Linear instability of broad baroclinic zones

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SUMMARY

Baroclinic energy conversions play a central role in mid-latitude synoptic-scale dynamics, drawing energy from the meridional temperature gradient and feeding it into the eddy dynamics. The physical mechanisms underlying these conversions can be investigated through the theory of small-amplitude instabilities growing on a steady basic state. In a recent study the author analysed the instability associated with a boundary temperature transition concentrated in a narrow front. This paper looks at the complementary problem of a broad baroclinic zone. The solution is applicable both to the Eady model and to the Phillips two-level quasi-geostrophic model. The reduction in the growth rate relative to an infinitely broad, homogeneous baroclinic zone is inversely proportional to the width of the baroclinic zone, as found in previous studies. The constant of proportionality is expressed in a simplified and more general form, related to the derivative with respect to zonal wave number of the complex phase speed of the homogeneous problem. If the flow is purely baroclinic the maximum growth rate is reduced compared with that of a homogeneous, infinitely broad frontal zone by a factor $1 - (2k_{\text{max}}r)^{-1}$, where $k_{\text{max}}$ is the wave number of the most unstable mode of the homogeneous problem and $r$ is the width of the baroclinic zone. The width, $r$, is defined in terms of the second meridional derivative of the boundary temperature gradient evaluated at the maximum of the latter. Results in the literature give consistent, though less general, expressions for the growth rate, but various predictions for the structure of the waves. The numerical solutions show that as the jet width is reduced a new type of mode appears in which critical lines at the lid (tropopause) play an important role.

KEYWORDS: Atmospheric dynamics  Fronts

1. INTRODUCTION

This work is a continuation of Juckes (1998a,b) (hereafter JA and JB respectively), in which the instability of narrow fronts was analysed, and extends the mathematical techniques developed there. The growth rate of the instability was found to be proportional to the temperature change across the baroclinic zone, with a correction of order $r^2$, where $r$ is the width of the baroclinic zone, non-dimensionalized, by the Rossby deformation radius. Comparison with numerical solutions showed that the asymptotic solution was highly accurate for $r < 0.25$, but for $r > 0.5$ it broke down completely. Here the complementary problem of a broad baroclinic zone is investigated.

Stone (1969) investigated the broad-jet limit with the two-layer quasi-geostrophic model (hereafter, Phillips model, following Phillips (1954)), using the WKBJ method (e.g. Olver 1974). This approach assumes that the length scale of meridional variation in the jet is much greater than that of the waves. The latter assumption is discussed in more detail in section 8. Simmons (1974a) solved the same problem by reducing the equations for the meridional structure to the form of the harmonic oscillator. The solutions of this equation are Hermite polynomials modulated by a Gaussian function. As discussed further in Stone (1975) and Simmons (1975) these two methods give different predictions about the meridional structure of the instability. The reasons for this disagreement will be clarified below. Gent (1974, 1975) looks at the problem in the continuous, as opposed to two-level, model (Eady 1949). He also notes the discrepancy between the Gaussian-type solution and the WKBJ approach, favouring the latter. The WKBJ approach was also used by Ioannu and Lindzen (1986, 1990).

A different analytic approach to the non-separable baroclinic-instability problem was developed by McIntyre (1970). The studies cited above are all based on asymptotic analyses for extreme values of the width, $r$, of the baroclinic zone. McIntyre considered

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a zonally symmetric basic flow with uniform baroclinicity apart from a perturbation of finite width and small amplitude. The meridional extent of the baroclinic waves was determined by fixed side walls, rather than being determined by the dynamics. This disadvantage is offset by the ability to deal with a jet of finite meridional scale.

Moore and Peltier (1989, 1990) compared the various approximate analytic solutions with accurate numerical solutions. They found good agreement for jet widths of the order 2 Rossby radii or more. The present study differs in three respects. Firstly, the modification of the Eady problem is restricted to changes in the boundary potential-temperature gradients. The tropospheric potential vorticity is kept constant. This restriction reduces the computational cost of the numerical solutions considerably. Secondly, the meridional structure of the waves is studied in more detail. It is found, as argued by Simmons (1974a,b, 1975), that the WKBJ method cannot be used to describe the meridional structure of the waves, even though it gives the correct growth rates. Thirdly, open meridional boundary conditions are used rather than periodicity. It is found that critical lines play an important role in determining the structure of normal modes on jets of realistic widths. In previous studies this feature may have been masked by the proximity of the critical lines to meridional boundaries.

The approach followed below is closest to that used by Simmons (1975). The results are in agreement with Simmons, but more general. The method used here applies to any system described by the advection of a conserved quantity on two levels. The conserved quantity can represent the boundary potential temperature in a uniform potential-vorticity model or the potential vorticity in the Phillips model. The results given below are expressed in terms of Green's functions which describe the relationship between the stream function and the conserved quantity. The Green's function formulation allows issues related to the structure of the basic state to be separated from issues related to the dynamical system. This makes it possible to derive a result with a large degree of generality.

2. THEORETICAL FORMULATION

The formulation follows JA and JB closely. In the present case the geostrophic-coordinate transformation, which is necessary when dealing with narrow frontal zones, does not play a significant role. Geostrophic coordinates are nevertheless used to preserve continuity with the earlier parts of this study. The most important symbols are summarized in Table 1, other symbols will be defined as they occur.

As noted above, there have been a number of previous studies looking at baroclinic disturbances on slowly varying basic states. Some of these studies have used the Eady model and others have used the Phillips two-layer model. The main results presented here apply to both models. The derivation is made as general as possible by expressing it in terms of Green's functions. Both the Phillips and the Eady model can be expressed in terms of the horizontal advection of a conserved quantity, \( C_\mu \), on two 'levels', \( \mu = 0, 1 \). In the Eady model the two levels are the upper and lower boundaries, in the Phillips model they represent the upper and lower halves of the troposphere. In the former, \( C_\mu \) is taken to be potential temperature (with a minus sign at \( \mu = 1 \), see below), in the latter, \( C_\mu \) is taken to be half the quasi-geostrophic potential vorticity. In both cases, the velocities which advect the conserved quantities are linearly related to the conserved quantities via a \( 2 \times 2 \) matrix of Green's functions. When results are specific to one model this will be indicated by superscripts 'P' and '8' for the Phillips and Eady models respectively. \( C \) can be thought of as the vertical integral of a potential-vorticity anomaly generalized to include boundary potential temperature (Bretherton 1966b). The factor
TABLE I.

<table>
<thead>
<tr>
<th>Primary constants</th>
<th>Symbol</th>
<th>Typical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tropopause height</td>
<td>$H$</td>
<td>8 km</td>
</tr>
<tr>
<td>Reference potential temperature</td>
<td>$\theta_\theta$</td>
<td>300 K</td>
</tr>
<tr>
<td>Vertical change in $\theta$ in the troposphere</td>
<td>$\theta_c$</td>
<td>24 K</td>
</tr>
<tr>
<td>Gravity</td>
<td>$g$</td>
<td>$10 \text{ m s}^{-2}$</td>
</tr>
<tr>
<td>Coriolis force</td>
<td>$f$</td>
<td>$10^{-4} \text{ s}^{-1}$</td>
</tr>
<tr>
<td>Non-dimensional strength of the baroclinicity</td>
<td>$E_{fr}$</td>
<td></td>
</tr>
<tr>
<td>Non-dimensional width of baroclinic zone</td>
<td>$r$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Secondary constants</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entropy stratification</td>
<td>$\Gamma' \equiv \theta_{ec}/H$</td>
</tr>
<tr>
<td>Brunt-Väisälä frequency squared</td>
<td>$N^2 \equiv g\Gamma/\theta_\theta$</td>
</tr>
<tr>
<td>Horizontal scale</td>
<td>$x_{ec}$</td>
</tr>
<tr>
<td>Vertical scale</td>
<td>$z_{ec}$</td>
</tr>
<tr>
<td>Time scale</td>
<td>$t_{ec}$</td>
</tr>
<tr>
<td>Inverse deformation radius</td>
<td>$\gamma$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>Scaling</th>
<th>Typical value of scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geostrophic coordinates</td>
<td>$(x, y)$</td>
<td>$x_{ec}$</td>
</tr>
<tr>
<td>Height</td>
<td>$z$</td>
<td>$H$</td>
</tr>
<tr>
<td>Time</td>
<td>$t$</td>
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<tr>
<td>Horizontal wave numbers</td>
<td>$(k, l)$</td>
<td>$x_{ec}^{-1}$</td>
</tr>
<tr>
<td>Total horizontal wave number</td>
<td>$\lambda$</td>
<td>$x_{ec}^{-1}$</td>
</tr>
</tbody>
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<tr>
<th>Field variables</th>
<th></th>
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<tr>
<td>Cross-frontal structure</td>
<td>$\Lambda$</td>
<td></td>
</tr>
<tr>
<td>Potential temperature</td>
<td>$\theta$</td>
<td>$\theta_{ec}$</td>
</tr>
<tr>
<td>Geostrophic stream function</td>
<td>$\psi$</td>
<td>$x_{ec}^2 x_{ec}^{-2}$</td>
</tr>
<tr>
<td>Quasi-geostrophic potential vorticity</td>
<td>$q$</td>
<td>$r_{ec}^{-1}$</td>
</tr>
<tr>
<td>Meridional displacement</td>
<td>$\eta$</td>
<td>$x_{ec}$</td>
</tr>
<tr>
<td>Conserved quantity</td>
<td>$\xi$</td>
<td>$G$</td>
</tr>
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<table>
<thead>
<tr>
<th>Annotations</th>
<th>Range</th>
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<tbody>
<tr>
<td>Level number</td>
<td>$\mu v$</td>
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<tr>
<td>Index of moment expansion</td>
<td>$\sigma_{(en)}$</td>
</tr>
<tr>
<td>Eady model</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>Phillips model</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>Pertaining to the homogeneous, $r = \infty$, problem</td>
<td>$\sim$</td>
</tr>
<tr>
<td>Fourier transform in $x$</td>
<td>$\sim$</td>
</tr>
<tr>
<td>Fourier transform in $x$ and $y$</td>
<td>$\sim$</td>
</tr>
<tr>
<td>Convolution over $y$</td>
<td>$\circ$</td>
</tr>
</tbody>
</table>

The primary constants define the reference atmosphere together with the strength and scale of the baroclinic zone. The secondary constants can be expressed in terms of the primary constants. The rest of the table shows coordinates and variables, together with the constants used to non-dimensionalize them and a typical value of that constant.

