A comprehensive two-moment warm microphysical bulk scheme. I: Description and tests

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SUMMARY

A bulk microphysical scheme which predicts the concentrations and mixing ratios of cloud droplets and raindrops is presented. The scheme draws its originality from the use of generalized gamma law as basis functions for the drop size distributions and also from the attention paid to performing analytical integrations of most of the microphysical transfer rates. The numerical representation of each process has been reviewed throughout and specific tests have been made to evaluate separately the accuracy of the scheme compared with a bin-size model. The scheme, which depends on the specification of a few input parameters shaping the activation spectrum, is incorporated in a three-dimensional non-hydrostatic model with some applications given in a companion paper (Part II).

KEYWORDS: CCN activation Cloud physics Coalescence Modelling Sedimentation Two-moment scheme

1. INTRODUCTION

The trend in advanced warm microphysical studies is to combine the dynamics of three-dimensional (3D) mesoscale/large-eddy simulation (LES) numerical models with the most explicit description of the aerosol/water-drop spectra in fixed bin-size frameworks. Such a technique in the representation of the microphysics imposes, however, severe computing limitations owing to the huge number of scalar variables (one per bin) to be carried on through the course of the model. To the authors’ knowledge, only Kogan et al. (1995) published results of complete LES simulations of ‘cloud condensation nuclei (CCN) active’ stratocumulus by integrating a minimum of 19 + 22 microphysical variables (CCN and cloud-droplet categories) in a $40^3$ computational domain for 2–3 hours. So it is not surprising that at mesoscale the bin approach has not received much attention. Some reasons for this are that less sophisticated microphysical schemes are probably sufficient in most cases, but also that current simulations, made now over typical horizontal domains larger than $100^2$ grid points for several hours, make the use of bins clearly prohibitive.

These arguments are strong enough to encourage the development of bulk microphysical schemes covering the whole condensed water spectrum with two, four or six prognostic variables corresponding to statistical moments of basis functions (Clark 1976). From this concept, the classical Kessler scheme (Kessler 1969), with a revised autoconversion rate (see Wyant et al. 1997), remains the most simple parametrization for the explicit modelling of precipitating clouds. In its essence, the scheme predicts only the mixing ratios (third-order moments) of the cloud droplets and raindrops. The scheme ignores details about cloud-droplet spectra and assumes a Marshall–Palmer distribution law for rain ($n_r = N_0 \exp(-\lambda_r D)$) with a constant intercept parameter $N_0$ thus linking the raindrop concentration to the rain mixing ratio through the slope parameter $\lambda_r$, with $\alpha_r = \nu_v = 1$ and $N_0 = N_r \lambda_r$ in Eq. (1). The latter constraints are very limiting and can lead to inaccurate results in the case of weakly and highly precipitating warm clouds (see Feingold et al. 1998).

The next step is to consider bulk schemes with prediction of at least three (Richard and Chaumerlie 1989, hereafter RC89) or four prognostic microphysical variables

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(Ziegler 1985, hereafter Z85; Chaumerliac et al. 1987), so including concentrations (zeroth-order moments) and mixing ratios of each of the two categories of condensed water. RC89 did not predict the cloud-droplet concentrations but the parametrization they used contains an explicit dependence on the droplet spectrum when computing the autoconversion and accretion rates which were parametrized according to Berry and Reinhart (1974), hereafter BR74. RC89 showed that adding a prognostic equation for raindrop concentration provides a better location of the precipitation downwind of a plateau due to wind drifting. An even greater flexibility in bulk warm microphysical schemes is offered by considering the time evolution of cloud-droplet concentration. Besides its impact in the formation of the first raindrops by autoconversion, the prediction of cloud-droplet number concentration is especially important in cloud chemistry and radiative transfer. For example, LES studies on the stability of stratocumulus clouds are an important avenue or theme of active research (Kogan et al. 1994; Stevens et al. 1996) for a climate perspective, so the inclusion of a prediction equation for cloud-droplet concentration seems desirable. A higher degree of development in bulk microphysical schemes would be to incorporate more information about the spectral width of cloud-droplet and raindrop size distributions. This parameter is usually kept constant in bulk schemes because it is rather unclear how to parametrize the effect of nucleation and condensation processes. For instance, spectral broadening of cloud-droplet distributions is still an open question with possible effects of mixing and turbulence (see Brenguier (1998) for a review). As Clark and Hall (1983) concluded that little was gained by adding the prediction of a third moment, it seems wise to consider a two-moment scheme to simplify the closure and some problems of realizability because concentrations and mixing ratios are not fully independent variables.

Another point of concern is that a lot of research has been done on getting analytical solutions to equations approximating to a mathematical representation of the microphysical processes. As microphysics is very complex (full Köhler equation and water-vapour condensation, collection kernels and coalescence, etc.), it is out of the question to look for exact solutions to each process. However, the strength of analytical solutions is great (Verlinde et al. 1990) and can be evaluated against numerical solutions of the most accurate representation of a process (Z85). This is an interesting aspect of this work, showing that it is possible to get analytical solutions even for complex distributions describing cloud-droplet and raindrop size distributions and to test them against purely numerical solutions.

This paper is organized as follows. First, we discuss the functional representation of liquid water. The bulk microphysical scheme is then presented and results of specific tests are analysed. A list of symbols used is given in appendix D. Other results obtained by running the full scheme in 2D cases are discussed in Part II of this paper (Cohard and Pinty 2000).

2. THE LIQUID WATER IN BULK MICROPHYSICAL SCHEMES

It is customary in bulk microphysical schemes to consider two modes in which liquid water is distributed, thus providing a natural partition between cloud and rainwater. Roughly speaking, these two categories are characterized by an equivalent mixing ratio (mass of condensate scaled by the mass of dry air) but by very different number concentrations, from a few tens or hundreds per cubic centimetre for the cloud droplets down to a few units or tens per litre for the raindrops. This will have some consequences in the parametrization of some transfer rates as shown in the next section.
Experimental evidence and numerical studies legitimately raised the question of whether it is necessary to consider more details such as submodes within each of the two water categories. For example, Brenguier and Grabowski (1993) illustrated the bimodal behaviour of non-homogeneous entraining cloud parcels in active cores of shallow cumulus. They showed that, at their resolved scale of 25 m, the differing origin of droplet populations can explain the persistence of droplet distributions with multiple degrees of spectral broadening. This suggests that such fine-scale fluctuations of droplet size distributions cannot be easily captured at larger scales without minimal observational information such as that given in Hudson and Svensson (1995) or in Hudson and Yum (1997). At the other end of the large droplet/drop spectrum, studies such as those of Low and List (1982a,b) and Brown (1997), which are related to the various types of collision break-up of the raindrops, led to the conclusion that several peaks could be detectable in 'equilibrium' rain spectra. However, these results are not supported by one-dimensional simulations of Hu and Srivastava (1995), and even contradict the data interpretation of Sauvageot and Lacaux (1995) who found that secondary peaks are much less pronounced and even not detectable in observed drop size distributions. This short discussion leads to the conclusion that, for our macroscopic approach, two modes are probably sufficient to describe liquid-water distributions and consequently the various conversion processes acting on them.

