropy (not shown).
Figure 6. Time evolution of the local Kolmogorov entropy, $K$, (dashed lines) together with: (a) thermal entropy dissipation, $P_q(t)$, and (b) viscous entropy dissipation, $P_v(t)$ (full lines), showing a phase opposition between $K$ and these quantities. (c) Time evolution of $P_q(t)$ (dotted line), $P_v(t)$ (dashed line) and linear velocity $v_1(t)$ (full line) showing that these quantities are nearly in phase. Parameter values as in Fig. 1 but with a sampling time $\Delta t = 0.001$ units.
Figure 7. (a) Thermal entropy dissipation, $P_q(t)$; (b) viscous entropy dissipation, $P_v(t)$; and (c) local Kolmogorov entropy, $K$, each versus the sum of the local negative Lyapunov exponents, $\sigma_i$. Parameter values as in Fig. 1.
As in section 3, both thermal and viscous dissipation become more vigorous when all three local $\sigma$'s are negative, in which case their sum is minus the phase-space-volume contraction rate, Eq. (14). Furthermore, both types of dissipation are in phase with the local linear velocity, as in Fig. 6(c).

We have also explored the quite different range of values of $r$ for which the Lorenz equations admit simple, 2-circuit, periodic solutions as, for example, when $r = 350$. All conclusions summarized in this section and in section 3 hold true in this case as well.

In the light of the above results it seems legitimate to conclude that we have been able to identify a generic property of the entropy production, as far as its connection with the quantifiers of the complexity of the dynamics in phase-space is concerned.

5. Transient Behaviour of Entropy Production

So far we have been concerned with the behaviour of entropy production once the system has attained its asymptotic regime on the attractor. On the other hand, in many instances the atmosphere is subjected to perturbations of various kinds that momentarily remove its state from the attractor; but once these perturbations have disappeared, there is a transient stage of evolution tending to restore the attractor. In this section we are concerned with the behaviour of entropy production during this stage.
The transient evolution of entropy production from an initial stage, not on the attractor, is also closely related to the existence, or not, of an extremum property. This point is illustrated on Prigogine's minimum entropy production theorem (Prigogine 1947), asserting that in the linear range of irreversible processes, and insofar as the phenomenological coefficients are constant and obey Onsager's reciprocity relation, the steady state is a state of minimum entropy production, $P_s = P_{\text{min}}$. It follows that the entropy production $P(0)$ of a nonstationary state, viewed as an 'initial' state, would be larger than $P_{\text{min}}$; since this must be true for any state of this kind it is concluded that there exists a $t_0$ such that, for any $t > t_0$

$$\frac{dP(t)}{dt} \leq 0, \quad P(t \to \infty) = P_s = P_{\text{min}}$$

(17)

An important point, not sufficiently realized in the literature, is that the validity of extremum principles in general, and of Prigogine's theorem in particular, also depends crucially on the type of constraint exerted on the system. In Prigogine's theorem inertial effects are excluded, and the thermodynamic forces acting on the system must be such that the boundary conditions are fixed or zero. These are exactly the boundary conditions applied to the Boussinesq equations of a fluid layer heated from below. Accordingly,
Prigogine's theorem is valid in the regime of pure conduction, since this regime is purely dissipative and belongs to the linear range of irreversible processes. It fails beyond the thermal convection instability, despite the fact that the constraints remain the same, since inertial effects come now into play. This failure is manifested already in the regime of stationary convection (Nicolis 1979), well below the onset of chaos.

Naturally, the atmosphere can never be in a steady state except in a statistical sense. In the context of the model considered in the present paper such a state will correspond to taking long-time (ergodic) averages over the chaotic attractor. By virtue of (13a) and (13b) this procedure will generate two well-defined values $\tilde{P}_q$ and $\tilde{P}_v$, which will be representative of the 'climatological mean' of the thermal and viscous dissipation on the attractor.

