On the vertical velocity in an isentropic layer

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SUMMARY

We investigate the dynamics of an isentropic layer of air, governed by the hydrostatic primitive equations. In such a layer the horizontal velocity is independent of height if this is assumed to be the case at some initial time. From the mass-conservation equation in pressure coordinates it follows that \( \omega \), the material derivative of the pressure, varies linearly with pressure. From the mass-conservation equation in height coordinates the vertical velocity \( u \) is obtained. The resulting expression is shown to be in accordance with Richardson's equation for the vertical velocity in a general hydrostatic atmosphere, when particularized to an isentropic layer.

KEYWORDS: Isentropic coordinates Layered models Richardson's equation Vertical velocity

1. INTRODUCTION

The use of isentropic coordinates, i.e. of specific entropy or, equivalently, potential temperature as a vertical coordinate, was first proposed by Starr (1945). Starr's proposal was followed by several studies of numerical models based on isentropic coordinates, like those of Eliassen and Raustein (1968, 1970), Raustein (1969) and Bleck (1973, 1974, 1984). Hsu and Arakawa (1990) discuss the advantages and disadvantages of isentropic coordinates in the light of recent modelling experience.

One of the advantages listed by Hsu and Arakawa (1990) is that the hydrostatic primitive equations assume a form analogous to the shallow-water equations. As a result, the potential vorticity is simply proportional to the product of the (isentropic) absolute vorticity and the lapse rate of potential temperature with pressure. Furthermore, a vertical discretization immediately suggests itself: a division of the atmosphere into a finite number of isentropic layers, i.e. layers with uniform potential temperature. As pointed out by Pedlosky (1987), the advantage of such a discretization is that it leads to a simplified—although realizable—physical system that is an exact solution of the basic equations in the absence of forcing/friction and heating/cooling.

For an isopycnic layer of water, i.e. a layer of water with uniform density, Pedlosky (1987) showed that the basic equations are satisfied exactly by deriving an expression for the vertical velocity inside the layer. The use of this expression transcends the merits of formal consistency. Indeed, such an expression is necessary if the output of models based on isopycnic layers is used to advect tracers in a way that is consistent with mass conservation. The same considerations apply to atmospheric models based on isentropic layers. If the mass-conservation equation is used with pressure as a vertical coordinate, then Pedlosky's analysis can be applied directly to an isentropic layer. The corresponding analysis for the mass-conservation equation in height coordinates is less straightforward and has, to the author's knowledge, not been documented in the published literature. Going through this analysis is the purpose of the present article.

A summary of the hydrostatic primitive equations for a rotating sphere is given in section 2. In section 3 it is first recalled that in an isentropic layer the assumption of hydrostatic equilibrium implies that the absolute temperature decreases linearly with height, following the dry adiabatic lapse rate. It then follows that the density and pressure decrease with height according to simple power laws. It is further noticed that, as a result, the horizontal pressure-gradient force per unit mass is independent

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of height. Therefore, if the horizontal velocity is independent of height at some initial time, then this will be true at all later times. In this case, expressions for both $\omega$ (the material derivative of pressure) and $w$ (the vertical velocity) can be obtained that satisfy the mass-conservation equation exactly at all points within the layer. The vertical velocity, $w$, can be obtained alternatively by solving Richardson's (1922) equation for the vertical velocity in a general hydrostatic atmosphere. Indeed, the horizontal velocity being independent of height simplifies this equation to such an extent that it can be solved analytically. The last part of section 3 contains a short discussion of the interface conditions between two isentropic layers and the conditions that apply at the uppermost and lowermost interfaces of a layered isentropic model. Section 4 concludes the article.

2. THE HYDROSTATIC PRIMITIVE EQUATIONS

Atmospheric motion is described by equations that express the conservation of thermodynamic energy, mass and momentum. We state these equations in terms of a coordinate system, $(\lambda, \phi, z)$, that is fixed to the earth, with $\lambda =$ longitude, $\phi =$ latitude and $z =$ height. Here $z = r - a$ with $a$ the radius of the earth and $r$ the distance from the centre of the earth, the earth being assumed perfectly spherical. At every point $r$, given by the coordinates $(\lambda, \phi, z)$, we define a right-handed set of unit vectors $i$, $j$ and $k$, where $i$ points in the direction of increasing $\lambda$, $j$ points in the direction of increasing $\phi$ and $k$ points in the direction of increasing $z$. The expressions that are given are approximate in the sense that the metric coefficients in the horizontal vector operators and material derivatives are approximated by replacing $r$ by $a$. This is justified by the observation that the atmosphere is 'shallow' in the sense that the vertical scale of the motion is much smaller than the radius of the earth, see Phillips (1966).