One half enters into the Phillips model because the layer depth in the non-dimensional coordinates is one half. With this scaling the two models become equivalent at large scales.

The stream functions at the two levels, $(\psi_0, \psi_1)$, are related to the corresponding $(\xi_0, \xi_1)$ by a matrix of Green’s functions:

$$
\begin{pmatrix}
\psi_0 \\
\psi_1
\end{pmatrix} = 4\pi^2 \hat{G} \begin{pmatrix}
\xi_0 \\
\xi_1
\end{pmatrix}, \quad \text{where} \quad \hat{G} = \begin{pmatrix}
\hat{G}_{00} & \hat{G}_{01} \\
\hat{G}_{10} & \hat{G}_{11}
\end{pmatrix}.
$$

(1)

The ‘\(\sim\)’ denotes a horizontal Fourier transform as in JA.
For the Boussinesq Eady problem (see JA)

\[
\mathcal{F}^e = -\frac{1}{4\pi^2} \begin{pmatrix} \coth \lambda / \lambda & 1 / (\lambda \sinh \lambda) \\ 1 / (\lambda \sinh \lambda) & \coth \lambda / \lambda \end{pmatrix},
\]

where \( \lambda = \sqrt{k^2 + l^2} \) is the modulus of the horizontal wave number, \((k, l)\). As noted above the conserved quantity, \( C_1 \), is defined to be minus the potential temperature on the lid. The corresponding Green’s functions, therefore, differ from those used in JA by a minus sign.

For the Phillips model (presented in more detail in appendix A) the Green’s functions are given by:

\[
\mathcal{F}^p = -\frac{1}{4\pi^2} \begin{pmatrix} 1 / \lambda^2 + 1 / (\lambda^2 + 2\gamma^2) & 1 / \lambda^2 - 1 / (\lambda^2 + 2\gamma^2) \\ 1 / \lambda^2 - 1 / (\lambda^2 + 2\gamma^2) & 1 / \lambda^2 + 1 / (\lambda^2 + 2\gamma^2) \end{pmatrix},
\]

where \( \gamma \) is the inverse deformation radius for baroclinic disturbances.

Figure 1 shows the Green’s functions multiplied by \(-4\pi^2 \lambda^2\). The cross terms, \( \mathcal{G}_{01} \), of the two models are very similar. The self-interaction terms, \( \mathcal{G}_{00} \), differ markedly in that the Eady model has much stronger small-scale dynamics. Since the range of wave numbers of interest here is \( 0 < \lambda < 2.5 \), this difference between the two models does not have a very great impact on the results of this study.

The dynamics of both models can be expressed in terms of the displacement of material contours. Let \( \eta_\mu(x, y, t) \) be the meridional displacement of an air parcel on level \( \mu \) at the position \((x, y)\). The evolution is determined by the kinematic equation:

\[
\frac{\partial \eta_\mu}{\partial t} + \bar{u}_\mu \frac{\partial \eta_\mu}{\partial x} - v_\mu = 0, \quad \mu = 0, 1 \tag{4a}
\]

\[
C_\mu = \eta_\mu C_{fr}(\Lambda_\mu - \beta), \quad \mu = 0, 1 \tag{4b}
\]

where \( C_{fr}\Lambda_\mu(y) = -\frac{\text{d} C_\mu}{\text{d} y}, \quad (u, v) = (-\psi_y, \psi_x) \tag{4c} \)
and $C_{fr}$ is a constant defining the strength of the baroclinicity. $\beta$ is non-dimensionalized and scaled, being related to $\beta^T$, the meridional gradient of planetary vorticity, by
\[
\beta = \frac{\beta^T x_{sc} f_{sc}}{C_{fr}}.
\]
In contrast to JA, the meridional structure function, $\Lambda$, appearing in Eq. (4b) is non-dimensionalized to facilitate the simultaneous treatment of the Phillips and Eady models. For the Eady model, further, $\theta = \psi_z$. Equation (4a) describes the linearized advection of material contours on the two levels and Eq. (4b) relates the anomalies in $\zeta$ to the resulting displacements. The stream function can be calculated from Eq. (1). The $\beta$ term is generally omitted in the Eady model. Its inclusion in this form would imply that potential-vorticity anomalies created by meridional advection of the planetary vorticity gradient, $\beta$, immediately become localized at the upper and lower boundaries. This is somewhat unphysical, but not particularly more so than the implicit assumption in the two-layer model that the potential temperature is constant on the upper and lower boundaries. The present study will nevertheless follow convention and set $\beta = 0$ when dealing with the Eady model.

The method can, in principle, be applied to the non-Boussinesq flow and non-rigid lid generalization of the Eady model given in Juckes (1994), but this introduces an irregularity in the Green’s functions at $\lambda \to 0$. This irregularity causes technical problems, so this paper restricts attention to the Green’s functions given above.

In JA it was shown how this type of problem could be cast as a single evolution equation for the boundary contour displacement, $\eta_\mu(x, y, t)$, (Eq. (28) in that paper). The only differences here are the inclusion of $\beta$ in the term describing the creation of eddy potential vorticity by meridional advection and the fact that $\Lambda$ is here allowed to vary between levels 0 and 1. There is also an additional constant factor because $\Lambda$ is now non-dimensionalized. These minor changes lead to the following equation:
\[
\frac{\partial \tilde{\eta}_\mu(k; y, t)}{\partial t} = -2\pi i k C_{fr} \sum_{\nu=0,1} \left[ \tilde{G}_{\mu\nu}(0; y-y') \circ \Lambda_\nu(y') \tilde{\eta}_\mu(k; y, t) \right.
- \tilde{G}_{\mu\nu}(k; y-y') \circ \left( (\Lambda_\nu(y') - \beta) \tilde{\eta}_\nu(k; y', t) \right)].
\]
(5)

The symbol ‘$\circ$’ denotes a convolution over $y'$.

The terms on the right-hand side of the first line of Eq. (5) describe the advection by the basic-state zonal wind and those in the second line describe the advection of the basic state by the eddy meridional wind. In general, $\Lambda$ is a function of $y$, so a separable solution is not possible. In JA an asymptotically separable solution was found in the limit $r \to 0$. Here, the opposing limit $r \to \infty$ will be examined. For reference, the Eady solution is obtained when $\tilde{u} = C_{fr}(z - 1/2), \tilde{v} = -C_{fr} y$, for a constant $C_{fr}$. In this case, the eigenmodes have the structure $\exp[i(kx + ly - \omega t)]$, and the evolution is given by
\[
\frac{\partial \tilde{\eta}_\mu}{\partial t} = -ik C_{fr} \sum_{\nu=0,1} M_{\mu\nu}^{\infty, \epsilon} \tilde{\eta}_\nu
\]
(6a)
where
\[
(M_{\mu\nu}^{\infty, \epsilon}) = \begin{pmatrix}
-1/2 + (\coth \lambda)/\lambda & -1/(\lambda \sinh \lambda) \\
1/(\lambda \sinh \lambda) & 1/2 - (\coth \lambda)/\lambda
\end{pmatrix}.
\]
(6b)

The maximum growth rate, $k C_{fr}$ times the complex eigenvalue of the matrix in Eq. (6b), is $0.31 C_{fr}$ at $k \approx 1.6$. 

3. Solution by Moment Expansion

The asymptotic analysis below makes use of an expansion in terms of meridional moments, as described in JA. The first moment represents a mean value and successive higher moments represent successively finer-scale oscillations about the mean. The development is very close to that in JA, but there are a few minor changes to permit the simultaneous treatment of the Eady and Phillips models. Besides providing an extra solution, this gives a clearer separation of the model-dependent aspects of the result from those that are dependent on the cross-frontal flow structure.