Figures 11(a) and 11(b) summarize the results of numerical simulations of the evolution of the thermal and viscous dissipation toward these asymptotic values, starting from initial states not on the attractor. In each case one considers averages over an initial non-equilibrium ensemble, i.e. a set of points distributed in phase-space according to a density different from the invariant density. Subsequently, each of these points evolves according to (5), and upon averaging over these image points the successive values depicted in Fig. 11 are obtained. Since the system has good ergodic properties one
expects that averaging over a sufficiently short sliding time-window would produce a similar result.

As can be seen, the system sustains states whose entropy production can be lower or higher than the asymptotic values. In particular, in both $P_q(t)$ and $P_v(t)$ one observes an overshoot when starting from a low-dissipation state. This implies that the system at hand displays no extremum property in the usual sense of the term.

It is worth noticing that the entropy production on the attractor itself undergoes substantial fluctuations around the asymptotic level of Fig. 11. Figures 12(a) and 12(b) depict the probability density of the thermal and viscous dissipation corroborating this point. Furthermore, while low dissipation is the most probable state of the viscous part, the thermal part is peaked around a finite value which is about half of the steady-state mean level.

A familiar example of transient behaviour, which is at the very heart of the problem of predicting the states of the atmosphere, is the growth of an initial small error $\epsilon$ induced by the instability of the dynamics, and its eventual increase to a value of the order of the attractor size (Nicolis 1992). One might wonder whether during this crucial stage
dissipation behaves in some unexpected way. Numerical study shows that this is actually not the case; both the error in entropy production

$$\Delta P_t = P(t; X_0 + \epsilon) - P(t; X_0),$$  \hspace{1cm} (18a)

and the entropy production of the error

$$\Delta P'_t = P(t; \epsilon),$$  \hspace{1cm} (18b)

where $X_0$ stands for the initial ‘reference’ state, follow exactly the same pattern as the error itself.

6. CONCLUSIONS

The study of the asymptotic and of the transient evolution of the entropy production in Lorenz’s thermal convection model has led us to two unexpected conclusions. Firstly, a locally highly unstable situation on the attractor corresponds to poor dissipation while, on the contrary, high dissipation values are realized in attractor regions in which the number of stable directions is maximum and the local Kolmogorov entropy nearly
vanishes. Secondly, the evolution on the attractor does not seem to correspond to any extremal property of entropy production; initial states outside the attractor may have both higher and lower entropy production values as compared to the mean entropy production on the attractor. Furthermore, as a rule this last value is approached in time non-monotonically.

The existence of a negative correlation between local entropy production and dynamical instability is, in some respects, natural. Indeed, entropy production is closely related to the dissipative character of a dynamical system, which in turn rests on the fact that the magnitude of the negative Lyapunov exponents is larger than that of the positive ones. It could also be of some practical value, since in a given problem it is far easier to identify regions of high or low dissipation, rather than regions of high or low local Lyapunov exponents or local Kolmogorov entropy. A chart providing the ‘hot’ and ‘cold’ spots of a system, relating to dissipation, would thus constitute a valuable characterization of this system from the point of view of its instability and hence of its predictability properties as well.

The conclusion that the attractor itself is not a regime of maximum or minimum dissipation is to some extent disappointing, since it seems to rule out the existence of a global organizing principle. Yet, in the author’s view, however useful such a principle might have been, its absence is the inevitable consequence of the lack of universality prevailing in the range of phenomena governed by complex, chaotic dynamics. There remains the remote possibility that variational properties might be recovered in some averaged sense using, for example, an appropriate time-window, particularly when the system is constrained by its thermodynamic forces in a different way than that shown in the present problem.

The work reported here can be extended in several directions. The most obvious one is to carry out the analysis on the full-scale Boussinesq equations in the range of spatio–temporal chaos. Of more interest would be to consider realistic atmospheric circulation models, such as quasi-geostrophic models. In these models dissipation is often introduced in a phenomenological way through friction type terms. One difficulty, that will have to be overcome at the outset, would be to achieve a thermodynamically consistent form of entropy production, involving the traditional expressions of the thermodynamic fluxes and forces as displayed in (8b).

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