We now examine the practical choice of an empirical distribution law that is able to describe cloud-droplet and raindrop spectra. Two functional forms have been proposed in past studies. The log-normal law was adopted by Clark (1976), Clark and Hall (1983), Nickerson et al. (1986) and RC89 with a good observational basis for raindrops provided by Feingold and Levin (1986). The use of the gamma distribution law was promoted by Clark (1974), ZB85, Flatau et al. (1989) and Walko et al. (1995) for modelling studies of precipitating clouds, and by Ulbrich (1983) for data analysis of precipitations. From our modelling point of view, it is important to select a mathematical function that permits the exact quadrature of difficult to integrate functions (Verlindt et al. 1990). This is in favour of gamma-like distribution laws for several reasons. Recently, arguments from statistical theory applied to idealized atmospheric-particle systems led Liu and Hallett (1998) to suggest that a generalized gamma law would be preferable to describe cloud-droplet spectra. Furthermore, their study indicates that the special case of the Weibull distribution law is particularly well suited and thus supports earlier results of Williams and Wojtowicz (1982). Using different arguments, Considine and Curry (1996) found also that the generalized gamma law can describe the most probable shape of droplet spectra experiencing condensation growth but with different dispersion parameter settings. Note that, generally, gamma distributions are preferred in radiative-transfer studies of scattering media (Deirmendjian 1969). Therefore it appears that the generalized gamma distribution combines several advantages because it applies to both droplet and raindrop spectra, it enables analytical integrations (see for instance appendix B) and it offers greater flexibility of adjusting the spectral dispersion. The present scheme is based upon such a representation, so the droplet/drop size distributions are assumed to follow the normalized form:

$$n_i(D) = N_i \frac{\alpha_i}{\Gamma(v_i)} \lambda_i^{v_i} D^{v_i - 1} \exp\left\{-\left(\lambda_i D\right)^{\alpha_i}\right\},$$

where the index $i \in [c, r]$ stands for cloud or rain, respectively. Our strategy is to predict only two of the most significant moments of Eq. (1) that possess a clear physical
meaning, namely the zeroth, \( N_i \), and third-order moments

\[
    r_i = \frac{1}{\rho_0} \int_0^\infty (\pi/6) \rho_w D^3 n_i(D) \, dD.
\]

As these two moments are determined from Eq. (1), the variable slope parameter \( \lambda_i \) can be deduced from

\[
    \lambda_i = \left\{ \frac{\pi}{6} \rho_w \frac{\Gamma(v_i + 3/\alpha_i)}{\Gamma(v_i)} \frac{N_i}{\rho_0 r_i} \right\}^{1/3},
\]

whereas the remaining parameters \( \alpha_i \) and \( v_i \), that are mostly related to the spectral breadth of Eq. (1), are held fixed for the moment. Equation (2) is an application of a general formula to compute the \( p \)-moment of Eq. (1), that is

\[
    \int_0^\infty D^p n_i(D) \, dD = \frac{N_i \Gamma(v_i + p/\alpha_i)}{\lambda_i^p \Gamma(v_i)} = N_i M_i(p).
\]

We expect that future versions of the scheme will incorporate simple parametric dependences of the dispersion such as proposed by Hudson and Yum (1997) for cloud droplets and by Ulbrich and Atlas (1998) for raindrops.

3. The Bulk Microphysical Scheme

(a) System of equation

The continuity equations of the condensed phaser, described in terms of concentration and mixing ratio, are written in symbolic form as follows:

\[
    \frac{\partial N_c}{\partial t} = \sum \left( \frac{\partial N_c}{\partial t} \right)_{\text{NMT}} + \text{CVHENC} - \text{CCACCR} - \text{CCSCOC},
\]

\[
    \frac{\partial r_c}{\partial t} = \sum \left( \frac{\partial r_c}{\partial t} \right)_{\text{NMT}} + \text{RVHENC} + \text{RVCONC} - \text{RCAUTR} - \text{RCACCR},
\]

\[
    \frac{\partial N_r}{\partial t} = \sum \left( \frac{\partial N_r}{\partial t} \right)_{\text{NMT}} + \text{CCAUTR} - \text{CRSCOR} - \text{CSED},
\]

\[
    \frac{\partial r_r}{\partial t} = \sum \left( \frac{\partial r_r}{\partial t} \right)_{\text{NMT}} + \text{RCAUTR} + \text{RCACCR} - \text{RREVAV} - \text{RSED}.
\]

The subscript NMT refers to non-microphysical tendencies (advection, turbulence, numerics and other physical processes) while the meaning of the other symbols is given in Table 1 and detailed in section 3(b). In addition to Eqs. (4a)–(4d), an equation of conservation for \( N_a \), the number concentration of the activated CCN, is necessary to keep track of the CCN upon which cloud droplets have already nucleated as explained later. This additional equation is given by

\[
    \frac{\partial N_a}{\partial t} = \sum \left( \frac{\partial N_a}{\partial t} \right)_{\text{NMT}} + \text{CVHENC}.
\]

Finally the system is closed by expressing the conservation of the water-vapour mixing ratio \( r_v \) and of the dry potential temperature \( \theta \):

\[
    \frac{\partial r_v}{\partial t} = \sum \left( \frac{\partial r_v}{\partial t} \right)_{\text{NMT}} + \text{RREVAV} - \text{RVCONC} - \text{RVEHNC},
\]

\[
    \frac{\partial \theta}{\partial t} = \sum \left( \frac{\partial \theta}{\partial t} \right)_{\text{NMT}} + \frac{L_v}{\Pi C_p\theta} (\text{RVCONC} - \text{RREVAV}).
\]
The coefficients appearing in the formulae that follow are expressed in SI units unless specified otherwise.

(b) Nucleation and condensation

The CCN activation and the condensation growth of cloud droplets are the dominant processes distinctly affecting the number concentration, $N_c$, and the mixing ratio, $r_c$, of cloud liquid water at the early stage of the cloud lifetime. They strongly influence the associated radiative properties of clouds as well as their ability to precipitate by triggering coalescence between droplets. These processes are, however, difficult to model explicitly because they depend upon the maximum (CCN activation) and mean (condensation) local supersaturation values, $s_{v,w}$, in contact with the CCN and the droplets. This quantity, which depends on the vertical velocity and on the thermodynamics (temperature, pressure, water-vapour mixing ratio), is not generally well captured in 3D cloud models because it is highly variable due to the non-homogeneous character of the mixing processes in clouds (Latham and Reed 1977) and also because it is subject to spurious, non-physical, numerical bursts at cloud boundaries (Stevens et al. 1996). Moreover, the activation process results from an unstable thermodynamical equilibrium at short time-scales in competition with the condensation growth that tends to absorb any excess supersaturation. Therefore, computing $s_{v,w}$ at grid scale to treat the droplet nucleation and condensation processes is questionable and other methods must be explored.

