Nonlinear equatorial Kelvin waves and CISK. I: Small-amplitude approximation and the trailing edge of a cloud region

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SUMMARY

Kelvin wave–CISK (Conditional Instability of the Second Kind) is studied as a possible explanation for the intraseasonal oscillations of the equatorial troposphere. The small-amplitude approximation will be made in part I of this paper. The governing equations of the small-amplitude approximation are nonlinear, however, because of the cumulus parametrization scheme used. It was found that: (1) Kelvin waves when coupled with cumulus heating are never unstable if the cumulus heating rate is smaller than the adiabatic cooling rate; (2) the solution is not equatorially confined if the cumulus heating rate is larger than the adiabatic cooling rate; (3) since a Kelvin wave is not a dispersive wave, the method of characteristics is used to solve the governing equations when the cumulus heating rate is smaller than the adiabatic cooling rate. It was found that the cumulus parametrization scheme often used in theoretical studies of CISK is incompatible with the so-called ‘jump condition’ of the generalized solutions often required by hyperbolic systems of equations.

KEYWORDS: Convection Intraseasonal oscillation Tropical dynamics

1. INTRODUCTION

The present study was motivated by Lim et al. (1990) who pointed out the intrinsic nonlinearity of the cumulus parametrization scheme usually used in theoretical studies of CISK (Conditional Instability of the Second Kind). We will show in section 3 that this kind of intrinsic nonlinearity is incompatible with the so called ‘jump condition’. A jump condition is a conservation law when applied to a discontinuity. It will be explained in section 3.

This study was also motivated by the fact that equatorial Kelvin wave–CISK has been proposed as a possible mechanism responsible for the equatorial intraseasonal oscillations in the atmosphere since the discovery of the oscillation by Madden and Julian (Madden and Julian 1971; Chang and Lim 1988; Lim et al. 1990; Lau and Peng 1987; Lau et al. 1989; Emanuel 1987; Neelin et al. 1987, Sui and Lau 1989; Wang and Rui 1990; Cho et al. 1994).

Let us first clarify the meaning of a few terms that are used throughout this study. CISK will mean Conditional Instability of the Second Kind, no more and no less. In theoretical studies of the CISK process, there is a simple cumulus parametrization scheme that is often used which I also adopt in this study. In the scheme one assumes that cumulus heating is proportional to the rate of convergence in the boundary layer, usually represented approximately by $\omega^*$, the mean vertical velocity at the cloud-base level. The superscript * denotes values of variables at the height of the cloud base, $p^*$. The vertical distribution of latent-heat release is assumed to be given by a function $f(p)$ of height. Different vertical-distribution functions can be found in the literature; it is a function to be specified by the authors. Recently Cho and Pendlebury (1997) discussed sensitivities of CISK analysis to the $f(p)$ used. To summarize this simple parametrization scheme, we write

$$-Q(p) = \epsilon\sigma \omega^* f(p), \quad (1)$$

where $\epsilon\sigma$ is used to denote a proportional constant and $\sigma$ is used to represent the vertical stability parameter $-\frac{RT}{p} \frac{\partial \ln \theta}{\partial p}$, where $R$ is the universal gas constant, $T$ is temperature

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and $\theta$ potential temperature. The proportional constant $\varepsilon$ is a function of atmospheric moisture content, for example. In the literature this simple scheme is often referred to as the CISK parametrization, but I avoid using such a term.

What surprised me most among the results of this study is the conclusion that Kelvin waves and cumulus heating will not lead to instability. The details of this conclusion will be found in section 3 for small-amplitude Kelvin waves, and in Part II when the wave amplitude is finite. In section 2 let us first briefly discuss the governing equations.