The derivation is essentially as in JA, with the addition of some extra subscripts to allow for differing upper and lower meridional structures. The slightly modified definitions are given in appendix B. Taking moments of the evolution equation and assuming exponential time dependence gives the following evolution equation for the $m$th moment:

$$\left(\sigma - i\omega\right)\tilde{\eta}^{(m)}_{\mu}(k, t) \equiv \frac{\partial \tilde{\eta}^{(m)}_{\mu}(k, t)}{\partial t} = ik\mathcal{E} \sum_{n, \nu} M^{(mn)}_{\mu\nu} \tilde{\eta}^{(n)}_{\nu}$$

(7)

where $\tilde{\cdot}$ denotes a Fourier transform with respect to $x$ (as in JA) and

$$M^{(mn)}_{\mu\nu} = 4\pi^2 \int_{-\infty}^{\infty} \left( \tilde{G}^{(mn)}_{\mu\nu}(k, l) \tilde{A}^{(mn)}_{\mu\nu}(k, l) - \delta_{\mu\nu} \sum_{\nu'} \tilde{G}^{(nm)}_{\mu\nu'}(0, l) \tilde{B}^{(mn)}_{\nu\nu'}(k, l) \right) \, \text{d}l$$

(8)

with

$$\tilde{A}^{(mn)}_{\mu\nu}(k, l) = 2\pi \tilde{P}^{(m)}_{\mu} \circ (\tilde{\Lambda}_\nu - \beta),$$

and

$$\tilde{B}^{(mn)}_{\mu\nu}(k, l) = 2\pi (\tilde{P}^{(m)}_{\mu} \circ \tilde{P}^{(n)}_{\nu}) \tilde{\Lambda}_\nu.$$  

(9)

The functions $\tilde{A}$ and $\tilde{B}$ depend on $l$ explicitly but also on $k$, since the scales of the structure functions, $R^2$ and $b_\mu$, will depend on $k$. In this formulation the functions $A$ and $B$ contain all the information about the geometry of the flow and the dynamical information is contained in the Green’s functions. The functions $P$ and $P^*$ are defined in appendix B.

4. The Zonal Flow

Following Juckes (1995) the reference state is defined by prescribing $\Lambda$ (Eq. (4c)). Only the maximum value of the gradient and its second derivative contribute to the problem in the broad-jet limit, but it is convenient to specify the structure in the form of a Gaussian:

$$\Lambda_\mu(y) = \left\{ \Lambda^{(0)}_\mu + r^{-2} \Lambda^{(4)}_\mu H^{(4)} \left( \frac{y}{R_\mu} \right) + \cdots \right\} \exp \left( -\frac{y^2}{2r_\mu^2} \right),$$

(10)

where $r = (r_0 + r_1)/2$. The scaling of the structure function is such that it has a maximum value, $\Lambda^{(0)}_\mu$, which is constant in the limit $r \to \infty$. This contrasts with the scaling in JA, where the integral, $\int_{-\infty}^{\infty} \Lambda \, \text{d}y$, was constant in the limit $r \to 0$. In both cases the scaling is such that the growth rates tend to a finite value in the limit under consideration. The value of $R_\mu$ (which enters the problem through the definition of $P$ in appendix B) is not known a priori but will be determined below. The Fourier transform
of Eq. (10) gives:

\[ \widehat{\Lambda}_\mu(l) = \frac{r_\mu}{\sqrt{2\pi}} \left( \Lambda^{(0)} + \mathcal{O}(r^{-2}) \right) \exp \left( -\frac{l^2 r_\mu^2}{2} \right) \]  

(11)

where \( \mathcal{O} \) indicates an error term of magnitude bounded by a constant times the argument in the limit under consideration. It will be seen below that regularity at \( l = 0 \) requires \( \widehat{\Lambda}_0(0) + \widehat{\Lambda}_1(0) = 0 \). This is satisfied if

\[ \Lambda^{(0)} = (-1)^\mu \frac{r}{r_\mu}. \]  

(12)

Similarly, the \( \beta \) effect must be expanded (as in Simmons 1974b), even if \( \beta \) is constant:

\[ \beta = \left\{ \beta^{(0)} + r^{-1} \beta^{(2)} H^{(2)} \left( \frac{y}{R_\mu} \right) + \ldots \right\} \exp \left( -\frac{y^2}{2r_\mu^2} \right). \]

The second moment of the \( \beta \) expansion is, in general, non-zero, and this will enter into the calculation below. Fourth and higher moments of both \( \Lambda \) and \( \beta \) can be neglected in the limit \( r \to \infty \). Expanding about \( y = 0 \) (and using \( H^{(2)}(s) = 1 - s^2 \)) gives \( \beta = \beta^{(0)} + \beta^{(2)}/r - y^2/(2r^2)(\beta^{(0)} + 2r\beta^{(2)}/R_\mu^2) + \mathcal{O}(y^2) \). For constant \( \beta \) this implies that

\[ \beta^{(2)} = -\beta^{(0)} R_\mu^2/(2r) \]

and hence

\[ \beta^{(0)} = \beta \left( 1 + \frac{R_\mu^2}{2r^2} \right). \]  

(13)

The zonal flow is given by (see JA)

\[ \widehat{u}_\mu = 4\pi^2 C_{fr} \sum_{\nu=0,1} \widehat{G}_{\mu\nu} \widehat{\Lambda}_\nu. \]  

(14)

Earlier studies have generally prescribed the \( \vec{u} \) structure. Here the structure of \( \vec{c} \)

is prescribed and \( \vec{u} \) is derived from there. The relation between the two, expressed in
Eq. (14), is generally non-trivial, but does have a simple form in the limit \( r \to \infty \).
This is clearest when the barotropic and baroclinic components (subscripts ‘bt’ and ‘bc’
respectively) of the flow are considered separately:

\[ \widehat{u}_{bt} = 4\pi^2 C_{fr} \widehat{G}_{bt} \widehat{\Lambda}_{bt} \]

\[ \widehat{u}_{bc} = 4\pi^2 C_{fr} \widehat{G}_{bc} \widehat{\Lambda}_{bc} \]

where \( \widehat{u}_{bt} = (\widehat{u}_0 + \widehat{u}_1)/2 \) and \( \widehat{u}_{bc} = (\widehat{u}_1 - \widehat{u}_0)/2 \), with similar definitions for \( \Lambda \). (Note that \( \Lambda_{bt} \) is defined here, for the Eady model, to be minus the temperature gradient on the
lid, so that a flow with equal temperature gradients at \( z = 0 \) and \( z = 1 \) has \( \widehat{\Lambda}_{bt} = 0 \).) The
baroclinic and barotropic Green’s functions are given by

\[ \widehat{G}_{bc}^e = -(cosh \lambda - 1)/(4\pi^2 \lambda \sinh \lambda) \]

\[ \widehat{G}_{bt}^e = -(cosh \lambda + 1)/(4\pi^2 \lambda \sinh \lambda) \]

\[ \widehat{G}_{bc}^p = -1/(2\pi^2 (\lambda^2 + 2\gamma^2)) \]

\[ \widehat{G}_{bt}^p = -1/(2\pi^2 \lambda^2). \]  

(15)

This representation makes the singularity of the barotropic component clear: both \( \widehat{G}_{bt}^e \)
and \( \widehat{G}_{bc}^p \) are \( \mathcal{O}(\lambda^{-2}) \) as \( \lambda \to 0 \). The velocity remains well behaved only if the barotropic
component of $\mathcal{\Lambda}$ satisfies

$$\mathcal{\Lambda}_{bt} = \mathcal{O}(l^2), \quad \text{as } l \to 0.$$  

As noted above, this condition is satisfied automatically by the $\Lambda^{(0)}(y)$ defined in Eq. (12). The barotropic component in the flow is related to the sum $\Lambda_0 + \Lambda_1$, which is in turn related to the difference between $r_0$ and $r_1$. Let $c_{bt} = r(r_0 - r_1)$, so that:

$$r_0 = r + \frac{c_{bt}}{2r}, \quad r_1 = r - \frac{c_{bt}}{2r}.$$  

(16)
Substituting Eq. (16) in Eq. (11) and evaluating the baroclinic and barotropic components gives

\[ \tilde{\Lambda}_{bc} \rightarrow \frac{r}{\sqrt{2\pi}} \]

and \[ \tilde{\Lambda}_{bt} \rightarrow \frac{r}{\sqrt{2\pi}} \frac{l^2 c_{bt}}{2} \] as \( r \rightarrow \infty, l \rightarrow 0 \).

The resulting zonal flows have magnitudes at \( l = 0 \) given by

\[ \left( \frac{\tilde{x}_0}{\tilde{u}_1} \right) \rightarrow \frac{c_{fr} r}{\sqrt{2\pi}} \left( \frac{-u_{bc} + u_{bt}}{u_{bc} + u_{bt}} \right), \quad \text{as } r \rightarrow \infty, \]
where

\[(u_{bc}, u_{bt})^\delta = \left(\frac{1}{2}, 2c_{bt}^\delta\right),\]

\[(u_{bc}, u_{bt})^{\varphi} = \left(\frac{1}{\gamma^2}, 2c_{bt}^{\varphi}\right).\]

A flow with near-zero surface wind is obtained when \(u_{bc} = u_{bt}\), which holds if \(c_{bt}^{\delta} = 1/4\) and \(c_{bt}^{\varphi} = 1/(2\gamma^2)\) for the Eady and Phillips models respectively. Earlier related work has largely concentrated on either purely baroclinic flow, with \(u_0 = -u_1\), or on flow with zero surface wind. Here \(c_{bt}\) is kept as a free parameter and this allows the roles of the baroclinic and barotropic components of the basic state to be separated.