The activation of the CCN, an initiating source term in Eqs. (4a)-(4b), is a complex process that results from the growth of wet aerosols past a critical particle size which, according to the Köhler theory (Pruppacher and Klett 1997, hereafter PK97), depends on their size, their chemical and hygroscopic properties and on some thermodynamical fields (with an emphasis on $s_{v,w}$). In bin models (Lee et al. 1980; Flossmann et al. 1985), an initial multi-modal aerosol distribution including condensation nuclei (CN) whose spectrum corresponds to the accumulation mode, is discretized over about 50 bins in order to follow growth beyond water-vapour saturation. When $s_{v,w} > 0$ locally, a diagnostic relation is applied to check the smallest critical radius that can be activated in the CN distribution (initially CCN and later newly formed cloud droplets) from background interstitial aerosols. Such a technique is deemed to be rigorous and accurate provided a good estimate of the $s_{v,w}$ maximum within the time step and a precise follow-up of the aerosol growth exist. It is, however, clearly disproportionate to 3D modelling.
requirements for the sole purpose of computing a nucleation rate. So, motivated by the simplicity of the diagnostic and integral approach of Twomey (1959), Chaumerladi et al. (1987) experimented with a parametrization based upon an activation spectrum described by \( N_{\text{CCN}} = C s_{v, w}^k \), where \( s_{v, w} \) is generally expressed as a percentage and where \( C \) and \( k \) are shape parameters to fit each aerosol type. Although it was derived from a simplified form of the Köhler equation, most of the criticism addressed towards Twomey’s original parametrization concerned the ability of such a simple power law to account for all the nucleation properties of complex aerosol populations. This led Cohard et al. (1998), hereafter CPB98, to look for a more general expression of the activation spectrum following a functional form:

\[
N_{\text{CCN}} = C s_{v, w_{\text{max}}}^k F(\mu, k/2, k/2 + 1; -\beta s_{v, w_{\text{max}}}^2).
\] (7)

This form enables an analytical integration in Twomey’s framework and also adheres more closely to observations of \( N_{\text{CCN}} \) at large \( s_{v, w} \). Note that in Eq. (7), \( F(a, b, c; x) \) is the hypergeometric function (see Press et al. (1992) for tabulation) and that \( C \) and \( k \) now have different numerical values from the ones of the preceding \( N_{\text{CCN}}(s_{v, w}) \) power law. The advantage of Eq. (7) has been confirmed recently by Cohard et al. (2000), who showed that it is possible to establish parametric relationships for each of the four unknowns in Eq. (7) based on observable properties of aerosols in terms of size distribution, chemical composition and solubility. Using Eq. (7) and following Twomey’s method, CPB98 showed that an estimate of the maximum supersaturation \( s_{v, w_{\text{max}}} \) is a solution of

\[
s_{v, w_{\text{max}}}^{k+2} F(\mu, k/2, k/2 + 3/2; -\beta s_{v, w_{\text{max}}}^2) = \frac{\rho_a (\psi_1 w)^{3/2}}{2kC \pi \rho_w \psi_2 (G)^{3/2} B(k/2, 3/2)}.
\] (8)

The accuracy of solving numerically for the root of Eq. (8) is analysed in the next section. Substituting values of \( s_{v, w_{\text{max}}} \) into Eq. (7) provides an estimate of the potentially activated CCN number concentration, so the production rate of newly nucleated droplets is given by comparison with CCN that have been already activated: that is

\[
\text{CVHENC} = \max(0, N_{\text{CCN}}(s_{v, w_{\text{max}}}) - N_a(t - \delta t)).
\] (9)

Finally the associated mixing-ratio tendency of the nucleation process is computed as if a monodisperse CCN distribution is assumed with a diameter equal to the critical diameter \( D_{\text{crit}} = 4A/3s_{v, w_{\text{max}}}; \)

\[
\text{RVHENC} = \text{CVHENC} \times \frac{\rho_w \pi}{\rho_a 6} \rho_w D_{\text{crit}}^3.
\] (10)

Equation (10) underestimates the RVHENC rate, but this process contributes very modestly to the cloud liquid-water content which depends essentially on the condensation rate in the absence of precipitation. Note also that, when a cloud dissipates by complete evaporation (see the diagnosis on \( r_c \) later), the \( N_a \) and \( N_c \) concentrations are reset to zero.

The reversible condensation/evaporation process is implicit and results from the non-iterative saturation adjustment (Langlois 1973) that is summarized in appendix C. This treatment is well justified by observations that show that the interior of clouds is nearly in thermodynamical equilibrium (\( s_{v, w} < 1\% \), PK97). An estimation of the condensation rate can be obtained by solving for \( T \) the equation of the first law of thermodynamics:

\[
(T - T^*) + \frac{L_v(T)}{C_{ph}} (r_{vs}(T) - r_{v}^*) = 0,
\] (11)
where $T^*$ and $r_v^*$ are the temperature and vapour mixing ratio of an intermediate state after integrating all the other explicit processes. The condensation rate is given by

$$\text{RVCONC} = \max(-r_c, r_v^* - r_{vs}(T)).$$

(12)

In the case of condensation growth ($r_v^* > r_{vs}$), the number concentration of cloud droplets is not affected, whereas if partial (complete) evaporation occurs ($r_v^* < r_{vs}$), some (or all) of the droplets will evaporate completely and so the number concentration ought to be reduced in principle. In fact the problem is more complicated because, for instance, the current theory considers partial droplet evaporation as the result of non-homogeneous mixing of undersaturated air entrained in clouds. The explicit modelling of such a process requires consideration of all microphysical aspects in the evaporating parcels of drier air embedded in cloudy grid boxes (Brenquier and Grabowski 1993). As a further subgrid-scale reactivation of the CCN from the completely evaporated droplets will probably replenish the initial droplet concentration (Baker et al. 1980), it is reasonable to assume that $N_c$ remains constant in the case of partial evaporation but obviously $N_c = 0$ whenever $\text{RVCONC} = -r_c$ (Eq. (12)). This sharp transition between ‘in-cloud’ and ‘out-of-cloud’ droplet number concentration at grid scale is believed to be justified if one is not interested in monitoring subgrid effects, leading to the idea of fractional cloudiness.

(c) Coalescence

Coalescence is the most demanding process to include in a warm microphysical scheme because it is responsible for the first hydrometeor formation and also because the numerical treatment of the continuous drop-spectrum evolution (with the problem of mass conservation and of artificial spectral acceleration and broadening) requires careful attention (Tzivion et al. 1987). Furthermore, uncertainties in collection kernels (see Beard and Ochs (1984) for a review) and turbulence effects (Pinsky and Khain 1997) hamper a purely deterministic computation of the growing drop spectra.

The driving equation of coalescence rate is the following (quasi-) stochastic collection equation (SCE):

$$\frac{\partial n(x, t)}{\partial t} = \frac{1}{2} \int_0^x K(y, x-y)n(y, t)n(x-y, t) \, dy - n(x, t) \int_0^\infty K(y, x)n(y, t) \, dy,$$

written in continuous and dimensionless form (PK97). In Eq. (13), $K(x, y)$ is the collection kernel of a drop of size $x$ colliding with a drop of size $y$, and $n(x, t)$ is the mass size distribution of drops at time $t$. If we consider the droplet/drop spectrum as the composition of two generalized gamma functions $n_c$ and $n_r$, Eq. (13) can be rewritten as

$$\frac{\partial n(x, t)}{\partial t} = \text{SCE}(n_c) + \text{SCE}(n_r) + \int_0^x K(y, x-y)n_c(y, t)n_r(x-y, t) \, dy$$

$$- n_r(x, t) \int_0^\infty K(y, x)n_c(y, t) \, dy - n_c(x, t) \int_0^\infty K(y, x)n_r(y, t) \, dy$$

$$= \text{SCE}(n_c) + \text{SCE}(n_r) + \text{ACC}(n_r) + \text{ACC}(n_c),$$

where $\text{SCE}(n_i)$ is the SCE relative to $n_i$ ($i \in [c, r]$). The development of Eq. (14) was made possible because of the symmetry of the kernels $K(x, y)$. As first explained by
Z85, if the distribution functions \( n_c \) and \( n_r \) do not overlap too much, i.e. \( n(x) \approx n_c(x) \) for \( D \leq 82 \mu m \) and \( n(x) \approx n_r(x) \) for \( D \geq 82 \mu m \), then SCE\((n_c)\) corresponds to the cloud-droplet self-collection, SCE\((n_r)\) to the raindrop self-collection, the following two integrals in Eq. (14), grouped in ACC\((n_c)\), correspond to the accretion of cloud droplets by raindrops affecting the raindrop spectrum, while the last term, ACC\((n_c)\), is the loss of cloud droplets by accretion. This interpretation of Eq. (14) is possible if the mass distribution of total liquid water is clearly bimodal. In the absence of raindrops, Eq. (14) reduces to the SCE\((n_c)\) term, so it is necessary to parametrize the steady production of very few big droplets, a fraction of the SCE\((n_c)\) term, because it is the only way to initiate a raindrop spectrum. The most difficult problem in a semi-spectral framework is to conceive such an unambiguous partition starting from a single distribution \( n_c \) or, in other words, to represent the autoconversion process.