2. Governing equations for nonlinear Kelvin waves

The nonlinear equations for equatorial disturbances written on the equatorial $\beta$-plane are

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} + \frac{\partial \Phi}{\partial x} - \beta y u = 0 \tag{2a}
\]
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} + \frac{\partial \Phi}{\partial y} + \beta y u = 0 \tag{2b}
\]
\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \left( \frac{\partial \Phi}{\partial p} \right) + \sigma \omega = -\frac{\alpha}{C_p} \frac{dx}{dt} \tag{2c}
\]
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \tag{2d}
\]
\[
\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{RT}{p} \tag{2e}
\]

Here $u$, $v$ and $w$ are conventional velocity components in cartesian coordinates $x$, $y$, $p$; $t$ is time, $C_p$ the specific heat of air at constant pressure, and $\Phi$ is the geopotential. The right-hand side of (2c) represents heating by cumulus convection where $s$ is the entropy; $\sigma$ is the vertical stability parameter which is assumed to be a constant. Equation (2d) is the mass continuity equation and (2e) is the hydrostatic relation. Since in equatorial Kelvin waves the $y$-component of the velocity is zero, the $y$-component of the equation of motion becomes a diagnostic relation describing geostrophic balance in that direction. The governing equations for equatorial Kelvin waves then become

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \omega \frac{\partial u}{\partial p} + \frac{\partial \Phi}{\partial x} = 0 \tag{3a}
\]
\[
\frac{\partial}{\partial t} \left( \frac{\partial \Phi}{\partial p} \right) + u \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial p} \right) + \sigma \omega = Q. \tag{3b}
\]

Here we have kept only the two prognostic equations. The three diagnostic relationships are the geostrophic balance condition in the $y$-direction

\[
\frac{\partial \Phi}{\partial y} + \beta y u = 0, \tag{3c}
\]

together with the mass continuity equation (2d) and the hydrostatic relation (2e). The term of vertical advection of zonal momentum in (3a) is often ignored. This can be justified by assuming $u \sim 10 \text{ m s}^{-1}$, $\omega \sim 5 \times 10^{-5} \text{ mb s}^{-1}$, a horizontal scale
\( L \sim 10^7 \) m, and a vertical scale \( 10^3 \) mb. This gives

\[
\frac{\partial u}{\partial x} \sim 10^{-5}
\]

\[
\frac{\partial u}{\partial p} \sim 5 \times 10^{-7}.
\]

We shall adopt (1) to parametrize cumulus convection. We assume that \( f(p) \) is a normalized vertical distribution of cumulus heating:

\[
\frac{1}{\Delta p} \int_{p_t}^{p_s} f(p) \, dp = 1,
\]

where \( p_t \) and \( p_s \) denote the pressures at the tropopause and at the surface. \( \Delta p = p_s - p_t \). At the present time we keep the function \( f \) as general as possible. As pointed out in Lim et al. (1990) the values of \( \varepsilon \) should depend on the values of \( \omega^* \). For example, if \( \omega^* < 0 \), then we should have convection and therefore \( \varepsilon > 0 \); on the other hand, if \( \omega^* > 0 \), then convection should be suppressed and we have \( \varepsilon = 0 \), i.e.

\[
\varepsilon = \begin{cases} 
\varepsilon_0 = \text{const.} > 0 & \text{if } \omega^* < 0 \\
0 & \text{if } \omega^* > 0.
\end{cases}
\]

As pointed out by Lim et al. (1990) this makes \( \varepsilon \) a nonlinear function of \( \omega^* \). Furthermore, this nonlinear dependence does not depend on the amplitude of \( \omega^* \) and hence cannot be linearized.

3. \textbf{Small-amplitude approximation}

If one assumes the amplitudes of the perturbations from a resting atmosphere are very small, then the second terms on the left-hand sides of (3a) and (3b), as well as the third term on the left-hand side of (3a), can be neglected. The governing equations in this case become

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial \Phi}{\partial x} &= 0 \\
\frac{\partial}{\partial t} \left( \frac{\partial \Phi}{\partial p} \right) + \sigma \omega &= Q \\
\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial p} &= 0 \\
\frac{\partial \Phi}{\partial y} + \beta y u &= 0 \\
\frac{\partial \Phi}{\partial p} &= -\alpha = \frac{RT}{p}.
\end{align*}
\]

(4)