The prognostic equations state the conservation of thermodynamic energy, mass and horizontal momentum and are given by:

$$
\frac{D_z \theta}{Dt} + w \frac{\partial \theta}{\partial z} = \frac{\theta}{c_p T} Q,
$$

$$
\frac{D_z \rho}{Dt} + w \frac{\partial \rho}{\partial z} = -\rho \nabla_z \cdot \mathbf{v} - \rho \frac{\partial w}{\partial z},
$$

$$
\frac{D_z \mathbf{v}}{Dt} + w \frac{\partial \mathbf{v}}{\partial z} = -f \mathbf{k} \times \mathbf{v} - \frac{1}{\rho} \nabla_z p + \mathbf{F}.
$$

Here $\theta$ is the potential temperature, $c_p$ is the specific heat of dry air at constant pressure, $\rho$ is the density and $\mathbf{v}$ is the horizontal component of the velocity field. The full velocity field, $\mathbf{u}$, is written as $u i + v j + w k$, so that the horizontal velocity field, $\mathbf{u}$, is defined by $u i + v j$ and the vertical velocity field by $wk$. Furthermore, $T$ is the absolute temperature and $p$ is the pressure. The system is driven by a thermodynamic heating/cooling, $Q$, and a mechanical forcing/friction, $\mathbf{F}$, both per unit mass. The parameter $f$ in the horizontal momentum equation is the Coriolis parameter, given by $f = 2 \Omega \sin \phi$, where $\Omega$ is the angular velocity of the earth's rotation. The vector operators and material derivatives have been split up into a horizontal part, in which $z$ is kept constant and denoted by a subscript $z$, and a vertical part. Expressions of the horizontal parts in terms of $\lambda$ and $\phi$ can be obtained from the general formulas given by Haltiner and Williams (1980) and are given in the appendix.
VERTICAL VELOCITY

\[ u \rightarrow z_u, p_u \]
\[ \theta, M, V \]
\[ l \rightarrow z_l, p_l \]

Figure 1. A layer of air with constant potential temperature, \( \theta \). Variables referring to the lower boundary of this layer are denoted by a subscript \( l \), variables referring to the upper boundary of this layer are denoted by a subscript \( u \). It is demonstrated in section 3 that the Montgomery potential, \( M \), within such a layer does not depend on height. It can therefore be assumed that the horizontal velocity, \( V \), in the layer is also independent of height if this is assumed to be the case at some initial time.

The diagnostic equations are the definition of potential temperature, the ideal-gas law and the hydrostatic approximation, given below:

\[ \theta = T \left( \frac{p_r}{p} \right)^{\kappa} \quad (4) \]
\[ \rho = \frac{p}{RT} \quad (5) \]
\[ \frac{\partial p}{\partial z} = -\rho g \quad (6) \]

Here the non-dimensional parameter \( \kappa \) is defined by \( R/c_p \), where \( R \) is the gas constant of dry air. Furthermore, \( g \) is the acceleration due to gravity and \( p_r \) is a reference pressure of 1000 hPa. Note that the equation for hydrostatic equilibrium has replaced the conservation equation of vertical momentum.

3. DYNAMICS OF AN ISENTROPIC LAYER

In this section we will make two further assumptions, namely that the rate of heating/cooling, \( Q \), per unit mass in (1) is zero and that the term \( F \) in (3) is zero as well. As the latter term usually stands for friction processes, this means that we will consider atmospheric flows that are adiabatic and frictionless. In this section we will study the dynamics of a layer of air in which the potential temperature, \( \theta \), is assumed to be uniform. The height of the lower boundary of this layer is denoted by \( z_l \), the height of the upper boundary is denoted by \( z_u \), see Fig. 1. Although it is not strictly necessary for what follows, we will assume that \( z_l \) and \( z_u \) are single-valued functions of the horizontal coordinates \( \lambda \) and \( \phi \). Within a layer of uniform potential temperature the equation for conservation of thermodynamic energy (1) is trivially satisfied. Furthermore, if we make sure that the upper and lower boundaries of this layer move with the local wind velocity, then conservation of thermodynamic energy is also satisfied at the upper and lower boundaries and thus for the whole layer. So, in the following we need to consider only the two prognostic equations, (2) and (3), and the three diagnostic equations, (4), (5) and (6).