(i) *Autoconversion.* It is convenient to discriminate two regimes in the autoconversion parametrization. The first one identifies the ‘initiation stage’, when the raindrop mode is hardly distinguishable from the droplets, and the second one corresponds to the ‘feeding stage’, when raindrops are already present.

Autoconversion is an efficient collection process for generating a copious amount of mid-size drops (90 \( \mu m \leq D \leq 200 \mu m \)) resulting from collisions between the largest cloud droplets (see the representative values of \( r_H \) in Table 1 of BR74). This observation led Z85 to define a criterion for rain initiation based on the number concentration of these intermediate drops. On the other hand, when the raindrop mode exists, these intermediate drops are again numerous but now they are clearly smaller than, say, the mean-volume drop diameter and so they tend to exaggerate the value of raindrop concentration \( N_r \) without increasing the rain mixing ratio \( r_t \) much. This effect is illustrated in Fig. 1 where a bin model was used to integrate Eq. (13) numerically for specific initial conditions on \( n_c(r, t) \). The spectra generated by the aforementioned parametrization of Eq. (13) is shown for comparison. Results are presented in log–log scale for both mass and number concentration distributions. In Fig. 1(a), the mass spectra computed by the bin-size model is close to the one obtained by integrating the parametrization with two generalized gamma distribution laws. The maxima of the distribution are well recovered, but the gap between the droplet and the raindrop regimes is exaggerated on the spectrum produced by the parametrization. This effect shows up in Fig. 1(b) where the number concentration of the drops in the 50–200 \( \mu m \) range is not well captured by the two generalized gamma functions. Changing the set of dispersion parameters, \( v_r \) and \( a_r \) would be helpful in fitting the number concentration but detrimental for the mass distribution which fits nicely for the largest raindrops. In fact, the freshly autoconverted drops in this 50–200 \( \mu m \) mid-range are in a transient growing stage. This drop reservoir is rapidly depleted by the large precipitating raindrops through accretion. Moreover, it has been already stressed by Ferrier (1994) that predicting the number concentration is less crucial than preserving the basic spectral characteristics of the particle distributions. Therefore, it is important to ensure for this regime of mid-size drops that the autoconversion process does not hinder the development of the raindrop spectrum. This is why the parametrization of its effect on \( N_r \) must be considered specifically in this case.

The initiation of raindrop spectra has been parametrized by many authors (Cotton 1972; BR74; Manton and Cotton 1977; Beheng 1994). In the present scheme, only the BR74 formulation is considered because the estimated autoconversion rates are explicit functions of detailed local properties of cloud-droplet spectra. In fact the BR74 parametrization is built on the observation that a characteristic water content, \( L \), of small
drops develops steadily over a characteristic time-scale, $\tau$. These two positive quantities are expressed in the ranges $20 \mu m \leq D_c \leq 36 \mu m$ and $0 \leq \nu_c \leq 3$ by

$$L = 2.7 \times 10^{-2} \rho_a r_c \left( \frac{1}{16} \times 10^{20} \sigma_c^3 D_c - 0.4 \right),$$  

$$\tau = \frac{3.7}{\rho_a r_c} \left( 0.5 \times 10^6 \sigma_c - 7.5 \right)^{-1},$$

where, using Eq. (3),

$$D_c = \left\{ \frac{M_c(0)}{M_c(3)} \right\}^{1/3} = \frac{\lambda_c}{\sigma_c} \left\{ \frac{\Gamma(\nu_c + 3/\alpha_c)}{\Gamma(\nu_c)} \right\}^{1/3},$$  

$$\sigma_c = \left\{ \frac{M_c(6)}{M_c(3)} - \left( \frac{M_c(3)}{M_c(0)} \right)^2 \right\}^{1/6} = \frac{\lambda_c}{\sigma_c} \left\{ \frac{\Gamma(\nu_c + 6/\alpha_c)}{\Gamma(\nu_c)} - \left( \frac{\Gamma(\nu_c + 3/\alpha_c)}{\Gamma(\nu_c)} \right)^2 \right\}^{1/6}$$

are the mean-volume drop diameter and the standard deviation of the cloud-droplet size distributions, respectively. So, for given cloud conditions, a mean-mass autoconversion rate $L/\tau$, can be computed only if $\sigma_c > 15 \mu m$. Following BR74, the number concentration of the newly formed raindrops at time-scale $\tau$ is simply expressed by $N_r = 3.5 \times 10^9 L$. The following formula summarizes the autoconversion rates:

$$RCAUTR = - \max(L/\tau, 0),$$  

$$CCAUTR = -3.5 \times 10^9 \frac{\rho_a L}{\tau}.$$  

As noticed by many authors (Z85; Beheng 1994), this kind of parametrization includes all possible collection processes occurring simultaneously during the early stage of
autoconversion. This means that accretion and raindrop self-collection have to be excluded, or limited at least, to avoid double counting. This explains why it is important to wait for the emergence of a distinct hump in the rain spectrum before activating these processes. BR74 indicated that this feature is apparent after an estimated time of $1.2 \times \tau$. As the autoconversion process still produces rain at the same mean rate during this extra time, accretion and raindrop self-collection are turned off until the rain mixing ratio $r_t$ reaches $1.2 \times L$.

Once the raindrop spectrum is initiated, autoconversion switches into the feeding regime. As explained above, the number of drops that are produced by autoconversion in Eq. (18) is not representative for $N_r$ although the mass of these drops contributes to the development of the raindrop spectrum. Now looking at the tendency equation for $N_r$ with the moment formulation, it can be observed that at steady state and for a homogeneous column of precipitations the autoconversion rate CCAUTR is balanced by the self-collection rate CRSCOR. These terms are, however, of different amplitude ($\text{CCAUTR} \gg \text{CRSCOR}$) if CCAUTR is formulated by Eq. (18). In such a case, due to the presence of numerous drops, the mean-volume drop diameter $D_r$ will culminate around only some hundred micrometres. Note that it is not because autoconversion continuously provides drops to the rain spectra that bigger raindrops cannot form by self-collection. As a matter of fact, these two processes do not affect the raindrop number concentration in the same range of diameters: autoconversion acts on the drop concentration while self-collection is significant for large-size raindrops. So, keeping in mind that $r_t$ and $D_r$ are of greater interest than the number concentration $N_r$ for the production of precipitation, it has been assumed that in the feeding regime the mean-volume drop diameter is not modified by autoconversion. Keeping the same mass production rate Eq. (17), the number-concentration tendency is rewritten as