Note that the thermodynamic equation is still a nonlinear equation because of the term \( Q \). All the left-hand side terms have been linearized, however. Further we can assume the separability of the three spatial coordinate axes as follows:

\[
\begin{align*}
&u = u_x(x, t) u_y(y) u_p(p) & \Phi = \Phi_x(x, t) \Phi_y(y) \Phi_p(p) & \omega = \omega_x(x, t) \omega_y(y) \omega_p(p).
\end{align*}
\]

From the third and the fourth equations in (4) we can assume that \( u_p = \Phi_p = \omega_p' \), \( u_y = \omega_y \), and \( \partial u_x / \partial x + \omega_x = 0 \), where in this section the prime denotes the vertical
derivative. If one assumes that \( u_p = \Phi_p = g(p) \) and \( \omega_p = h(p) \), then the second equation in (4) implies that

\[
\frac{\partial \Phi_x}{\partial t} - h''(p) \left[ \frac{\partial u_x}{\partial x} \right] + \left( \varepsilon\sigma h(p^*) f(p) - \sigma h(p) \right) = 0.
\]

Divide the above equation by \( h''(p) \) at height levels where \( h''(p) \) is not zero, it is obvious then that

\[
-\frac{h(p) \left( \sigma - \varepsilon\sigma h(p^*) f(p) \right)}{h''(p)} = \text{const.} = b^2,
\]

(5a)

since \( \Phi_x \) and \( u_x \) are functions of \( x \) and \( t \) only. Note that \( b^2 \) defined in the equation is a real constant in each of the cloud regions or the cloud-free region, and its value is proportional to the difference between the adiabatic cooling term and the cumulus heating term in the thermodynamic equation. Because of the requirement that \( \omega = 0 \) at \( p = p_c \) and \( p_i \), the ratio \( -h(p)/h''(p) \) is usually a positive quantity. Therefore \( \varepsilon = 0 \) and \( b^2 > 0 \) in a cloud-free region. In the cloud region, \( b^2 \geq 0 \) when the adiabatic cooling effect is larger than or equal to the cumulus heating term at all height levels. Furthermore, as we shall see later in this section, \( \pm b \) is the propagation speed of the hyperbolic waves governed by the equations. If, in the cloud region, the cumulus heating is larger than adiabatic cooling at all levels then \( b^2 < 0 \), i.e. \( b \) is a purely imaginary number in the cloud region. From (5a) a differential equation can be derived for \( h \):

\[
h''(p) + \frac{\sigma}{b^2} h(p) = \frac{\varepsilon\sigma}{b^2} h(p^*) f(p).
\]

(5b)

The solution must satisfy the requirement that at the cloud-base level, \( h(p) \) must be equal to \( h(p^*) \) on the right-hand side of the equation. This condition can be used to determine the value of \( b^2 \). For details of this analysis see appendix A. Note that in appendix A, \( b^2 \) is determined from an integral relationship. In previous linear studies people were looking for complex values for \( b^2 \). In this paper we merely point out that \( b^2 \) is physically realistic only when it is a real number. Further, as shown in appendix A, \( b^2 \) is larger in the cloud-free region than in the cloudy region.

From the second equation of (4):\n
\[
\frac{\partial \Phi_x}{\partial t} + b^2 \frac{\partial u_x}{\partial x} = 0.
\]

(6)

When this result is coupled with the first equation of (4) we obtain

\[
\frac{\partial u}{\partial t} + \frac{\partial \Phi}{\partial x} = 0
\]

\[
\frac{\partial \Phi}{\partial t} + b^2 \frac{\partial u}{\partial x} = 0.
\]

(7)

We have dropped the subscript \( x \) in these equations, and this convention will be adopted throughout this small-amplitude analysis.