Let us first consider the diagnostic equations. From the definition of potential temperature (4) it follows that we have for the temperature, \( T \), as a function of the pressure, \( p \), in an isentropic layer

\[ \frac{T}{\theta} = \left( \frac{p}{p_r} \right)^{\kappa} \quad (7) \]
By differentiation with respect to $z$ we obtain from this expression
\[
\frac{\partial}{\partial z} \left( \frac{T}{\theta} \right) = \kappa \left( \frac{p}{p_r} \right)^\kappa \frac{1}{\rho} \frac{\partial p}{\partial z} .
\] (8)

From the assumption of hydrostatic equilibrium (6), the ideal gas law (5), and (7) we have
\[
\frac{1}{p} \frac{\partial p}{\partial z} = - \frac{1}{\kappa H} \left( \frac{p_r}{p} \right)^\kappa ,
\] (9)
where
\[ H \equiv \frac{c_p \theta}{g} . \] (10)

The quantity $H$ has the dimension of length$^*$ and is referred to as the scale height of the layer. Substituting expression (9) into (8) we find
\[
\frac{\partial}{\partial z} \left( \frac{T}{\theta} \right) = - \frac{1}{H} ,
\] (11)
from which we conclude that in a layer of uniform potential temperature the temperature decreases linearly with height. In fact, the linear decrease of temperature is independent of the value of $\theta$ because it follows from (10) and (11) that
\[
\frac{\partial T}{\partial z} = - \frac{g}{c_p} ,
\] (12)
which we recognise as the dry adiabatic lapse rate; see also Hess (1979, section 6.3).

The solution of (11) can be written alternatively as
\[
\frac{T}{\theta} = \frac{T_1}{\theta} + \frac{z_1 - z}{H} ,
\] (13a)
\[
\frac{T}{\theta} = \frac{T_u}{\theta} + \frac{z_u - z}{H} ,
\] (13b)
where $T_1$ is the temperature at the lower boundary and $T_u$ is the temperature at the upper boundary. From (7) it follows that
\[
\frac{p}{p_r} = \left( \frac{T}{\theta} \right)^{1/\kappa} ,
\] (14)
which, in combination with (13), gives the dependence of the pressure, $p$, on height. To obtain the dependence of the density, $\rho$, on height, we first note that it follows from the ideal-gas law that
\[
\frac{\rho}{\rho_r} = \frac{p}{p_r T} ,
\] (15)
where $\rho_r$ is the density of air at the reference pressure $p_r$, i.e.
\[
\rho_r = \frac{p_r}{R \theta} .
\] (16)

$^*$ With $c_p = 1005\text{ J K}^{-1}\text{kg}^{-1}$ and $g = 9.80616\text{ m s}^{-2}$ (Dutton (1986), appendix 3) and a typical value of 300 K for $\theta$ we obtain for $H$ a value of around 30 km.
We thus have from (14) and (15)

\[
\frac{\rho}{\rho_r} = \left( \frac{T}{\theta} \right)^{1/\kappa - 1}.
\]

(17)

Equations (13), (14) and (17) give the temperature, pressure and density in a layer of uniform potential temperature as a function of height, \(z\), given the value of the temperature, \(T_1\), at the lower boundary or the value, \(T_u\), at the upper boundary. These relationships ensure that the diagnostic equations, (4), (5) and (6), are satisfied in the layer.

Let us now continue with the prognostic equation (3). The linear dependence of absolute temperature with height, as expressed by (12), has an important consequence. This is that in an isentropic layer we may assume that the horizontal velocity, \(v\), is independent of height, \(z\), if this is the case at some initial time. The proof of this and its further consequences is inspired by Pedlosky's (1987, section 3.3) derivation of the shallow-water equations. We first note that it follows from (14) and (17) that we have for the pressure-gradient term in the horizontal-momentum equation, (3):

\[
-\frac{1}{\rho} \nabla_z p = -\frac{p_r}{\rho_r \kappa} \nabla_z \left( \frac{T}{\theta} \right) = -\nabla_z (c_p T) = -\nabla_z M.
\]

(18)

In the second equality we used expression (16) for \(\rho_r\) and the fact that \(\kappa = R/c_p\). In the third equality we used the definition of the Montgomery potential, \(M\) (Montgomery 1937)

\[
M \equiv g z + c_p T.
\]

(19)

We see that both terms in the definition of the Montgomery potential are linear functions of height, \(z\). In fact, it follows from (12) that \(M\) is independent of height so that

\[
M = M_1 = M_u,
\]