$$\text{CCAUTR} = \frac{N_r}{r_t} \text{RCAUTR}. \quad (19)$$

(ii) **Accretion and self-collection.** The processes of accretion and self-collection are integrated analytically in the present scheme. Z85 first showed that this was possible with a polynomial fitting of the collection kernels (Long’s 1974) and a gamma law for mass distributions. This function is equivalent to imposing $\alpha = 1$ in Eq. (1). In this case, the gain term of the SCE (first term on the right-hand side (r.h.s.) of Eq. (13)) can be interpreted as a convolution product between functions $F_1 = n(y, t)$ and $F_2 = K(x, y)n(y, t)$. So, using the convolution theorem

$$\mathcal{L}[F_1 * F_2] = \mathcal{L}[F_1] \times \mathcal{L}[F_2],$$

where $\mathcal{L}$ is the Laplace transform and $\ast$ denotes the convolution product, and the moment theorem

$$\mathcal{L}[x^n F] = (-1)^n d^n \mathcal{L}[F]/ds^n,$$

where $s$ is the variable associated with the $\mathcal{L}$ transform, the method of Z85 provides an elegant way of integrating the zeroth- and third-order moments of some partial terms in Eq. (14). In this work (see appendix A and the next discussion), it is shown that such mathematical manipulation is clearly unnecessary. Much more simple analytical integration of these moments can be established even for any value of $\alpha$, thus allowing a broader use of Eq. (1) in Eq. (13). It is worth stressing that Eq. (1) is not an exact solution of the SCE in the sense given by PK97, but the function described by Eq. (1) allows for an analytical evaluation of some useful moments in the r.h.s. of Eq. (13).
The collection kernels of Long’s (1974), already used by Z85, are considered in this study:

\[ K(D_1, D_2) = \begin{cases} 
K_2(D_1^5 + D_2^5), & \text{if } D_1 \leq 100 \text{ } \mu\text{m}, \\
K_1(D_1^3 + D_2^3), & \text{if } D_1 > 100 \text{ } \mu\text{m},
\end{cases} \]  

(20)

with \( K_2 = 2.59 \times 10^{15} \text{ m}^{-3}\text{s}^{-1} \) and \( K_1 = 3.03 \times 10^3 \text{ m}^{-3}\text{s}^{-1} \). PK97 reported that these approximate kernels are accurate enough if employed in place of more explicit ones, to get numerical solutions to Eq. (13). This is somewhat surprising because Long’s kernels were designed to apply to the whole water-drop spectrum, but with such a simple form that even analytical solutions are possible to SCE problems. A comparison, not shown here, between Eq. (20) and collection kernels compiled by Hall (1980) from several sources, reveals noticeable differences, however. So, while it can be concluded that the use of Eq. (20) is promising, a numerical evaluation of the analytical solution to Eq. (13) is clearly needed. Note that there are possibilities of more elaborate formulas (higher degree polynomials, combinations of polynomials and exponentials) for improving Eq. (20) and the accuracy of an SCE analytical integration, but this question regarding \( K(x, y) \) is beyond the scope of the present work.

In the case of accretion (RCACCR term in Eqs. (4b)–(4d)), it is sufficient to concentrate on the ACC\( (n_c) \) term of Eq. (14) which summarizes the accretion effect on the cloud-droplet spectrum \( n_c \). Due to the fact that the SCE (Eq. 13) is mass conserving (as are all the SCE terms in Eq. (14)), it is obvious that any mass change due to the ACC\( (n_c) \) term is exactly cancelled by a mass change due to the ACC\( (n_c) \) term. The advantage here is that the latter can be directly computed as shown in appendix B. Furthermore, it can be deduced from appendix A that the zeroth-order moment of ACC\( (n_c) \) is zero so the effect of accretion on \( n_c \) and \( n_t \) is also restricted to the computation of that ACC\( (n_c) \) moment to give the CCACCR term in Eq. (4a). In doing so, it is assumed that accretion produces raindrops exclusively, but never large droplets that could feed the trailing edge of the droplet spectrum. In summary, the accretion terms are computed with the following integral \( I \):

\[
I = C_n \int_0^{D_m} D^n \frac{d n_c(D)}{d t} \bigg|_{\text{ACC}} dD \\
\approx C_n \int_0^{D_m} D^n \text{ACC}(n_c(D)) dD \\
\approx C_n \int_0^{\infty} -D''_n n_c(D_1, t) \int_0^{\infty} K(D_1, D_2)n_t(D_2, t) dD_1 dD_2 \\
= -C_n 2I_{\text{ACC}},
\]

(21)

where \( C_n \) is 1 for \( n = 0 \) and equal to \( (\pi/6)(\rho_w/\rho_a) \) for \( n = 3 \). From appendix B and if \( D_r \geq 100 \text{ } \mu\text{m} \), one gets

\[
\text{CCACCR} = -K_1N_cN_t \left\{ \frac{\Gamma(v_c + 3/\alpha_c)}{\Gamma(v_c)\lambda_c^3} + \frac{\Gamma(v_r + 3/\alpha_r)}{\Gamma(v_r)\lambda_r^3} \right\},
\]

(22a)

\[
\text{RCACCR} = -\frac{\pi}{6} \frac{\rho_w}{\rho_a} K_1 \frac{N_cN_t}{\lambda_c^3} \left\{ \frac{\Gamma(v_c + 6/\alpha_c)}{\Gamma(v_c)\lambda_c^3} + \frac{\Gamma(v_c + 3/\alpha_c)}{\Gamma(v_c)} \frac{\Gamma(v_r + 3/\alpha_r)}{\Gamma(v_r)\lambda_r^3} \right\};
\]

(22b)
while, if \( D_r < 100 \, \mu m \),

\[
CCACCR = -K_2 N_c N_r \left\{ \frac{\Gamma(v_c + 6/\alpha_c)}{\Gamma(v_c) \lambda_c^6} + \frac{\Gamma(v_r + 6/\alpha_r)}{\Gamma(v_r) \lambda_r^6} \right\},
\]

(23a)

\[
RCACCR = -\pi \frac{\rho_w}{6} K_2 \frac{N_c N_r}{\lambda_c^3} \left\{ \frac{\Gamma(v_c + 9/\alpha_c)}{\Gamma(v_c) \lambda_c^6} + \frac{\Gamma(v_r + 3/\alpha_c)}{\Gamma(v_r) \lambda_r^6} \right\}.
\]

(23b)

Concerning the self-collections, it is easy to get an analytical solution to the SCE\((n_i)\) terms in Eq. (14). Reasoning as above, it is demonstrated in appendix A that the zeroth-order moment of the SCE gain integral (first term in the r.h.s. of Eq. (13)) is exactly half that of the loss integral. This property comes from the positivity of the \( n_i \) distribution functions. So applying results of appendix B to the trivial case of \( n_c = n_r (= n_i) \), one gets

\[
CiSCO_i = -K_1 N_i^2 \frac{\Gamma(v_i + 3/\alpha_i)}{\Gamma(v_i) \lambda_i^3}, \quad \text{if } D_i \geq 100 \, \mu m,
\]

(24)

\[
CiSCO_i = -K_2 N_i^2 \frac{\Gamma(v_i + 6/\alpha_i)}{\Gamma(v_i) \lambda_i^6}, \quad \text{if } D_i < 100 \, \mu m.
\]

(25)

The CCSCOC term in Eq. (4a) results from a direct use of Eq. (25) with \( i = c \).