If we let

\[
\psi = \begin{pmatrix} u \\ \Phi \end{pmatrix},
\]
Then (7) can be written as:

\[ \frac{\partial \psi}{\partial t} + A \frac{\partial \psi}{\partial x} = 0, \tag{8} \]

where \( A \) is the matrix

\[ A = \begin{pmatrix} 0 & 1 \\ b^2 & 0 \end{pmatrix}. \]

The eigenvalues of \( A \) are

\[ \lambda_1 = +b, \quad \lambda_2 = -b. \tag{9a} \]

These are the propagating speed \( c \) of the hyperbolic waves if \( b^2 \) is positive. The eigenvectors are

\[ r_1 = \begin{pmatrix} 1 \\ b \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ -b \end{pmatrix}. \tag{9b} \]

Note that because \( A \) is not symmetric, the eigenvectors are not mutually perpendicular. This can be resolved in two ways. First by changing the scales for time and length so that the unit for velocity becomes \( b \). In such a set of scales, we have

\[ b^2 \frac{\partial u}{\partial t} + b^2 \frac{\partial \phi}{\partial x} = 0 \]

\[ b^2 \frac{\partial \phi}{\partial t} + b^2 \frac{\partial u}{\partial x} = 0. \]

We have assumed that \( b \neq 0 \). The equation can be written as

\[ \frac{\partial \psi}{\partial t} + B \frac{\partial \psi}{\partial x} = 0, \]

where \( \psi \) is the same as defined earlier. \( B \) is given by

\[ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Note that \( B \) is a symmetric matrix.

The second approach is based on the fact that Kelvin waves are only allowed to propagate eastward (explained later in this section). Since only one mode is allowed to propagate, we will use the ordinary units.

The properties of the governing equations clearly depend on the sign of \( b^2 \). If \( b^2 > 0 \), (i.e. when the adiabatic cooling rate is larger than the cumulus heating rate) then the governing equations are a hyperbolic system. If \( b^2 < 0 \), then the eigenvalues are purely imaginary, and hence not allowed because the solutions are not equatorially confined (see later in this section).

From (9b) we can see that

\[ \phi_{1,2} = \pm bu_{1,2}. \]

This can be substituted into (3c) to obtain the \( y \)-structure of the eigenvectors:

\[ \phi_i = \phi_0 \exp \left( -\frac{\beta}{2c} y^2 \right), \]
where

\[ c = \pm b \quad \text{for} \quad i = 1, 2. \quad (10) \]

Therefore, an equatorially trapped mode is possible only when \( b^2 > 0 \) and only for the first eigenmode. Since we are dealing with equatorial Kelvin waves, we will assume the case of \( b^2 > 0 \) in the remainder of this paper.

The standard method of solving hyperbolic sets of equations is the method of characteristics. Let the initial value of the solution \( \psi(x, t = 0) = \psi_0(x) \). One can make the expansion if \( r_i \) for \( i = 1, 2 \) are mutually perpendicular:

\[ \psi_0 = \sum_{i=1}^{2} v_i(x, 0) r_i \]

and

\[ \psi(x, t) = \sum_{i=1}^{2} v_i(x, t) r_i = \sum_{i=1}^{2} v_i(x - \lambda_i t, 0) r_i \]

if the system is hyperbolic. The lines in the \( (x, t) \) plane are given by

\[ \frac{dx}{dt} = \lambda_i \]

and are called the characteristics of the hyperbolic system of equations. We shall keep only the first eigenmode in the solution to represent equatorially trapped waves. Or in other words, we shall keep only

\[ \frac{\partial v_1}{\partial t} + \lambda_1 \frac{\partial v_1}{\partial x} = 0. \quad (11) \]

Since

\[ r_1 = \begin{pmatrix} 1 \\ b \end{pmatrix}, \]

it is obvious that \( v_1 = u \) and \( \Phi_1 = bu \). We can also write (11) in a flux form by defining

\[ F(u) = bu = \Phi_1. \quad (12) \]

Equation (11) then becomes

\[ \frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0. \quad (13) \]

Since we consider only one eigenmode, the system reduces to a first-order conservation law.