(20)

where \(M_1\) is the Montgomery potential evaluated at the lower boundary and \(M_u\) is the Montgomery potential evaluated at the upper boundary. We note, incidentally, that from (7), (19) and (20) it follows that within an isentropic layer the difference between \(z\) and the lower boundary, \(z_1\), is given by

\[
\frac{z - z_1}{H} = \left( \frac{p_1}{p_r} \right)^\kappa - \left( \frac{p}{p_r} \right)^\kappa,
\]

(21)

where \(p\) and \(p_1\) are the pressures at heights \(z\) and \(z_1\), respectively. As a result of (18), the horizontal pressure-gradient force is independent of height in an isentropic layer. As the Coriolis parameter, \(f\), is also independent of height and because it was assumed that \(\textbf{F} = 0\), this means that the horizontal acceleration is independent of height as well. If we, therefore, prepare our isentropic layer in a state in which the horizontal velocity does not depend on height, this will remain so in the course of time. We conclude that in an isentropic layer

\[
\frac{\partial M}{\partial z} = 0,
\]

(22)

and that we may consistently assume also that

\[
\frac{\partial \mathbf{v}}{\partial z} = 0
\]

(23)
in this layer. As a result, the second term on the left-hand side of the equation for conservation of momentum, (3), vanishes. This equation becomes, therefore,

$$\frac{D_z v}{Dt} = - f k \times v - \nabla_z M,$$

(24)

where we recall that $F$ was assumed to be zero.

We now consider Eq. (2) for the conservation of mass. This equation will be used to find an expression for the vertical velocity, $w$, in terms of fields already known. Before calculating $w$ we first consider $\omega$, the material derivative of the pressure*. Using the transformation formulas between alternative vertical coordinates as given by Haltiner and Williams (1980, sections 1-9), we obtain from (2)—for a hydrostatic atmosphere—the familiar linear form of the mass-conservation equation:

$$\frac{\partial \omega}{\partial p} = - \nabla_p \cdot v.$$

(25)

Here the subscript $p$ means that the horizontal derivatives are to be taken with $p$ constant instead of $z$. Now, because $\mathbf{v}$ is independent of height—see (23)—and, therefore, independent of pressure, the horizontal divergence $\nabla_p \cdot \mathbf{v}$ is also independent of pressure. Equation (25) can thus be integrated straightforwardly. The transformation formulas given by Haltiner and Williams (1980) show that, due to (23),

$$\nabla_p \cdot \mathbf{v} = \nabla_z \cdot \mathbf{v}.$$

(26)

Also due to (23) we have that $\omega$, when evaluated at the lower boundary, is given by $\omega_l = D_z p_l / Dt$, so that the solution of (25) can be written as

$$\omega = \frac{D_z p_l}{Dt} + (\nabla_z \cdot \mathbf{v})(p_l - p).$$

(27)

The value of $\omega$ at the upper boundary should be equal to $\omega_u = D_z p_u / Dt$ so that it follows from (27)

$$\frac{D_z}{Dt}(p_l - p_u) + (\nabla_z \cdot \mathbf{v})(p_l - p_u) = 0.$$

(28)

This is a familiar equation, see Rossby (1940). We notice that $(p_l - p_u)/g$ is the mass per unit horizontal area in a vertical column of air in hydrostatic equilibrium, bounded by upper and lower boundaries at pressures $p_u$ and $p_l$, respectively. Therefore, the equation above states that this mass is conserved in an isentropic layer following the horizontal motion of the column. A shortcut to this result from general mass conservation is to note that the mass contained in a vertical column of air cannot change in time because—due to the fact that the horizontal velocity is independent of height—no air can enter or leave the column from the sides and—due to the material character of the upper and lower boundaries—no air can enter or leave the column from the top or the bottom. Essential in this argument is that the horizontal velocity is independent of height and that the upper and lower boundaries are material surfaces. Rossby (1940) also noted that the mass-conservation equation, as just derived, can be used to eliminate the horizontal divergence term from the conservation equation for absolute vorticity. The result is the well-known material conservation law, $D_z P / Dt = 0$, in which the potential vorticity is defined by $P = (f + \zeta)/(p_l - p_u)$, where $\zeta = k \cdot \nabla_z \times \mathbf{v}$ is the relative vorticity.