(iii) Raindrop break-up. Collisional break-up is the only break-up effect that is considered in the following for its significant contribution to the slope parameter \( \lambda \) at the edge of raindrop distributions (Srivastava 1978). This process has been investigated in depth by Low and List (1982a,b) who determined three modes of drop disruption and parametrized them (Brown 1997). When put together, break-up and coalescence lead to a new governing SCE that is so complex that it is preferable to interpret the break-up process as a perturbation affecting the formulation of the raindrop self-collection term CRSCOR in Eqs. (24)–(25). Doing so, Z85 and Verlinde and Cotton (1993) suggested the introduction of a bulk collection efficiency \( E_c \) that reduces the overall coalescence efficiency in CRSCOR with

\[
E_c = \begin{cases} 
1, & \text{if } D_r < 600 \, \mu m, \\
\exp\{-2.5 \times 10^3(D_r - 6 \times 10^{-4})\}, & \text{if } 600 \, \mu m \leq D_r < 2000 \, \mu m, \\
0, & \text{if } D_r \geq 2000 \, \mu m.
\end{cases}
\]

(26)

Values of the cut-off diameters in Eq. (26) are subjected to specific tests; the complete scheme is discussed in Part II.

(d) Sedimentation and evaporation

Due to the fact that the terminal velocity of drops depends on their diameter, the gravitational sedimentation of hydrometeors is selective if one considers the whole range of raindrop spectra. This leads to an efficient size-sorting process which needs to be reproduced in the case of weak precipitations (Feingold et al. 1998), but also in the case of warm cumuli where some drops can be large enough to fall while smaller ones are delayed and maintained in updraught cores for further growth. This differential settling between drops is accounted for in two-moment schemes because both sedimentation
fluxes of $N_r$ and $r_r$ are computed:

$$RSEDR = \frac{1}{\rho_a} \frac{\partial}{\partial z} \int_0^{\infty} \frac{\pi}{6} \rho_w D^3 V(D) N(D) \, dD$$

$$= \frac{\partial}{\partial z} \left\{ \frac{\rho_{00}}{\rho_a} \Gamma(\nu_r + \{d + 3\}/\alpha_r) \right\}. \tag{27}$$

$$CSEDR = \frac{\partial}{\partial z} \int_0^{\infty} V(D) N(D) \, dD$$

$$= \frac{\partial}{\partial z} \left\{ \frac{\rho_{00}}{\rho_a} \Gamma(\nu_r + d/\alpha_r) \right\}. \tag{28}$$

In Eqs. (27)–(28), a simple power-law dependence in diameter including air-density effect (Foote and Du Toit 1969) has been assumed:

$$V(D) = \left( \frac{\rho_{00}}{\rho_a} \right)^{0.4} c D^d,$$ \tag{29}

but a more accurate formula such as the one employed by Ferrier (1994) could be used instead. This framework applies also to cloud droplets whose sedimentation in stratus clouds is comparable to transport by subsidence. A numerical test is presented in the next section to illustrate the ability of Eqs. (27)–(28) to capture the evolution of the mean drop diameter by sedimentation.

The evaporation rate of a raindrop population, falling in an undersaturated environment ($s_{v,w} \ll 0$), is obtained after performing an analytical integration over the whole drop-mass spectrum. According to PK97, the size evolution of a single evaporating drop of diameter $D$ is given by

$$\left. \frac{dD}{dt} \right|_{EVA} = \frac{4s_{v,w} \bar{f} G}{D}, \tag{30}$$

where the ventilation factor $\bar{f}$, an empirical function of the Reynolds number $Re = V D / \nu_{cin}$ of the flow, is given by

$$\bar{f} = 1 + F Re^{0.5} = 1 + 0.22 \left\{ \left( \frac{\rho_{00}}{\rho_a} \right)^{0.4} c D^{d+1} / \nu_{cin} \right\}^{0.5}, \tag{31}$$

after substituting from Eq. (29). Integrating Eq. (30) with Eq. (31) and Eq. (1) leads to an analytical expression of the evaporation rate: that is

$$\text{RREVAV}$$

$$= \frac{1}{\rho_a} \int_0^{\infty} \frac{\pi}{2} \rho_w D^2 \left. \frac{dD}{dt} \right|_{EVA} n_r(D) \, dD \tag{32}$$

$$= \frac{2\pi s_{v,w} N_r G \rho_w}{\lambda \Gamma(\nu_r)} \rho_a \left\{ \frac{\rho_{00}}{\rho_a} \right\}^{0.2} \left( \frac{c}{\nu_{cin}} \right)^{0.5} \left( \frac{\Gamma(\nu_r + \{d + 3\}/2\alpha_r)}{\lambda_r^{(d+1)/2}} \right).$$

Concentrations $N_r$ are not affected by evaporation in the present scheme. However, if the mean-volume drop diameter becomes smaller than 82 $\mu$m then all raindrops are converted into cloud droplets, i.e. $N_c = N_c + N_r, r_c = r_c + r_r$ and $N_r = r_r = 0.$
4. Validation Tests

A warm microphysical scheme has been developed in the preceding section with emphasis put on the analytical integration of most of the processes. However, as the scheme predicts only two moments of the generalized gamma distribution law, here assumed to fit the droplet and raindrop spectra separately, it is necessary to perform specific tests to verify that some critical processes (such as coalescence and sedimentation) are well enough described by such a bulk scheme. This section also gives an opportunity for a short study on the practical use of the activation scheme.

(a) Nucleation

The parametrization of cloud-droplet nucleation that is implemented in the present scheme relies on the credibility of the mathematical approach of Twomey (see PK97). To improve Twomey's method, CPB98 promoted the use of Eq. (7) to describe natural activation spectra more realistically. Then, using data generated by a model of aerosol growth and activation, CPB98 showed that a nonlinear adjustment can perform well to determine the four free coefficients in Eq. (7). This is illustrated in Fig. 2, reproduced from CPB98, where the excellent agreement found on the plots proves that the choice of Eq. (7) is a good approximation to either maritime or continental activation spectra originating from log-normal distributed aerosols. Moreover, and as already mentioned, recent work of Cohard et al. (2000) shows that additional interest can be taken in Eq. (7) because simple parametric relationships can be established to link the microphysical properties of the activated aerosols and the unknown coefficients of the activation spectra.
At this point, a potential problem that can be encountered for a practical use of the nucleation scheme is the way to extract an accurate root $s_{v_w,m}$ in Eq. (8). In order to avoid a complex root-finding algorithm, a simple successive-approximation method has been tested. It consists of computing the hypergeometric factor in Eq. (8) with the last value of the root obtained at the most recent iteration. This factor is set to unity, the first term of the hypergeometric series, at the first iteration. Figure 3 shows the curves of the CCN number concentration as a function of the vertical velocity for two types of activation spectra and for several numbers of iteration of Eq. (7). The thermodynamical condition for this test are $T = 283$ K and $P = 800$ hPa. As expected, the method converges and two iterations are sufficient for the continental-like activation spectrum, while about four are necessary in the maritime case because of the greater effect of the hypergeometric factor when approaching the asymptotic regime.

\[(b)\] Coalescence

It is necessary to perform a global test for all the coalescence processes including autoconversion, accretion and self-collection because their parametrization is of mixed character (with a BR74 parametrization coupled to analytical integrations) and also because of the strong nonlinearity of the processes. The effects of break-up are studied independently through dedicated simulations by integrating the whole scheme in Part II of this paper.