If there is abrupt change in \( \varepsilon \), say at the boundary between the clear area and cloudy region where \( \varepsilon = 0 \) in the clear area and \( \varepsilon \) has a finite value in the convective region, then there is also an abrupt change in the propagation speed, \( c \), of the wave. Therefore, weak solutions should be considered for the present system of equations (see, for example, LeVeque 1992). We will consider two different parts of the of the cloud region: namely the trailing edge and the leading edge where there are discontinuities in \( c \). Let us first consider in this part of the paper only the trailing edge in detail.
(a) Shock wave

Keep in mind that the vertical profile, say $h(p)$ for $\omega_p$, is different to the left of a cloud region from in the cloud region itself if the cloud region is propagating eastward, in addition to the difference between the values of $\epsilon$. In this case the air on the left side of the trailing edge of a cloudy region should be clear, and $\lambda = \lambda_L = +b_\ell$ where $b_\ell$ corresponds to $\epsilon = 0$ and the vertical profile $h(p)$ in the clear area. On the right side, $\lambda = \lambda_r = +b_r$ where $b_r$ corresponds to $\epsilon > 0$ and the vertical profile $h(p)$ in the cloud area. It is not difficult to deduce that $\lambda_L > \lambda_r$ (see appendix A). Since the wave on the left-hand side will catch up with the waves on the right-hand side of the boundary region, a discontinuity (or a ‘shock’) will form. The propagation speed $s$ of the discontinuity is given by the Rankine–Hugoniot jump condition (see, for example, LeVeque 1992):

$$S = \frac{F(u_L) - F(u_r)}{u_L - u_r} - \frac{[F]}{[u]},$$

(14)

where $[\cdot]$ is used to denote the jump of $\cdot$ across the discontinuity and $F$ is the flux of $u$. This jump condition is merely a restatement of the conservation law as given in (13) when applied to a discontinuity; namely, any local change in $u$ must be balanced by the convergence or divergence of $F$. This is illustrated schematically in Figure 1 where a jump in $u$ is shown across the discontinuity. Due to the propagation of the discontinuity, there is a jump in the content of $u$. This jump, if $u$ is conserved, has to be supplied by a discontinuity in the flux of $u$, namely $[F]$. This can also be seen by integrating (13) across a discontinuity in $u$, assuming the propagation speed of the discontinuity is $s$. Equation (13) indicates that

$$\frac{d}{dt} \int_{x_2}^{x_1} u \, dx + F(x_1) - F(x_2) = 0.$$  

Let $x_0$ be the point of discontinuity in $u$ with $x_1$ to the right and $x_2$ to the left of $x_0$. We get

$$\frac{d}{dt} \int_{x_2}^{x_1} u \, dx = \frac{d}{dt} \int_{x_2}^{x_0^-} u \, dx + \frac{d}{dt} \int_{x_0^+}^{x_1} u \, dx$$

$$= \int_{x_2}^{x_0^-} \frac{\partial u}{\partial t} \, dx + \int_{x_0^+}^{x_1} \frac{\partial u}{\partial t} \, dx + \{u(x_0^-) - u(x_0^+)\} \cdot S = F(x_2) - F(x_1).$$

In the limit as both $x_2$ and $x_1$ approach $x_0$; i.e., $x_2 \uparrow x_0$, $x_1 \downarrow x_0$, we get the jump condition (14). With (12), (14) can be written as

$$S = \frac{b_\ell u_L - b_r u_r}{u_L - u_r}.$$  

At the trailing edge of a cloud region, we must have $\partial u/\partial x < 0$ to the right and $\partial u/\partial x > 0$ to the left (as illustrated schematically in Figure 2). If $u$ is continuous before the formation of the shock, at the trailing edge $\partial u/\partial x = 0$ and $u$ has a local maximum $u_m$. Consider a time $t = 0$ when the discontinuity is just forming at the trailing edge. At this time $u_L \rightarrow u_L \rightarrow u_m$, therefore $S = (b_\ell - b_r) u_m/(u_L - u_r)$. Since there is a finite jump in $\epsilon$, we have $b_\ell > b_r$. The jump condition leads to a physically unacceptable situation where $S = \infty$. Therefore the discontinuity in $\epsilon$ seems to be too harsh a condition for the cumulus parametrization scheme at $t = 0$ if the initial $u$ is continuous.
Figure 1. Schematic diagram showing the jump condition when there is a discontinuity in wind velocity, \( u \); \( u_L \) and \( u_R \) are velocities to the left and right of the discontinuity, respectively. If the discontinuity propagates with a speed \( S \), then there is an increase in the content of \( u \) with time; this is supplied by a corresponding discontinuity in the flux of \( u \), \( F(u) \), if \( u \) is conserved.