Figure 2 shows the basic flow in the Eady model for widths \(r = 4, r = 2\) and \(r = 1\), both with and without barotropic shear. (Here barotropic is used in the sense normal in the literature on the two-layer model, implying \(\int u\, dz \neq 0\), rather than in that sometimes associated with the Eady model where it refers to \(u(0) \neq 0\).) Superimposed are the Eliassen-Palm (Andrews and McIntyre 1976; Edmon et al. 1980) flux vectors of the most unstable normal mode, defined as

\[(F_{(y)}, F_{(z)}) = (-\bar{u}\bar{v}, \bar{v}\bar{\theta}).\]

The upward component of the flux indicates a down-gradient heat transport and the horizontal component an up-gradient momentum flux, reinforcing the barotropic component of the jet. The structure of the wave will be discussed further below.

5. HEURISTIC DERIVATION OF THE MERIDIONAL SCALE OF BAROCLINIC WAVES

Previous studies have shown that the width of the most unstable mode varies as the square root of the width of the baroclinic region. This section presents a heuristic discussion of the factors that lead to this result, the next section will use the details of the moment expansion to put these ideas into a more rigorous mathematical form.

Consider a disturbance of Gaussian meridional structure with length scale \(b\):

\[\eta \propto \exp\left(-\frac{y^2}{2b^2}\right).\]  \hspace{1cm} (17)

The magnitude of \(b\) (assuming, for now, that the same scale applies at both \(z = 0\) and \(z = 1\)) can be determined by comparing the scales of the various anomaly fields. The potential-temperature anomaly is the product of the displacements and the mean baroclinicity, giving

\[\theta' \propto \exp\left(-\frac{y^2}{2}\frac{r^2}{r^2b^2}\right) = \exp\left(-\frac{y^2}{2r_\theta^2}\right),\]

where \(r_\theta^2 = (r^2b^2)/(r^2 + b^2)\). The velocity is determined by a convolution of the potential-temperature anomaly with the Green's function which has a characteristic scale of, say, \(R_k\). This convolution can be estimated by approximating the Green's function, for heuristic purposes, with a Gaussian: \(\exp(-y^2/2R_k^2)\). It follows that

\[v \propto \exp\left(-\frac{y^2}{2(R_k^2 + r_\theta^2)}\right).\]  \hspace{1cm} (18)
The operations of multiplication and convolution are related by the fact that a multiplication in physical space is a convolution in spectral space and vice versa (the convolution theorem). A multiplication tends to lead to a more peaked distribution, while convolution tends to smooth the distribution out.

In the normal mode the meridional structure and scale of the velocity must match those of the displacement field. Comparing the coefficients of \( y^2 \) in Eqs. (17) and (18) then solving for \( b \) gives

\[
b^2 = \frac{R_k^2 + \sqrt{R_k^4 + 4r^2 R_k^2}}{2}.
\]

Thus for small \( r \) the displacement scales with the characteristic scale of the Green's function, \( (R_k) \), and at large \( r \) it scales with the geometric mean of \( R_k \) and the jet width. Equation (19) gives the scaling law for \( b \). By considering the competing factors which affect the growth rate it is possible to replace the somewhat vaguely defined \( R_k \) with a more precise quantity.

The primary interest of this paper is in problems for which the fastest mode in the homogeneous problem has unlimited horizontal extent. This preference for large meridional extent is counteracted by the fact that the gradients are localized. The larger the disturbance the smaller the mean baroclinicity across the region spanned by the disturbance. If it is assumed that \( \Lambda \) varies quadratically in the vicinity of its maximum value, \( \Lambda^{(0)} \), and has a second derivative, \( \Lambda^{(0)} r^{-2} \), then the mean baroclinicity over the interval \( y \in (-b, b) \) is \( (1 - b^2/(6r^2)) \Lambda^{(0)} \). Based on this we can estimate a growth-rate reduction proportional to \( \sigma^\infty b^2 r^2 \), where \( \sigma^\infty(k, l) \) is the growth rate of the homogeneous problem. If we further assume that the decrease in growth rate due to finite meridional scale is equivalent to replacing \( l = 0 \) by \( l = b^{-1} \) we obtain, for \( b \gg 1 \), a reduction proportional to \((1/b^2)(\partial^2 \sigma^\infty/\partial l^2)\). It follows that the net reduction in growth rate is, to leading order, proportional to

\[
c_{un} \frac{b^2}{r^2} \sigma^\infty + \frac{1}{b^2} \frac{\partial^2 \sigma^\infty}{\partial l^2},
\]

where \( c_{un} \) is an undetermined constant. The reduction is minimized with respect to \( b \), giving the maximum growth rate when \( b \) satisfies

\[
c_{un} \frac{b^4}{r^2} = \frac{1}{\sigma^\infty} \frac{\partial^2 \sigma^\infty}{\partial l^2}.
\]

Thus, we again find the scaling \( b \propto r^{1/2} \), but the heuristic scale, \( R_k \), has been replaced by a well defined scale related to the growth rate in the homogeneous problem. The value of \( b \) is determined by a balance between the tendency for the mode to seek out an infinite meridional scale on the one hand and the tendency for it to seek out the largest gradients on the other hand. This argument is not rigorous, it is intended to give a heuristic motivation for the mathematics that follow. The value of \( c_{un} \) is determined by the following rigorous analysis.

6. LARGE \( r \) ASYMPTOTIC EXPANSION

As in JA, the asymptotic solution is determined by requiring that the shearing of the disturbance by the basic-state wind be balanced, to leading order in the asymptotic expansion, by the eddy advection of the meridionally varying basic-state temperature
gradients. For narrow fronts this led to the condition that the projection scale should equal the width of the frontal zone, $a = r$. This choice proves convenient in the broad-jet limit, but it is not sufficient to determine the solution.

The calculation will be carried out at first without any assumption about the specific form of the Green’s functions, so that the results apply both to the uniform potential vorticity and to the Phillips model.

The heuristic analysis above suggests the scaling $b \propto r^{1/2}$. We shall adopt this scaling, and it will be shown below that the error term is then $\mathcal{O}(r^{-1})$ smaller than the leading-order correction, thus confirming the validity of the scaling.

The matrix elements defining the interaction of the zeroth moments are given by Eq. (8) with $(mn) = (00)$. The functions $\tilde{A}^{(00)}_{\mu\nu}$ and $\tilde{B}^{(00)}_{\mu\nu}$ are both of Gaussian form and as $r \to \infty$ the Gaussians become increasingly peaked at $l = 0$. The argument of the exponential is given by (see appendix C)

$$-\frac{l^2}{2} \left( r^2_{\mu} + \frac{r^2_{\nu} b^2_{\nu}}{r^2_{\mu} + b^2_{\nu}} \right) \approx -\frac{l^2}{2} (r^2_{\mu} + b^2_{\nu}) + \mathcal{O}(l^2), \quad r \to \infty. \quad (20)$$

The order $l^2$ term can be neglected because the integral is dominated by the wavenumber range $l = \mathcal{O}(r^{-1}) \ll 1$. The rapid decay of the Gaussian function away from $l = 0$ implies that the integrals that determine the interaction matrix are determined by the region near the origin. To be precise, for a general function $f(l)$ with $f'(0) = 0$,

$$\frac{r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-r^2_{\mu} + b^2_{\nu} l^2/2\} f'(l) \, dl = \frac{r}{\sqrt{r^2_{\mu} + b^2_{\nu}}} f'(0) + \mathcal{O}(r^{-2} r^2) f''$$

$$= \left( 1 - \frac{b^2_{\nu}}{r^2_{\mu}} \right) f'(0) + \mathcal{O}(r^{-2}),$$

where $b = (b_0 + b_1)/2$ (it will be shown below that $|b_0 - b_1| \ll |b|$ in the limit $r \to \infty$).

The matrix elements are consequently given by

$$M^{(00)}_{\mu\nu} = \left( 1 - \frac{b^2_{\nu}}{2r^2} \right) M^{\infty}_{\mu\nu} |\beta^{(0)} + \mathcal{O}(r^{-2})$$

(21)

where $M^{\infty}_{\mu\nu}$ is the interaction matrix for the homogeneous problem, here evaluated at $\beta = \beta^{(0)}$. It follows that the dispersion relation is

$$(\omega + i\sigma) = \left( 1 - \frac{b^2_{\nu}}{2r^2} \right) (\omega^{\infty} + i\sigma^{\infty}) |\beta^{(0)} + \mathcal{O}(r^{-2})$$

$$= \left( 1 - \frac{b^2_{\nu}}{2r^2} \right) (\omega^{\infty} + i\sigma^{\infty}) |\beta + \frac{b^2_{\nu}}{2r^2} \frac{\partial}{\partial \beta} (\omega^{\infty} + i\sigma^{\infty}),$$

(22)

using Eq. (13) for $\beta^{(0)}$. It will be shown below that $b$ is generally complex, so that $\sigma$ depends not only on $\sigma^{\infty}$ but also on $\omega^{\infty}$.