* I am grateful to the associate editor, Dr. G. J. Shat, for his suggestion to determine $\omega$ as a function of $p$. 
Rossby, in the same article, remarked that a similar conservation law can be derived for a continuously stratified atmosphere. Indeed, this is the material conservation law of potential vorticity for the hydrostatic primitive equations in isentropic coordinates. Here \( P = -g(f + \zeta_\theta)(\partial \theta / \partial p) \), where \( \zeta_\theta \) is the relative vorticity calculated with \( \theta \) constant. The latter definition can—in fact—also be used in a layer with constant \( \theta \). In that case \( P \) will be zero within the layer and concentrated in thin sheets at the upper and lower boundaries.

To obtain the vertical velocity, \( w \), from the mass-conservation equation in height coordinates, i.e. directly from (2), we first rewrite this equation as

\[
\frac{\partial w}{\partial z} + \left( \frac{1}{\rho} \frac{\partial \rho}{\partial z} \right) w = -\nabla_z \cdot \mathbf{v} - \frac{1}{\rho} \frac{D_z \rho}{D t},
\]

(29)

Like (25) this is a first-order linear inhomogeneous equation in the vertical velocity \( w \). We repeat that, due to (23), the horizontal divergence term on the right-hand side is independent of \( z \). For the \( z \)-dependence of the second terms on the left- and right-hand sides of (29) we have, using (17)

\[
\frac{1}{\rho} \frac{\partial \rho}{\partial z} = \left( \frac{\theta}{T} \right) \left( \frac{1 - \kappa}{\kappa} \right) \frac{\partial}{\partial z} \left( \frac{T}{\theta} \right), \tag{30}
\]

\[
\frac{1}{\rho} \frac{D_z \rho}{D t} = \left( \frac{\theta}{T} \right) \left( \frac{1 - \kappa}{\kappa} \right) \frac{D_z}{D t} \left( \frac{T}{\theta} \right). \tag{31}
\]

We have from (13a)

\[
\frac{\partial}{\partial z} \left( \frac{T}{\theta} \right) = -\frac{1}{H}, \tag{32}
\]

\[
\frac{D_z}{D t} \left( \frac{T}{\theta} \right) = \frac{D_z}{D t} \left( \frac{z_1 + T_1}{H} \right) \tag{33}
\]

from which we deduce that the \( z \)-dependence of (30) and (31) resides in the factor \( (\theta/T) \). A solution of the homogeneous part of problem (29) is

\[
\frac{w}{H} = A \left( \frac{T}{\theta} \right)^{1-1/\kappa}, \tag{34}
\]

where \( A \) is a field that only depends on the horizontal coordinates. A particular solution of the same problem is

\[
\frac{w}{H} = \kappa (\nabla_z \cdot \mathbf{v}) \left( \frac{T}{\theta} \right) + \frac{D_z}{D t} \left( \frac{z_1 + T_1}{H} \right). \tag{35}
\]

The general solution of (29), therefore, is

\[
\frac{w}{H} = A \left( \frac{T}{\theta} \right)^{1-1/\kappa} + \kappa (\nabla_z \cdot \mathbf{v}) \left( \frac{T}{\theta} \right) + \frac{D_z}{D t} \left( \frac{z_1 + T_1}{H} \right). \tag{36}
\]

If we require that the vertical velocity at the lower boundary, \( w_1 \), is equal to the material vertical velocity of the lower boundary, \( D_z z_1/D t \), i.e.

\[
\frac{w_1}{H} = \frac{D_z}{D t} \left( \frac{z_1}{H} \right), \tag{37}
\]
then it follows that

$$\begin{align*}
A &= -\left( \frac{T_i}{\theta} \right)^{1/\kappa - 1} \left\{ \frac{D_z}{Dt} \left( \frac{T_i}{\theta} \right) + \kappa (\nabla_z \cdot \mathbf{v}) \left( \frac{T_i}{\theta} \right) \right\}. 
\end{align*}$$

(38)

Using (7) this expression can be simplified to

$$A = -\kappa \left\{ \frac{D_z}{Dt} \left( \frac{p_l}{p_r} \right) + (\nabla_z \cdot \mathbf{v}) \left( \frac{p_l}{p_r} \right) \right\},$$