Figure 2. The distribution of wind speed, \( u \), as a function of \( x(x-u) \) plane at a particular instant at a low level. The region where \( \partial u / \partial x < 0 \) is the cloudy area. Also shown in the diagram are the characteristics lines (in the \( x-t \) plane) under the small-amplitude approximation. The figure shows that the lines originated in the cloud area are steeper than those originated in the clear area.

A simple remedy for this situation may be to assume, for example, that the proportional constant in the parametrization (epsilon), instead of being a constant, is itself proportional to \( \partial u / \partial x \) when there is convergence and zero when we have a divergent flow. The only drawback of this procedure is that CISK is then a truly nonlinear problem.

(b) Rarefaction wave

The situation is entirely different at the leading edge of a cloud region, because here the propagation speed on the right side is larger than the propagation speed on the left side:

\[ \lambda_L < \lambda_R. \]
At such a boundary, it has been shown that the solution takes the form of a rarefaction wave (Courant and Friedrichs 1948; LeVeque 1992):

$$\psi = \chi(x/t) \quad \text{for } \lambda_e \leq x/t \leq \lambda_t.$$

However, because of the linearization we have made, this is not a genuinely nonlinear problem (Lax 1957). The magnitude of $u$ cannot be determined in the region of a rarefaction wave. For details see the finite-amplitude analysis in Part II.

One surprising thing about these small-amplitude results is that there is no instability or growth of the wave as it propagates. Since $u$ is conserved along characteristics, there is no exponential amplification as predicted according to the linear CISK theory. This is independent of heating profile $f(p)$. In a linear CISK analysis on equatorial waves, Cho and Pendlebury (1997) found the sensitivity of the results to the vertical profile of cumulus heating. We note from (5a) that $b^2$ is always a real quantity, either positive or negative. When $b^2$ is negative, we do not have an equatorially trapped wave. In the linear CISK analysis, $b$ is a complex number if Kelvin waves are unstable. We suggest that this is not a physically reasonable result.

4. Propagation speeds of Kelvin waves

It is well known that the propagation speeds obtained for moist Kelvin waves are much smaller than those for dry Kelvin waves. This is because when the adiabatic cooling rate is larger than the cumulus heating rate, the Kelvin wave is equivalent to propagating in an environment with a much smaller dry static-stability parameter. This is not unlike a wave propagating in a spring system with a much smaller spring constant. When a wave is propagating in a medium with less stiff springs, the wave phase speed decreases.

The propagation speeds of a cloud region in Kelvin waves predicted in this study are comparable with, but smaller than, those predicted in the usual linear analyses. In the linear analyses the wave phase speeds, depending on the assumptions, are generally of the order of 10–40 m s$^{-1}$. However, because of the smallness of temperature perturbations in tropical systems, the geopotential perturbations are only of the order of 100 m$^2$s$^{-2}$. We have

$$bu \sim \Phi \sim 100 \text{ m}^2\text{s}^{-2}.$$  

The propagation phase speed is of the order 10 m s$^{-1}$, assuming that $u$ is of the order 10 m s$^{-1}$. We cannot explain the propagation speeds obtained in the linear analyses, because of its unphysical nature.

We would like to point out that the propagation speed just estimated applies only to a cloudy region. At the trailing edge of the cloudy region, because $\lambda_e > s > \lambda_t$, the boundary will erode the cloudy region which, as a result, will decrease in size with time. Eventually the cloud area will disappear altogether. In this paper we will not consider what happens after the cloud region has disappeared.