In order to evaluate $b_{\mu}$ we consider the second moment. The zeroth moment forces the second moment. In the limit of large $r$ the eigenvalues of the second-moment matrix, $M^{(22)}_{\mu\nu}$, are the same as those of the zeroth moment, since both have a meridional scale which is tending to $\infty$. In this situation there will in general be resonant forcing of the second moment which would violate the ordering of the expansion. This is avoided by an appropriate choice of $b$. An alternative approach, used by Simmons (1974a) for
the Phillips model, is to simplify the evolution equations, using the scaling introduced above. A solution of the form of Eq. (17) is then possible and the meridional scale is determined by the coefficients of the reduced evolution equations. Here we stay with the perturbation methodology developed in JA and JB.

The condition of no resonant forcing is

$$\sum_{\mu=0,1}^{\nu=0,1} e_\mu^+ M_{\mu\nu}^{(20)} e_\nu = 0, \quad (23)$$

where $M_{\mu\nu}^{(20)}$ defines the interaction between the zeroth moment and the second moment, $(e_0, e_1)$ is the unstable eigenvector in the homogeneous problem and $(e_0^+, e_1^+) = (e_0^*, -e_1^*)$ is its adjoint. (The adjoint is determined by the condition that it is perpendicular to the decaying eigenvector, $(e_0^*, e_1^*)$, which is the complex conjugate of the growing eigenvector.) When $r \to \infty$, the integrals are dominated by the contributions from small $l$, so the Green’s functions for $k \neq 0$ can be expanded about $l = 0$ as follows:

$$\hat{G}_{\mu\nu}(k, l) = \hat{G}_{\mu\nu}(k, 0) + \frac{l^2}{2} \hat{G}_{\mu\nu}''(k, 0) + \mathcal{O}(l^4), \quad (24)$$

where the prime denotes a derivative with respect to $l$.

At $k = 0$ there is a singularity, so the leading order terms are given by:

$$\hat{G}_{\mu\nu}(0, l) = G_{\mu\nu}^{(-2)} l^{-2} + G_{\mu\nu}^{(0)} + \mathcal{O}(l^2), \quad (25)$$

for constants $G_{\mu\nu}^{(-2)}$ and $G_{\mu\nu}^{(0)}$.

Appendix C gives the derivation of the interaction matrix. It will be shown later that the WKBJ method, which may appear more straightforward, does not give the correct meridional structure of the eigenmodes in this problem.

Let

$$g_{\mu\nu} = G_{\mu\nu} \left(1 - \frac{\beta^{(0)}}{\Lambda^{(0)}_\nu}\right).$$

With this definition the $\beta$ term is absorbed into the Green’s function and does not appear explicitly in the following algebra.

The interaction matrix is given by:

$$M_{\mu\nu}^{(20)} = \begin{pmatrix} 0 & (b_0^2 - b_1^2) \hat{g}_{01} \\ (b_0^2 - b_1^2) \hat{g}_{10} & 0 \end{pmatrix} + \left(\begin{array}{cc} -\chi \hat{g}_{00} - \hat{g}_{01}'' & \chi U_0 \\ -\chi \hat{g}_{10} - \hat{g}_{11}'' & \chi \hat{g}_{11} + \hat{g}_{01}'' \end{array}\right) + \mathcal{O}(r^{-1}) \quad (26a)$$

$$\chi = b^4/r^2, \quad b = (b_0 + b_1)/2, \quad U_\mu = 4\pi^2 G_{\nu}^{-1} \hat{g}_{\mu\nu}. \quad \text{The terms in Eq. (26a) drop out when the matrix is substituted into Eq. (23)} \quad \text{because } |e_0^2| \hat{g}_{01} = |e_0|^2 \hat{g}_{10}. \quad \text{The eigenvector, growth rate and phase speed of the homogeneous problem are given by:}

$$(e_0, e_1) = \{-\hat{g}_{01}, (\hat{g}_{00} + \hat{g}_{11} + U_0 - U_1)/2 - ik^{-1}\sigma\} \quad (27a)$$

$$\sigma^2 = k^2 \left(\frac{\hat{g}_{01}}{\hat{g}_{01}} + \hat{g}_{01} + U_0 - U_1\right)^2/4 \quad (27b)$$

$$\omega = k(\hat{g}_{00} - \hat{g}_{11} + U_0 + U_1)/2 \quad (27c)$$
Substituting Eq. (26) into Eq. (23), using Eq. (27), and solving for $\chi$ gives

$$\chi = \frac{(\hat{g}_{11} + \hat{g}_{00} + U_1 - U_0)(\hat{g}''_{11} + \hat{g}''_{00}) + \hat{g}_{10}\hat{g}''_{01} + \hat{g}_{01}\hat{g}''_{10} + i\sigma k^{-1}(\hat{g}''_{11} - \hat{g}''_{00})}{-(\hat{g}_{11} + \hat{g}_{00} + U_1 - U_0)^2 + \hat{g}_{10}\hat{g}_{01} + i\sigma \omega k^{-2}}.$$  

(28)

Since $\hat{g}$ is a function of $\lambda = \sqrt{k^2 + l^2}$, and the above derivatives are to be evaluated at $l = 0$, we can use

$$\hat{g}(k, l) \approx \hat{g}(k, 0) + \frac{l^2}{2}\hat{g}''(k, 0)$$

and

$$\hat{g}(k, l) \approx \hat{g}(k, 0) + \delta\lambda \frac{d\hat{g}}{d\lambda},$$

where $\delta\lambda = l^2/(2k)$, to find

$$\hat{g}''(k, 0) = \frac{1}{k} \frac{d\hat{g}}{d\lambda}.$$

Substituting into Eq. (28) and using the expressions 27(b) and (c) leads to the surprisingly simple result:

$$\chi = -\frac{1}{\sigma^\infty - i\omega^\infty} \frac{d}{dk} \left( \frac{\sigma^\infty - i\omega^\infty}{k} \right).$$  

(29)

The growth rate and frequency are, from Eq. (22), given simply by

$$\sigma - i\omega = \left(1 - \frac{\sqrt{\chi}}{2r}\right) \left(\sigma^\infty - i\omega^\infty\right) + \frac{\beta\sqrt{\chi}}{2r} \frac{\partial}{\partial\beta} (\omega^\infty + i\sigma^\infty).$$  

(30)

These expressions define the structure and changes in growth rate associated with a broad baroclinic zone purely in terms of the phase speed and growth rate of the homogeneous problem.

$\chi$ is generally complex. Since the meridional structure of the waves is given by $\exp\{-y^2/(2r\sqrt{\chi})\}$, the meridional scale is given by

$$\left\{\text{Re}(\chi^{-1/2})\right\}^{1/2}.$$

The imaginary component of $\chi^{-1/2}$ is related to the meridional phase shift of the wave.

As a consequence of the complex nature of $\chi$ both $\sigma^\infty$ and $\omega^\infty$ contribute to $\sigma$. It is slightly surprising that $\omega^\infty$ enters into the expression for the growth rate, since the system as a whole is Galilean invariant, so that adding a uniform velocity to the flow, and hence shifting the frequency, should not change the growth rate. However, the value of $\omega^\infty$ is the frequency appropriate to the frame of reference implied by the definition of the Green's functions. This leads to a uniquely defined frequency, $\omega^\infty$, which is independent of the frame of reference.

Figure 3 shows $1/\sqrt{\chi(k)}$ for the models considered here. If the flow has $\beta = 0$ and is purely baroclinic then $\omega^\infty$ vanishes and consequently $\chi$ is real. When barotropic shear is added $\omega^\infty$ becomes non-zero and $\chi$ gains an imaginary component which implies that the mode has a phase shift in the meridional direction. This phase shift is such as to feed momentum into the jet. That is, the wave is enhancing the barotropic kinetic energy, so that the growth of the wave is damped. As McIntyre (1970) says, the waves are deformed in 'the obvious sense' by the horizontal shear.
At short wavelengths the meridional phase structure is generated by the barotropic shear and only slightly modified by the $\beta$-effect. At long wavelengths, however, the $\beta$-effect becomes significant. This is reflected primarily in the greater meridional extent (lines (iv) and (vi) in Fig. 3(a)) of the normal modes.

The corresponding meridional structure of the finite-difference solutions is illustrated in Fig. 4, which shows the normal mode corresponding to the point $k = 1.4$ on curve (vi) in Fig. 3. Figure 4 is based on the finite-difference solution. A quantitative
Figure 4. The stream-function anomaly at $z = 0$ for the Phillips model with $c_0 = 0.125$, $\beta = 0.25$ and $k = 1.4$. Finite-difference solution. (a) $r = 4$, (b) $r = 1$. See text for further explanation.
comparison between finite-difference and asymptotic solutions will be carried out below.