The strategy of Z85 has been adopted in the design of the experiments presented here. It consists of comparing the droplet/raindrop spectra evolution by the present formulation with those obtained by running a detailed bin-size model (here containing
Figure 4. Mass-concentration distributions: comparisons between parametrized (left column) and finite difference (right column) coalescence models. Initial conditions are $r_c = 10^{-1}$ kg kg$^{-1}$, $a_c = 3$; and (a) and (b): $D_c = 24$ $\mu$m, $v_c = 3$; (c) and (d): $D_c = 24$ $\mu$m, $v_c = 1$; (e) and (f): $D_c = 28$ $\mu$m, $v_c = 1$. Symbols are explained in appendix D.

75 bins). The test is rather severe because it implies a direct comparison of results produced by a system of four prognostic equations to a numerical solution of the discrete version of the SCE Eq. (13), which requires the integration of more than 50 equations (here 75) to reach an acceptable level of accuracy. The algorithm used to get this high-resolution reference solution is close to the scheme introduced by BR74 with a nonlinear discretization of diameters (doubling every six bins from a starting diameter of 1.56 $\mu$m). A time step of 1 s has been adopted. The code has been checked and cross-validated with the two-moment algorithm developed by Tzivion et al. (1987) which can reach an equivalent accuracy with fewer and coarser bin intervals.

Both models work with the kernel $K(x, y)$ defined in Eq. (20). Three selected cases labelled (a), (b) and (c) have been analysed; they differ by the initial droplet size distributions. These are given by specifying $D_c$ from Eq. (16a) and $v_c$ ($a_c = 3$), in order to study the different regimes of raindrop production. For instance cases (a) and (b)
Figure 5. Same as in Fig. 4, but for the number-concentration distributions.

differ by the $v_c$ parameter ($v_c = 1$ in case (a), but 3 in case (b)) with $D_c = 24 \, \mu m$, while cases (a) and (c) differ by the choice of $D_c$ ($D_c = 28 \, \mu m$ in case (c)). In every case, however, the initial single mode of cloud droplets contains 1 g kg$^{-1}$ of water and the values $\alpha = 1$ and $v_r = 2$ have been set to characterize the raindrop spectra following Ferrier (1994).

Figures 4 and 5 present the development with time of mass and number-concentration distributions over a 30 min period (one plot each 5 min of integration) including the initial conditions. We first focus on the autoconversion process, i.e. the early stage of drop formation prior to the emergence of the raindrop mode. In each case the BR74 parametrization works well because the timing for the raindrop mode development is good. Obviously, the parametrization contains the necessary ingredients for following the formation of the first raindrops as is seen in the contrasting evolution of the spectra for the three cases presented in Fig. 4. It is worthwhile to note that introducing an artificial broadening, by increasing $v_c$ from 1 to 3 when moving from case (a) to case (b),
is sufficient to accelerate dramatically the formation of a raindrop spectrum. Also visible in Fig. 5 is the steady build-up of the drop spectra in the reference case in contrast to the abrupt formation of the same very few drops in the parametrized cases. This has, however, no apparent consequences on the mass distributions shown in Fig. 4. So it can be considered that the parametrization is efficient because no significant amount of rain is produced in case (a) after 30 min of simulation. Furthermore, the appearance of a raindrop spectrum occurs simultaneously for the parametrized and finite difference models in Fig. 4 after 20 min and 12 min of simulation in cases (b) and (c), respectively.

The second point to examine is the displacement of the raindrop mode towards larger diameters, i.e. the effect of accretion and self-collection processes on the raindrop growth. In case (c), for which the development of the raindrop spectrum is faster, a substantial delay is noticeable in the evolution of the parametrized spectra just at the end of the simulation (left column in Fig. 4). This is due to the fact that the parametrization generates a spectrum which is much narrower compared to the reference one, so the accretion and the self-collection tendencies are underestimated in this case. A better agreement could be obtained by increasing the values of $\alpha_r$ and $\nu_r$; however, we have some doubts as to the suitability of such modification because bin models are also increasingly susceptible to numerical diffusion (leading to spectral broadening) as the raindrop spectrum develops over ever coarser bin intervals. Another feature also to keep in mind is that collision break-up is probably efficient enough to limit this effect in real situations.

As a conclusion for these tests, it is reasonable to state that the present parametrization of coalescence processes is satisfactory in the context of Long's kernels. More experiments are necessary to improve the comparison with more elaborated kernels such as the ones of Hall (1980).

(c) Sedimentation

One-dimensional experiments have been performed to investigate the accuracy of drop sedimentation by the two-moment scheme. This study is suggested by the recent work of Feingold et al. (1998) where an analysis of drizzling boundary layers was presented.

Here three conceptual schemes are tested under the same conditions. These are Eqs. (27)–(28) of the present scheme, a bin-size model (with the raindrop spectra discretized over 40 classes and a starting diameter of 50 $\mu$m) and the analytical solution of a Lagrangian scheme. The experiment consists of following at ground level the evolution of precipitations generated by a stationary cloud with a base height at $z = 1000$ m. The time step is 1 s. The cloud is stationary and contains raindrops with the following specifications: $r_t = 1$ g kg$^{-1}$, $D_t = 600$ $\mu$m, $v_r = 1$ and $\alpha_r = 2$. No rain is initially present in the range $0 < z < 1000$ m. The Lagrangian formulation is taken as the reference solution. It relies on the determination of the diameter of the smallest drop $D_{\text{min}}^\text{lag}(t) = (1000/ct)1/d$ that can reach the ground after a given time of integration $t$. Then computation of the time evolution of rainwater mixing ratio and number concentration at the ground is straightforward: it proceeds from integration of Eq. (1) and its third moment, between $D_{\text{min}}^\text{lag}(t)$ and infinity. These integrals are numerically evaluated using the incomplete gamma function algorithm taken from Press et al. (1992). Other variants have been tested in the bin model (numerical scheme, bin representative diameter) but without discernible effect on the results.

The temporal integrations are made over a period of time of 1200 s and results are reproduced in Fig. 6 for vertical grid spacing $\Delta z$ of 50 and 200 m. In Fig. 6,
Figure 6. Instantaneous precipitation for (a) and (b) the number concentration, $N_r$; (c) and (d) the mixing ratio, $r_r$; and (e) and (f) the mean-volume diameter $D_r$. Computed with a 50 m (left column) and 200 m (right column) vertical mesh for a Lagrangian scheme (solid lines) and a forward Eulerian scheme applied to a bin-size liquid-water representation (short-dashed lines) and to the two-moment scheme (long-dashed lines). Symbols are explained in appendix D.
the time evolution of \( N_r, r_r \) and \( D_r \) is reproduced with a curve corresponding to each sedimentation scheme. Note that the computation of \( D_r \) in the bin-size model and for the Lagrangian scheme assumes the validity of Eq. (1) and Eq. (16a) but only for comparison with the present parametrization.