5. Conclusion

In this paper the propagation of the equatorial Kelvin waves with nonlinear heating is studied. In this first part of the paper we examined only the case with the small-amplitude approximation and the trailing edge of a cloud region. The study was motivated by the suggestions that: (i) equatorial Kelvin wave–CISK is a possible explanation for the intraseasonal oscillations observed in the tropical atmosphere; and (ii) the simple
parametrization scheme often used in theoretical studies of CISK (given in (1)) is intrinsically nonlinear (Lim et al. 1990). We found, to our surprise, that: (i) Kelvin waves are never unstable; (ii) there are no solutions for equatorially trapped waves when the cumulus heating rate is larger than the adiabatic cooling rate. This is very surprising as it throws doubt on the explanation of tropical intraseasonal oscillations using the equatorial Kelvin wave–CISK theory. Since Kelvin waves are non-dispersive waves, we examined their properties using the method of characteristics. In this study we have made the assumption that the wave perturbations relative to a resting atmosphere are very small, so that all terms with the exception of the cumulus parametrization scheme can be linearized. But the set of governing equations as a whole is still nonlinear because of the intrinsic nonlinearity of the parametrization scheme used (Lim et al. 1990). As is the case for nonlinear hyperbolic waves, generalized solutions should be considered for a general initial condition. I found that the parametrization scheme is incompatible with the so-called jump condition which is required by generalized solutions.

What then is the cause of the Madden–Julian oscillations? It seems that the issues of the validity of the analysis presented in this paper and the Kelvin wave–CISK theory of tropical intraseasonal oscillations are closely linked.

**APPENDIX A**

**Solution of vertical structure function h(p)**

The solution to (5b) can be expressed in terms of the Green’s function, as obtained by Cho et al. (1994) in the appendix of their paper. The Green’s function is the solution of the equation together with the boundary conditions of (5b) with the right-hand side term replaced by $\delta(p - p_t)$. In terms of this Green’s function $G(p, p_p, b^2)$ the solution to (5b) can be expressed as

$$h(p) = \frac{\varepsilon}{b^2} h(p^*) \int_{p_t}^{p_s} f(p_p) G(p, p_p, b^2) \, dp_p.$$  \hspace{1cm} (A1)

In this equation $p_s$ and $p_t$ are used to denote the values of the pressure at the surface and the tropopause levels. Since at $p = p^*$ $h(p) = h(p^*)$ the above equation becomes

$$b^2 = \varepsilon \int_{p_t}^{p_s} G(p, p_p, b^2) f(p_p) \, dp_p.$$  \hspace{1cm} (A2)

This is an integral equation for $b^2$ (the Green’s function depends on $b^2$; Cho et al. 1994). In the usual linear analysis, one looks for complex solutions for $b^2$. However, only real solutions for $b^2$ are physically realistic solutions.

One can also try to find the solution by expanding the function $f(p)$ into its Fourier series:

$$f(p) = \sum_{m=1}^{\infty} f_m(p) = \sum_{m} a_m \sin m\pi \frac{p - p_t}{p_s - p_t}.$$

The solution to (5b) is

$$h(p) = \sum_{m=1}^{\infty} h_m(p) = \sum_{m} c_m \sin m\pi \frac{p - p_t}{p_s - p_t},$$

where

$$c_m = \frac{\varepsilon \sigma h(p^*) a_m}{-b^2 m^2 \pi^2 (p_s - p_t)^2 + \sigma}.$$
We can assume without loss of generality that the same profile \( f(p) \) applies to cloud regions as well as to cloud-free regions, only that \( \varepsilon = 0 \) in cloud-free areas. If one makes this assumption, then

\[
\sum_m \frac{\alpha_m \sin m\pi (p^* - p_t)/(p_s - p_t)}{-b^2 m^2 \pi^2/(p_s - p_t)^2 + \sigma} \begin{cases} 
\approx \frac{1}{\sigma} & \text{if } \varepsilon = 0 \\
0 & \text{if } \varepsilon > 0.
\end{cases}
\]

Since \( a_1 > 0 \) is the dominant term, \( b^2 > 0 \) should be larger in the cloud-free regions than in the cloud regions.

REFERENCES