For the purely baroclinic Eady model the following asymptotic forms hold:

$$\chi \to \frac{4}{15} \left( 1 + \frac{8}{105} k^2 \right) \quad \text{as } k \to 0$$

and

$$\chi \to \frac{1}{2k_{\text{cut}}(k_{\text{cut}} - k)} \quad \text{as } k \to k_{\text{cut}},$$

where $k_{\text{cut}}$ is the cut-off wave number. The meridional scale diverges to $\infty$ as $k \to k_{\text{cut}}$. This is to be expected since the growth rate goes to zero at $k_{\text{cut}}$ and it is the growth of the disturbance which keeps it confined. That is, the tendency for a localized disturbance to disperse is balanced by the tendency for the instability to reinforce the localization. As a result, the scaling used here, which assumes that $\chi$ is of order unity, breaks down in the vicinity of $k = k_{\text{cut}}$. The growth rate also tends to zero as $k \to 0$, but so does the tendency to disperse (proportional to $\partial \left\{ (\sigma^\infty - i\omega^\infty)/k \right\} / \partial k$), so that the net effect is a finite limit for $\chi$.

When $\omega^\infty = 0$ it also follows from Eq. (29) that the value of $\chi$ at the wave number of maximum instability, $k_{\text{max}}$, for which $\partial \sigma^\infty / \partial k = 0$ by definition, is given simply by

$$\chi_{\text{max}} = \frac{1}{k_{\text{max}}^2}. \quad (31)$$

The meridional scale of the fastest growing mode and the corresponding growth rate are then given by

$$b = \left( \frac{r}{k_{\text{max}}} \right)^{1/2} \quad \text{and} \quad \sigma = \left( 1 - \frac{1}{2k_{\text{max}}r} \right) \sigma^\infty. \quad (32)$$

Equations (31) and (32) only apply when both $\beta$ and the barotropic component of the flow vanish, but since the latter two effects cause only weak modification to the most unstable mode these simple expressions also give a reasonable estimate for more general flows.

7. Comparison of Full Solution with Asymptotic Solutions

(a) The Eady model

Figure 5 shows the growth-rate curves for the Eady model, both with and without barotropic shear. The barotropic shear is sufficient to reduce the surface wind to near zero, as described in section 3. In the absence of barotropic shear the growth rates are reduced uniformly as the width, $r$, is reduced and no new features appear. With barotropic shear, however, there is a departure of the true solution from the asymptotic solution both at small and large $k$. Inspection of the structure of the modes shows that this is related to the influence of critical lines. In the purely baroclinic flow the critical line for all modes is at $z = 1/2$ and does not play any active role in the development. With barotropic shear, however, the critical line intersects the upper boundary at some finite value of $y$, $y_{\text{cl}}$ say. This is illustrated in Fig. 6, showing two basic states and Eliassen–Palm vectors of an associated normal mode.

For large $r$ we have $y_{\text{cl}} = \Theta(r)$. The meridional scale of the disturbance is $\Theta(\sqrt{r})$, so that for sufficiently large $r$ the critical line is well removed from the disturbance.
In Fig. 6 this is the case for the $r = 2$ mode. As $r$ is reduced the critical line moves in towards the disturbance. At $r = 1$, $k = 2$, the critical line is influencing the structure of the mode. Because the more slowly growing modes have greater meridional extent they are the first to be affected by the critical line. That is, as $k$ is increased towards the cut-off value the critical-line effects become more important. There is an enhancement of disturbance amplitudes near the critical line which tends to increase the growth rate.

It was shown above that the meridional phase shift of the wave is related to the barotropic component of the basic state. The meridional phase shift is important because it is related to the meridional momentum flux generated by the waves. In
Figure 6. Zonal-mean zonal wind (contour interval 0.1) and Eliassen–Palm flux vectors \((F_{(x)}, F_{(z)})\) for the corresponding \(k = 2\) mode. The dashed line shows the critical line for the mode (0.3883 in (a) and 0.4505 in (b)).

(a) \(r = 1\), (b) \(r = 2\). See text for further explanation.

the climatological-mean state there is a balance between the forcing generated by the divergence of the vertically integrated meridional eddy momentum flux and surface friction.

This momentum flux can be calculated from the leading-order stream function structure, with

\[
\tilde{\psi}_\mu \approx -4\pi^2 \sum_v \tilde{g}_{\mu v}(k, 0) \tilde{e}_v,
\]
approximating \( \tilde{g} \) by its value at \( l = 0 \). This leads to
\[
\tilde{\psi}_\mu \approx -4\pi^2 c_{fr} \sum_v (-1)^v \tilde{g}_{\mu v}(k, 0) \tilde{\eta}_v^{(0)} \exp \left( -\frac{y^2}{2b^2} \right),
\]
using \( b^2 \ll r^2 \), so that the meridional structure of \( \tilde{C}_\nu \) is dominated by the structure of \( \eta \). If the displacement coefficients are given by \( (\tilde{\eta}_0, \tilde{\eta}_1) = \epsilon \{ \exp(i\phi), \exp(-i\phi) \} \), for real constants \( \epsilon \) and \( \phi \), then
\[
F(y) = k c_{fr}^2 \epsilon^2 \left( \tilde{g}_{\mu 0}^2(k, 0) + 2 \cos \phi \tilde{g}_{\mu 0}(k, 0) \tilde{g}_{\mu 1}(k, 0) + \tilde{g}_{\mu 1}^2(k, 0) \right) y B_{im} \exp(-y^2 B_{re})
\]
(33)
where $\mathcal{B} = 1/b^2$ and subscripts 'im' and 're' denote imaginary and real parts respectively.

Figure 7 compares this result with accurate finite-difference calculations. At large $r$ there is good agreement. As $r$ approaches unity the meridional structure remains in agreement but the amplitude is substantially overestimated by Eq. (33).

Figure 8 shows the momentum fluxes for a normal mode closer to the cut-off wave number. As expected, the influence of the critical line is much stronger. The step-like structure in these figures, which is well resolved by the finite-difference approximation, clearly indicates a different physical structure to the modes shown in Fig. 7. So long as the growth rate of the disturbance is non-zero the critical line is not a singularity,
but when the growth rate is small there can be considerable amplification there. This amplification is related to the decrease in $|k \omega - \bar{u} - i k \sigma|$. Recent work on nonlinear baroclinic life cycles has placed emphasis on the role of shear in determining the structure of the breaking wave. There is, of course, a qualitative relation between the shear and the position of the critical line — with stronger shear the critical line will be close to the jet. While arguments based on shear give qualitative information, the position of the critical line is a quantitative aspect of the flow. This quantitative result is only strictly valid in linear theory, but is also surprisingly accurate for atmospheric diagnostics (Randel and Held 1991).
The most unstable mode is essentially an Eady mode with quantitative modifications due to the meridional structure in the basic state. It fits qualitatively into the ‘interacting Rossby wave’ conceptual model of Bretherton (1966b) and Hoskins et al. (1985), in which the instability is described in terms of a pair of counter-propagating waves, one based on the surface temperature gradients and a second on the tropopause temperature gradients. This conceptual model can be applied rigorously to the homogeneous Eady problem (Bishop 1993). In the present case, however, the meridional structure depends on the assumption of normal-mode growth. For non-modal waves the possibility of evolving meridional structure, as well as evolving phase differences would have to be allowed for. As the wavelength is decreased there is a transition to a qualitatively different regime. The maximum wave amplitude on the lid is not where the temperature
gradients are largest, but rather in the vicinity of a critical line on the flanks of the jet. In this case the upper disturbance does not resemble a discrete Rossby wave, but may be regarded as a perturbed element of the continuous spectrum which complements the incomplete set of discrete modes when there is a non-zero mean flow. The elements of the continuous spectrum have singularities, so they do not correspond to physical modes in the same way that the discrete waves do (e.g. Drazin et al. 1982). The possibility of baroclinic instability associated with critical lines has been raised by Bretherton (1966a), but no concrete example was given. McIntyre (1970) found a disturbance similar to that found here, at and beyond the short-wave cut-off, except that it was associated with a critical line in the mid troposphere.
Figure 9 illustrates the transition from the Eady-type mode at $k = 1.6$ with zero barotropicity, to the critical-line mode at $c_{bt} = 0.125$ and $k = 2.4$. As the barotropic component of the flow is increased the maximum displacements cease to occur at $y = 0$. The critical-line structure only becomes clearly dominant when the wave number is increased (Fig. 9(b)). These figures suggest, however, that the most unstable mode ($k = 1.6$) for a flow with zero surface wind contains significant critical-line features.

(b) The Phillips model

Figure 10 shows the curves corresponding to Fig. 5, but for the Phillips model. Also shown are growth-rate curves for $\beta = 0.25$. When $\beta = 0$ the curves are very similar to
those for the Eady model. As noted earlier, the growth rates of the asymptotic solution are identical to those given by Simmons (1974a). The effect of $\beta$ is to reduce growth rates, especially in the wave-number range $k = 0$ to 1.2. The non-dimensional $\beta$ is related to the dimensional variable by

$$\beta = \beta^\dagger x_{sc} t_{sc} = \beta^\dagger \frac{x_{sc}^2}{U_{sc}}.$$  

This means that for a given velocity scaling, $U_{sc}$, the stabilizing effect of $\beta$ is enhanced by an increase in $x_{sc}$. The values $x_{sc} = 10^6$ m, $U_{sc} = 40$ m s$^{-1}$ and $\beta^\dagger = 10^{-11}$ m$^{-1}$s$^{-1}$ give the value of $\beta$ used in Fig. 10.