(39)

where \( p_l \) is the pressure at the lower boundary. Using (13a) to write out the factor \( (T/\theta) \) behind the horizontal divergence term in (36), and (7) to express the factor \( (T/\theta) \) behind \( A \) in terms of pressure, we can write for (36)

$$\begin{align*}
\frac{w}{H} &= \frac{D_z}{Dt} \left( \frac{z_l}{H} \right) + \kappa (\nabla_z \cdot \mathbf{v}) \left( \frac{z_l - z}{H} \right) + \frac{D_z}{Dt} \left( \frac{T_i}{\theta} \right) + \kappa (\nabla_z \cdot \mathbf{v}) \left( \frac{T_i}{\theta} \right) \\
&\quad - \kappa \left( \frac{p_l}{p_r} \right)^{\kappa - 1} \left\{ \frac{D_z}{Dt} \left( \frac{p_l}{p_r} \right) + (\nabla_z \cdot \mathbf{v}) \left( \frac{p_l}{p_r} \right) \right\}. 
\end{align*}$$

(40)

Using (7) we can write for the sum of the third and fourth terms on the right-hand side of (40)

$$\frac{D_z}{Dt} \left( \frac{T_i}{\theta} \right) + \kappa (\nabla_z \cdot \mathbf{v}) \left( \frac{T_i}{\theta} \right) = \kappa \left( \frac{p_l}{p_r} \right)^{\kappa - 1} \left\{ \frac{D_z}{Dt} \left( \frac{p_l}{p_r} \right) + (\nabla_z \cdot \mathbf{v}) \left( \frac{p_l}{p_r} \right) \right\},$$

(41)

so that (40) becomes

$$\begin{align*}
\frac{w}{H} &= \frac{D_z}{Dt} \left( \frac{z_l}{H} \right) + \kappa (\nabla_z \cdot \mathbf{v}) \left( \frac{z_l - z}{H} \right) \\
&\quad + \kappa \left\{ \left( \frac{p_l}{p_r} \right)^{\kappa - 1} - \left( \frac{p_l}{p_r} \right)^{\kappa - 1} \right\} \left\{ \frac{D_z}{Dt} \left( \frac{p_l}{p_r} \right) + (\nabla_z \cdot \mathbf{v}) \left( \frac{p_l}{p_r} \right) \right\}. 
\end{align*}$$

(42)

This is the sought-for expression for the vertical velocity, \( w \). Again, we also require that the vertical velocity at the upper boundary, \( w_u \), is equal to \( D_z z_u / Dt \), i.e.

$$\frac{w_u}{H} = \frac{D_z}{Dt} \left( \frac{z_u}{H} \right).$$

(43)

Upon substituting this into (42) and using (21) with \( z_u \) and \( p_u \) substituted for \( z \) and \( p \), respectively, we see again that (28) must be valid. It should be emphasized that (42)---like (27)---is a purely diagnostic equation. It gives the vertical velocity that is needed to fulfill the mass-conservation equation and hydrostatic equilibrium in an isentropic layer. Later in this section we recall that a closed multi-layered model atmosphere can be formulated in terms of the pressures at the interfaces, without any need to know the vertical velocity inside the layers. This means that at any particular time all quantities needed to calculate the vertical velocity from (42) are known.

It is instructive to check the validity of (42) by investigating whether it is consistent with the equation for the vertical velocity derived by Richardson (1922, ch. 5/2) for general hydrostatic flow. We will use the equation in integral form as derived by Kasahara and Washington (1967). Allowing for an upper and lower material boundary denoted by subscripts u and l and putting the heating/cooling, \( Q \), equal to zero, their
Eq. (2.16a) reads:

\[ w = w_1 - \int_{z_1}^{z} (\nabla_z \cdot \mathbf{v}) \, dz' - \frac{1}{\gamma} \int_{z_1}^{z} \frac{B + J}{p} \, dz', \]

where (see their Eq. (2.15) and the remark after their Eq. (2.10))

\[ B = \frac{\partial p_u}{\partial t}, \quad (45a) \]

\[ J = \mathbf{v} \cdot \nabla_z p - g \int_{z}^{z_u} \nabla_z \cdot (\rho \mathbf{v}) \, dz' - g \rho_u w_u. \quad (45b) \]

Here \( \gamma = c_p/c_v \). Making use of the fact that \( \mathbf{v} \) is independent of height, it can be seen that we have

\[ B + J = \frac{D_z p_u}{Dt} + (\nabla_z \cdot \mathbf{v}) p_u - (\nabla_z \cdot \mathbf{v}) p. \quad (46) \]

This gives

\[ w = w_1 - \left(1 - \frac{1}{\gamma}\right) \int_{z_1}^{z} (\nabla_z \cdot \mathbf{v}) \, dz' - \frac{1}{\gamma} \left( \frac{D_z p_u}{Dt} + (\nabla_z \cdot \mathbf{v}) p_u \right) \int_{z_1}^{z} \frac{dz'}{p}. \quad (47) \]