At $k = 1.2$ the numerical solution appears to converge onto a value somewhat below $\sigma_{\infty}$. This is shown in Fig. 11. As $r$ is increased up to 100 the growth rate approaches the value of the $l = 0$ mode for homogeneous flow, which grows slightly slower than the most unstable mode $l = 0.563$. If the width is increased further the growth rate begins to increase again beyond $r = 100$. It is not possible to increase the width much beyond $r = 900$ with the numerical code being used here. At this value, of the order of 900 grid points are needed at each level to obtain results accurate to the fourth decimal place (necessary to resolve the behaviour in Fig. 11), which means finding eigenvalues of a $1800 \times 1800$ matrix.

The structure of the mode is shown in Fig. 12 at $r = 10$, $r = 40$ and $r = 400$. Between $r = 10$ and 40 the meridional scale of the wave varies as $\sqrt{r}$, as predicted by the asymptotic theory. At larger values of $r$, however, there is a transition to an internal mode with a meridional wavelength close to that of the fastest growing mode of the homogeneous problem. The change in structure is consistent with the interpretation of the growth rates given above.

It appears that the internal mode is more sensitive to shear than the external mode. This reveals potential pitfalls of both numerical and analytic techniques. An asymptotic solution based on the most unstable mode at $r = \infty$ would, in this case, not be valid for
a useful range of wave numbers. On the other hand, the numerical results for $r \leq 100$ could easily have been interpreted as showing convergence to a large $r$ limit.

8. THE WKBJ METHOD

The WKBJ method has been used for this problem by a number of authors, most recently by Ioannou and Lindzen (1986, 1990) (hereafter IL). The results are similar, but not identical to, those obtained by the moment expansion method, as discussed by Simmons (1974a). This section attempts to explain why.

In the limit $r \to \infty$ the growth rate depends only on the structure of the flow near $y = 0$. The above results apply to all flows of the form

$$\frac{\partial \bar{c}}{\partial y} \approx -C_{fr} \left( 1 - \frac{y^2}{2r^2} \right).$$

This allows direct comparison with the results of IL, setting $r^2 = L^2/q$, $L$ and $q$ as defined by IL. If we consider wave number $k$ and suppose the complex phase speed is $c_{\text{asym}}$, then it is assumed that the local meridional wave number, $l(y)$, can be derived from the dispersion relation for homogeneous flow with the local value of baroclinicity, $c^{\infty}(k, l, y)$:

$$c_{\text{asym}} = c(k, l) \approx c^{\infty}(k, 0, y) + l^2 c^{\infty''}/2,$$ (34)

where the prime denotes a derivative with respect to $l$ and it has been assumed that $l$ is small. We then define $y_1$ by $l(y_1) = 0$, so that Eq. (34) implies

$$c_{\text{asym}} = c^{\infty}(k, 0, y_1) = \frac{1 - y_1^2}{(2r^2)}c_0^{\infty},$$

where $c_0^{\infty} = c^{\infty}(k, 0, 0)$. Substituting in Eq. (34) gives

$$l^2 \approx \frac{2}{c^{\infty''}} \left( \Delta c - \frac{y^2}{2r^2} c^{\infty} \right),$$ (35)

where

$$\Delta c = c_0^{\infty} - c_{\text{asym}} = \frac{c_0^{\infty} y_1^2}{2r^2}.$$ (36)

Equations (35) and (36) show that $l^2$ is positive for $|y| < y_1$, giving a locally oscillatory solution, and negative for $|y| > y_1$, giving locally exponential solutions.

IL reduce the Eady instability problem with slowly varying baroclinicity to a standard WKBJ form:

$$\frac{d^2 W(y)}{dy^2} + l^2 W(y) = 0,$$ (37)

where $W$ is the amplitude of the disturbance.

The eigenvalues are determined by requiring that

$$\int_{-y_1}^{y_1} l \, dy = \left( n + \frac{1}{2} \right) \pi,$$ (38)
for some integer \( n \). In the WKBJ approximation this guarantees that the solution decays as \( |y| \to \infty \). Substituting Eq. (35) into Eq. (38) gives

\[
\left( n + \frac{1}{2} \right) \pi = \int_{-y_1}^{y_1} (y_1^2 - y^2)^{1/2} \, dy \sqrt{\frac{c_0^{\infty}}{c_0^{\infty''}}} r^2 = \frac{y_1^2 \pi}{2} \sqrt{\frac{c_0^{\infty}}{c_0^{\infty''}}} r^2 = \frac{\pi r \Delta c}{\sqrt{c_0^{\infty} c_0^{\infty''}}}.
\]

For the gravest mode, \( n = 0 \), it follows that

\[
\Delta c = \sqrt{\frac{c_0^{\infty}}{2r}}.
\]
This is identical to the result Eq. (30) for \( \beta = 0 \). It differs slightly from the result of IL because they used an empirical approximation of the dispersion relation to evaluate Eq. (38) instead of the asymptotic expression Eq. (35) used here. The difference is, however, insignificant.

The structure of the modes, as discussed by Simmons (1974a), is nevertheless different. The fact that a correct growth rate can be deduced with different structures is related to the fact that the growth rate depends only on the meridional scale of the mode.

The WKBJ solution is valid if \( l^2 \) is slowly varying, which in practice means

\[
\mathcal{F} = \frac{1}{g^{3/4}} \frac{d^2}{dy^2} \frac{1}{g^{1/4}} = \frac{5g''}{16g^3} - \frac{g'''}{4g^2} \ll 1
\]

(39)

where \( g = l^2 \) (Olver 1974). From Eq. (35) \( g''(0) = -2c_0^\infty/(r^2c_0^\infty') \) and \( g'(0) = c_0^\infty/(c_0^\infty' r^2) \). Evaluating Eq. (39) at \( y = 0 \) then gives

\[
\mathcal{F} = \frac{1}{2}.
\]

(40)

Thus, the error term remains order unity in the limit \( r \to \infty \). The problem is that although the \( l^2 \) varies slowly in the limit \( r \to \infty \), the magnitude of \( l^2 \) at \( y = 0 \) also tends to zero in that limit. The net effect of these two tendencies is that the error term tends to a constant. Thus, the WKBJ approach gives the correct eigenvalues but the incorrect structure, because the approximation is not locally valid. Apart from scaling, Eq. (37) is the equation for the harmonic oscillator. The solutions are of the form of Hermite polynomials multiplied by a Gaussian, as used in the moment solution above. The form of the error term shows that the WKBJ technique is not appropriate for this equation. It would be valid if a disturbance of a fixed meridional scale was considered, but the most unstable mode here generally has a scale which grows as the basic state is varied. The technique would be applicable for the large-scale waves with \( \beta \neq 0 \), but Eq. (35), used above to get an explicit expression for \( \Delta c \), is based on an expansion about \( l = 0 \), so the latter result is not valid in this case.
9. DISCUSSION

A simple formula, Eq. (29), has been derived for the meridional scale of baroclinic waves in a broad baroclinic zone. This can be written

\[ b^4 = r^2 \chi = \frac{r^2}{k^2} \left( 1 - \frac{c_g^\infty}{c^\infty} + \mathcal{O}(r^{-1}) \right), \]

where \( c_g^\infty \) and \( c^\infty \) are, respectively, the complex group and phase velocities of the homogeneous problem.

Comparison with accurate numerical solutions shows that this expression is accurate for \( r \geq 2 \) (the units here are Rossby radii). For the fastest growing mode it remains qualitatively accurate at smaller values of \( r \). The slower growing modes, however, are modified by encroaching critical lines. The distance from the centre of the jet to the critical line is of order \( r \), so that the critical line is, for large \( r \), well removed from the region occupied by the disturbance. Since the more slowly growing modes tend to have a larger meridional spread (small \( |c^\infty| \) ) they are the first to be affected when the critical line is moved closer to the jet centre.

Consequently, at \( r = 1 \) there is a significant difference between the structure of the modes at different wavelengths. The fastest growing mode is essentially confined by the localized baroclinicity, though influenced by the critical lines. The slower modes, on the other hand, are confined by the critical lines. The critical line also plays an active role in the disturbance, especially for the short waves.

APPENDIX A

The Phillips two-level model

The two-level model represents a quasi-geostrophic flow with zero potential-temperature anomalies on the upper and lower boundary and with the vertical structure approximated by a two-point discretization. The equations of motion can be written

\[ \frac{D_g}{Dt}(q_{\mu} + \beta y) = 0, \quad \mu = 0, 1, \]

where

\[ q_0 = \nabla^2 \psi_0 + \gamma^2 (\psi_1 - \psi_0) \]

and

\[ q_1 = \nabla^2 \psi_1 + \gamma^2 (\psi_0 - \psi_1). \]  

(A.1)

Here \( \gamma = f/(2NH) \). In the Fourier representation Eq. (A.1) becomes

\[ \begin{pmatrix} \hat{q}_0 \\ \hat{q}_1 \end{pmatrix} = - \begin{pmatrix} k^2 + l^2 + \gamma^2 & -\gamma^2 \\ -\gamma^2 & k^2 + l^2 + \gamma^2 \end{pmatrix} \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix}. \]

Inverting this matrix gives the Green’s function matrix given in Eq. (3).