Now we have:

\[ \int_{z_1}^{z} (\nabla_z \cdot \mathbf{v}) \, dz' = (\nabla_z \cdot \mathbf{v})(z - z_1). \quad (48a) \]

\[ \int_{z_1}^{z} \frac{dz'}{p} = -H \kappa \left( \frac{1}{\kappa - 1} \right) \left\{ \left( \frac{p}{p_t} \right)^{\kappa - 1} - \left( \frac{p_1}{p_t} \right)^{\kappa - 1} \right\}. \quad (48b) \]

Next we use that (recall that \( R = c_p - c_v \))

\[ -\left(1 - \frac{1}{\gamma}\right) = \frac{1}{\gamma} \frac{\kappa}{\kappa - 1} = -\kappa, \quad (49a) \]

so that we find for the vertical velocity:

\[ w = w_1 + \kappa (\nabla_z \cdot \mathbf{v})(z_1 - z) + \kappa H \frac{1}{p_t} \left\{ \left( \frac{p_1}{p_t} \right)^{\kappa - 1} - \left( \frac{p}{p_t} \right)^{\kappa - 1} \right\} \left( \frac{D_z p_u}{Dt} + (\nabla_z \cdot \mathbf{v}) p_u \right). \quad (50) \]

Dividing both sides of the equation by \( H \) and using (28) as well as (37), we verify that this result is identical with (42). This confirms that our expression for the vertical velocity is consistent with Richardson's equation.

We conclude this section by recalling that layered isentropic models can be constructed by stacking one or more of these layers on top of each other. The potential temperatures in these layers will generally be different as a result of which the temperature and density, as well as the Montgomery potential and the horizontal velocity, will be discontinuous at the interfaces. The vertical velocity and the pressure are, however, continuous at the interfaces. Applying expression (19) for the Montgomery potential at both sides of an interface gives a relation between the Montgomery potentials in the two adjacent layers in terms of the different values of the potential temperature of the layers.
and the pressure at the interface. The relation reads

$$M^{(i)} - M^{(i+1)} = c_p (\theta^{(i)} - \theta^{(i+1)}) \left( \frac{p_i}{p_f} \right)^k,$$

(51)

where the superscript \( (i) \) denotes fields in the layer above the interface, \( (i + 1) \) fields in the layer beneath the interface, and the subscript, \( i \), the value of a field at the interface itself.

For the uppermost and lowermost layers we need boundary conditions. Let the uppermost layer be denoted by the index 1 and the height and pressure of the upper boundary of this layer by \( z_0 \) and \( p_0 \), respectively. We can choose from two possible boundary conditions. Either we prescribe the height, \( z_0 \), of the upper boundary and let the pressure be free or we prescribe the pressure, \( p_0 \), of the upper boundary and let the height be free. Clearly, for an atmospheric model the first possibility would imply an external constraint. That leaves us with the second possibility. The choice

$$p_0 = 0$$

(52)

is then the most natural as it respects the material character of the upper boundary. Indeed, it now separates the uppermost layer from the vacuum of outer space. Note in this respect, that with this boundary condition we have that \( T_u^{(1)} = 0 \) so that also \( \rho_u^{(1)} = 0 \). Also note that, because the temperature in an isentropic layer decreases linearly with height, the height \( z_0 \) at which \( p_0 \) is zero is finite. For the lowermost layer the considerations are analogous. We denote this layer by the index \( N \) and the height and pressure of the lower boundary of this layer by \( z_N \) and \( p_N \), respectively. Also here we have two possible boundary conditions that we can impose: we can either fix the height \( z_N \) and let the pressure be free or we fix the pressure \( p_N \) and let the height be free. The appropriate choice here is, of course, to fix this height by the earth's orography \( z_B \), i.e. to require that

$$z_N = z_B.$$  

(53)

With Eq. (51) for the relation between two isentropic layers and Eqs. (52) and (53) for the upper and lower boundary conditions, a layered isentropic model is now a complete dynamical system. Indeed, for a model consisting of \( N \) layers we have \( N \) velocity fields \( \mathbf{v}^{(i)} \), \( N \) pressures \( p_i \) \( (p_0 = 0) \) and \( N \) Montgomery potentials \( M^{(i)} \), and, furthermore, \( N \) momentum equations, \( N \) mass-conservation equations, \( N - 1 \) relations between Montgomery potentials and one expression for the Montgomery potential in the lowermost layer, evaluated at the lower boundary. The temperature, pressure and density inside the layers are given by (13), (14) and (17). The vertical motion field inside the layers can be obtained from (27) or (42).

4. CONCLUSIONS

When potential temperature is used as a vertical coordinate, the conservation laws of mass and momentum in the hydrostatic primitive equations assume the same form as the corresponding laws in the shallow-water equations. As a consequence, the potential vorticity gets a particularly simple expression: it is proportional to the product of the (isentropic) absolute vorticity and the lapse rate of potential temperature with pressure, in close analogy with the corresponding expression for the shallow-water equations. Furthermore, isentropic coordinates naturally suggest a vertical discretization of the atmosphere in terms of isentropic layers. In this paper we have shown how the full
three-dimensional motion of air within a hydrostatic isentropic layer is determined by
the horizontal velocity in the layer and by the pressure at the upper and lower boundaries
of the layer. In particular we have shown how mass conservation in combination with
hydrostatic equilibrium determines the vertical motion in terms of these variables. In the
case of the vertical motion being expressed in terms of $\omega$—the material derivative of the
pressure $p$—this is straightforward and leads to a linear dependence of $\omega$ on pressure
inside an isentropic layer, see (27). In the case of the vertical motion being expressed in
terms of $w$—the material derivative of the height $z$—the result is more complicated and
given by (42), which expression is the main result of the present paper. It is shown that
expression (42) can also be obtained as the solution of Richardson’s (1922) equation for
the vertical velocity in a general hydrostatic atmosphere, when particularized to an isen-
 tropic layer. The results of the paper emphasize that an atmospheric model consisting
of a finite number of isentropic layers is a system that, although a simplification of the
real continuously stratified atmosphere, satisfies the basic equations exactly. This is not
only satisfying from a pedagogic point of view but is also relevant numerically because it
allows a dynamically consistent way of interpolating all fields—in particular the vertical
motion field—within the layers of an isentropic numerical model of the atmosphere.

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APPENDIX

Vector operators and material derivatives

In this appendix we give expressions for the horizontal components of the vector
operators and material derivatives of a scalar $A$ and a vector $A_h$. (Expressions (A.3)
and (A.4) are added for completeness.) The components of the vector $A_h$ along the unit
vectors $i$ and $j$ are denoted by $A_\lambda$ and $A_\phi$, respectively. The expressions below are based
on appendix A-3 of Haltiner and Williams (1980), where in the metric coefficients the
distance $r$ from the centre of the earth is replaced by the radius $a$ of the earth, as argued
by Phillips (1966). For the horizontal components of the vector operators we thus have:

$$\nabla_z A = \frac{1}{a \cos \phi} \frac{\partial A}{\partial \lambda} i + \frac{1}{a} \frac{\partial A}{\partial \phi} j,$$

(A.1)

$$\nabla_z \cdot A_h = \frac{1}{a \cos \phi} \frac{\partial A_\lambda}{\partial \lambda} + \frac{1}{a \cos \phi} \frac{\partial (\cos \phi A_\phi)}{\partial \phi},$$

(A.2)

$$\nabla_z \times A_h = \left( \frac{1}{a \cos \phi} \frac{\partial A_\phi}{\partial \lambda} - \frac{1}{a \cos \phi} \frac{\partial (\cos \phi A_\lambda)}{\partial \phi} \right) k,$$

(A.3)

$$\nabla_z^2 A = \frac{1}{a^2 \cos^2 \phi} \frac{\partial^2 A}{\partial \lambda^2} + \frac{1}{a^2 \cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial A}{\partial \phi} \right).$$

(A.4)

For the horizontal material derivative of a scalar $A$ and a vector $A_h$ we have

$$\frac{D_z A}{Dt} = \frac{\partial A}{\partial t} + \mathbf{v} \cdot \nabla_z A,$$

(A.5)

$$\frac{D_z A_h}{Dt} = \left( \frac{D_z A_\lambda}{Dt} - u A_\phi \frac{\tan \phi}{a} \right) i + \left( \frac{D_z A_\phi}{Dt} + u A_\lambda \frac{\tan \phi}{a} \right) j,$$

(A.6)
where the material derivatives in the latter expression are those for a scalar. Note that, apart from the fact that the scalars and vector fields are labelled by the vertical coordinate \( z \), these operators and material derivatives are identical to the corresponding operators and material derivatives for two-dimensional scalars and vector fields on the surface of a sphere with radius \( a \).

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