The solution for a meridionally uniform zonal flow can be obtained by setting \( \hat{\psi}_\mu = -\bar{u}_\mu y \). The equations for a disturbance proportional to \( \exp(ik(x - ct)) \) are then

\[ -(\bar{u}_1 - c)\{(\lambda^2 + \gamma^2)\hat{\psi}_1 - \gamma^2\hat{\psi}_0\} + (\beta + \gamma^2 U)\hat{\psi}_1 = 0, \]

\[ -(\bar{u}_0 - c)\{(\lambda^2 + \gamma^2)\hat{\psi}_0 - \gamma^2\hat{\psi}_1\} + (\beta - \gamma^2 U)\hat{\psi}_0 = 0, \]

where \( U = \bar{u}_1 - \bar{u}_0 \). The second term represents the meridional potential-vorticity gradient. In the upper layer \( \beta \) reinforces this gradient, in the lower layer it acts against the gradient.
It follows that the dispersion relation is
\[ c = k^{-1}(\omega^\infty + i\sigma^\infty) = \frac{\bar{u}_0 + \bar{u}_1}{2} - \frac{\beta(\gamma^2 + \lambda^2) \pm \sqrt{\gamma^4 \beta^2 - U^2 \lambda^4 (\gamma^4 - \lambda^4/4)}}{\lambda^2 (\lambda^2 + 2\gamma^2)}. \]

For \( \beta = 0 \) and \( \bar{u}_0 + \bar{u}_1 = 0 \) this simplifies to
\[ (\omega^\infty + i\sigma^\infty)^2 = \frac{k^2 U^2}{4} \left( \frac{k^2 - 2\gamma^2}{k^2 + 2\gamma^2} \right). \]

The most unstable wave number is then given by
\[ k_{\text{max}}^2 = 2(\sqrt{2} - 1)\gamma^2 \]
and the corresponding growth rate is
\[ \sigma_{\text{max}}^\infty = U\gamma \left( 1 - \frac{1}{\sqrt{2}} \right). \]

**APPENDIX B**

*The moment expansion*

The meridional displacement, \( \eta(x, y, t) \), is represented as in IA by the following moment expansion:
\[ \eta_\mu(x, y, t) = \sum_{m=0}^{\infty} \eta_\mu^{(m)}(x, t) P_\mu^{(m)}(y), \]
where
\[ P_\mu^{(m)}(y) = \frac{1}{m!} H^{(m)} \left( \frac{y}{R_\mu} \right) \exp \left( -\frac{y^2}{2a_\mu^2} \right). \]

\( R_\mu \) and \( b_\mu \) are constants defining the scale of the expansion polynomials and \( H^{(m)}(s) \) is the Hermite polynomial of order \( m \) as used in IA (Hochstrasser 1965). \( R_\mu \) may be considered to be an 'internal' scale, related to the wavelength of the meridional structure, and \( b_\mu \) an 'external' scale, giving the meridional extent of the disturbance. The \( m = 0 \) structure function is independent of \( R_\mu \) (since \( H^{(0)} \equiv 1 \)) and so has a single meridional scale \( b_\mu \).

It follows from the orthogonality properties of the Hermite polynomials that
\[ \eta^{(m)}(x, t) = \int_{-\infty}^{\infty} P_\mu^{(m)}(y) \eta(x, y, t) \, dy, \]
where
\[ P_\mu^{(m)}(y) = \frac{1}{R_\mu \sqrt{2\pi}} H^{(m)} \left( \frac{y}{R_\mu} \right) \exp \left( -\frac{y^2}{2a_\mu^2} \right), \quad R_\mu^{-2} = a_\mu^{-2} + b_\mu^{-2}. \]

The value of \( a_\mu \), the external scale of the projection functions, is determined by the orthogonality condition. Through this condition it is related to the two scales of the structure function. These will be determined by requiring that the zeroth moment decouples from the higher moments in the limit \( r \to \infty \).
# Table C.1

<table>
<thead>
<tr>
<th>$n = 0$</th>
<th>$(n = 2)/(n = 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}[P_{\mu}^{(n)} \exp(-y^2/(2r_{\mu}^2))]]$</td>
<td>$\frac{1}{\sqrt{2\pi}} \frac{b_{\mu} r_{\mu}}{b_{\mu}^2 + r_{\mu}^2} \exp \left( -\frac{i^2 b_{\mu}^2 r_{\mu}^2}{2(b_{\mu}^2 + r_{\mu}^2)} \right)$</td>
</tr>
<tr>
<td>$\mathcal{F}[P_{\mu}^{(n)}]$</td>
<td>$\frac{1}{2\pi} \frac{a_{\mu}}{R_{\mu}} \exp \left( -\frac{i^2 a_{\mu}^2}{2} \right)$</td>
</tr>
<tr>
<td>$\mathcal{F}[P_{\mu}^{(0)} P_{\mu}^{(n)}]$</td>
<td>$\frac{1}{\sqrt{2\pi}} \frac{1}{b_{\mu}^2 + r_{\mu}^2} \exp \left( -\frac{i^2 R_{\mu}^2}{2} \right)$</td>
</tr>
</tbody>
</table>

The first column lists the quantities evaluated, the second column gives the values for $n = 0$ and the last column gives the ratio of the $n = 2$ value to the $n = 0$ value, i.e. the $n = 2$ value is obtained by multiplying columns two and three. See text for further details.

# Appendix C

**Moment expansions and the limit $r \to \infty$**

Table C.1 is as Table D.1 from JA, except that subscripts are here retained. $\mathcal{F}[\ ]$ denotes a Fourier transform with respect to $y$.

Substituting the expressions from Table C.1 into Eq. (9) and setting $a_{\mu} = r_{\mu}$ gives:

$$\tilde{A}_{\mu v}^{(00)} = \frac{r_v}{\sqrt{2\pi}} \frac{b_v (r_{\mu}^2 + b_{\mu}^2)^{1/2}}{b_{\mu} (r_v^2 + b_v^2)^{1/2}} \exp \left\{ -\frac{i^2}{2} \left( \frac{r_v^2}{r_v^2 + b_v^2} \right) \right\}$$

$$\times \left( (\Lambda_v^{(0)} - \beta_v^{(0)}) + \frac{\beta_v^{(2)}}{r_v} \frac{i^2 r_v^2 b_v^2}{r_v^2 + b_v^2} \right)$$

$$\tilde{B}_{\mu v}^{(00)} = \frac{\Lambda_v^{(0)} r_v}{\sqrt{2\pi}} \exp \left\{ -\frac{i^2}{2} \left( \frac{r_v^2}{r_v^2 + b_v^2} \right) \right\}$$

$$\tilde{A}_{\mu v}^{(20)} = \frac{(\Lambda_v^{(0)} - \beta_v^{(0)}) b_v r_{\mu}^3 (r_{\mu}^2 + b_{\mu}^2)^{1/2}}{\sqrt{2\pi} b_{\mu}^3 (r_v^2 + b_v^2)^{1/2}} \{1 - (r_{\mu}^2 + b_{\mu}^2)^2 \}$$

$$\times \exp \left\{ -\frac{i^2}{2} \left( \frac{r_{\mu}^2}{r_{\mu}^2 + b_{\mu}^2} \right) \right\}$$

$$\tilde{B}_{\mu v}^{(20)} = \frac{-\Lambda_v^{(0)} r_v r_{\mu}^2 b_{\mu}^2}{\sqrt{2\pi} (r_{\mu}^2 + b_{\mu}^2)^{1/2}} \exp \left\{ -\frac{i^2}{2} \left( \frac{r_v^2}{r_v^2 + b_v^2} \right) \right\}.$$ 

In the large $r$ limit the integrals are dominated by the small $l$ behaviour of the Green's function. It follows that the integrals are dominated by the first and second moments of the functions $A$ and $B$. We need

$$\int \tilde{A}_{\mu v}^{(20)} \, dl = (\Lambda_v^{(0)} - \beta_v^{(0)}) \frac{b_v}{r_v b_{\mu}^3} \left( b_v^2 - b_{\mu}^2 - \frac{b_{\mu}^4}{r_v^2} \right),$$

$$\int i^2 \tilde{A}_{\mu v}^{(20)} \, dl = -(\Lambda_v^{(0)} - \beta_v^{(0)}) \frac{b_v}{r_v b_{\mu}^3}.$$
The coefficient $\hat{B}$ multiplies the $k = 0$ Green’s function which is singular at $l = 0$ with leading-order behaviour $l^{-2}$, the relevant integrals are

$$\int \mu \hat{B}_{\mu \nu}^{(20)} dl = -\frac{\Lambda^{(0)}_{\nu} b^2_{\mu}}{r^2_{\nu}} \left( 1 - \frac{b^2_{\mu}}{2r^2_{\nu}} - \frac{b^2_{\mu}}{r^2_{\mu}} \right),$$

$$\int \mu \hat{B}_{\mu \nu}^{(20)} dl = -\frac{\Lambda^{(0)}_{\nu} b^2_{\mu}}{r^2_{\nu} r^2_{\mu}} \left( 1 - \frac{3b^2_{\mu}}{2r^2_{\nu}} - \frac{b^2_{\mu}}{r^2_{\mu}} \right).$$

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