In Fig. 6 it is noticeable first that the Eulerian schemes (present formulation and the bin-size model) anticipate slightly the arrival of the precipitation at the surface, which is more pronounced in the \( \Delta z = 200 \) m case. This is interpreted as an effect of the numerical diffusion in both schemes. At time \( t \sim 500 \) s in the two-moment scheme, \( r_r \) appears clearly behind the reference solution while the evolution of \( N_r \) at this time is acceptable. This defect is a manifestation of the major limitation of the scheme which implies a remapping of the \( r_r - N_r \) couple (given values to \( v_r \) and \( \alpha_r \) to fit Eq. (1) at any time \( t \) and level \( z \). Such a technique redistributes implicitly the freshly fallen raindrops over the whole spectrum, while the bin-size model preserves a spectral truncation in such a way that only drops with a large enough diameter can fill the bins of the raindrop spectrum at the ground. So this bulk effect tends to lower \( D_r \) and to under(over)estimate the \( r_r(N_r) \) sedimentation flux. After \( t \sim 1000 \) s, an asymptotic regime is reached by \( r_r \) at \( 10^{-3} \) kg kg\(^{-1}\) while it is clearly anticipated by \( N_r \) (\( \sim 9 \times 10^3 \) m\(^{-3}\)).

Finally, to summarize the main features of this rather demanding test, it is instructive to examine the \( D_r \) curves in Fig. 6. A steady decrease of \( D_r \) with time, consistent with size sorting, can be seen. The comparison between the various \( D_r \) curves shows that the accuracy of the two-moment scheme is fair, so hopefully the errors inherent in the treatment of the sedimentation will not unduly perturb the other microphysical processes affecting the raindrops.

5. Conclusion

This paper reviewed the treatment of the main warm microphysical processes in the context of a two-moment framework. A scheme of such a type is believed to be a pragmatic compromise between over-simplified bulk parametrizations of precipitations as proposed by Kessler (1969) and very detailed bin models that are computationally too cumbersome for practical use in 3D mesoscale models. Inclusion of a prognostic equation for the number concentration of raindrops provides a better insight into the growth of large drops (size sorting by sedimentation, self-collection and break-up equilibrium) which in turn can only improve the time evolution of the mixing ratios. Another advantage is that monitoring the number concentration of cloud droplets, either from CCN spectra (CPB98) or directly from aerosol characteristics (Cohard et al. 2000), offers a practical way to differentiate several aspects of natural cloud properties (radiative transfer, physico-chemistry and precipitation release).

The scheme that is presented and analysed in this work allows analytical solutions to a reasonably simplified version of some microphysical processes (mostly nucleation and coalescence). This has been done with an elaborate function, the generalized gamma distribution that enables fine tuning of droplet/drop spectral shape through the adjustment of two free-dispersion parameters. All the tendencies, except the autoconversion of the cloud droplets, are parametrized on the basis of continuous integrals that encompass the whole range of drop diameters. With this method, the treatment of autoconversion (here the widely used BR74 formulation was adopted) remains clearly the weakest link in the scheme because this process acts precisely in the diameter range where the fuzzy transition between droplets and drops is hardly compliant with a bimodal and spectrally wide (from zero to infinity) representation of the drops. Nevertheless, specific tests made
with the BR74 formula show that autoconversion could control the formation of the first raindrops in the parametrization.

The validation of the present scheme has been partially undertaken with independent consideration of some key processes. A full examination of the behaviour of the complete scheme, including feedbacks between processes, is given in Part II. Firstly, as the accuracy of the proposed model of CCN activation relies, for the most part, on the proper fit of the activation spectra Eq. (7), it has been shown (CPB98) that this could be achieved basically for log-normal distributed aerosols. Furthermore, the coding of the scheme which requires finding a physical root of a highly nonlinear equation (Eq. (8)) remains quite simple. Secondly, numerical experiments put to the test an integration of the coalescence terms in the same way as did Z85. Our results corroborate his and so demonstrate that the essential features of drop growth by coalescence can be reproduced for realistic distributions of drop mass and number concentrations. Finally, simple one-dimensional tests have been performed to investigate the integration of the present (Eulerian) sedimentation scheme, as compared to the same scheme but for a bin model and also by comparison with the exact solution given by the Lagrangian approach. This test reveals that the two-moment scheme is competitive at the beginning stage of precipitation but results deteriorate later owing to an underestimation of the precipitation rate.

In conclusion, in Part I we have demonstrated that a prototype of a two-moment warm microphysical bulk scheme can be designed by removing most of the empiricism inherent in earlier approaches. Moreover, the use of generalized gamma distributions in describing droplet and raindrop spectra enhances the versatility of the scheme and so offers additional opportunity for finer tunings. Part II will illustrate the overall quality of the scheme and its robustness in modelling some extreme precipitation cases.

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APPENDIX A

Zeroth-order moment $M_0$ of the SCE

The integration of Eq. (13) over $x$ leads to:

$$\int_0^\infty \frac{\partial n(x, t)}{\partial t} \, dx = M_0$$

$$= \int_0^\infty \left\{ \frac{1}{2} \int_0^x K(y, x - y)n(y, t)n(x - y, t) \, dy \right\} \, dx$$

$$- \int_0^\infty \left\{ n(x, t) \int_0^\infty K(y, x)n(y, t) \, dy \right\} \, dx. \quad (A.1)$$

With the distribution $1_{[0,x]}(y)$ such as

$$\int_0^\infty 1_{[0,x]}(y)F(x - y) \, dy = \int_0^x F(x - y) \, dy,$$
and using the identity
\[ I_{[0,x]}(y) = I_{[y,\infty]}(x), \]

(A.1) is transformed into
\[
\mathcal{M}_0 = \int_0^\infty \left\{ \frac{1}{2} \int_y^\infty K(x, y-x)n(x, t)n(y-x, t) \, dx \right\} \, dy
- \int_0^\infty \left\{ n(x, t) \int_0^\infty K(y, x)n(y, t) \, dy \right\} \, dx.
\]  
(A.2)

The last variable change \( u = y - x \) leads finally to
\[
\mathcal{M}_0 = \int_0^\infty \left\{ \frac{1}{2} \int_0^\infty K(y, u)n(y, t)n(u, t) \, dy \right\} \, du
- \int_0^\infty \left\{ n(x, t) \int_0^\infty K(y, x)n(y, t) \, dy \right\} \, dx
= -\frac{1}{2} \int_0^\infty \left\{ n(x, t) \int_0^\infty K(y, x)n(y, t) \, dy \right\} \, dx.
\]  
(A.3)

Equation (A.3) is a trivial mathematical transcription of the fact that for each coalescence event, one drop is formed but two are lost, so the gain integral (first term in the r.h.s. of Eq. (13)) is exactly half the loss integral (second term in the r.h.s. of Eq. (13)).

**Third-order moment \( \mathcal{M}_3 \) of the SCE**

The starting equation is
\[
\int_0^\infty x \frac{\partial n(x, t)}{\partial t} \, dx = \mathcal{M}_3
\]
\[
= \int_0^\infty \left\{ \frac{x}{2} \int_0^x K(y, x-y)n(y, t)n(x-y, t) \, dy \right\} \, dx
- \int_0^\infty \left\{ xn(x, t) \int_0^\infty K(y, x)n(y, t) \, dy \right\} \, dx
= \int_0^\infty \left\{ \frac{1}{2} \int_0^x y K(y, x-y)n(y, t)n(x-y, t) \, dy \right\} \, dx
- \int_0^\infty \left\{ xn(x, t) \int_0^\infty K(y, x)n(y, t) \, dy \right\} \, dx.
\]  
(A.4)

With the same technique as above, Eq. (A.4) is transformed into
\[
\mathcal{M}_3 = \int_0^\infty \left\{ \frac{1}{2} \int_y^\infty y K(x, y-x)n(x, t)n(y-x, t) \, dx \right\} \, dy
- \int_0^\infty \left\{ xn(x, t) \int_0^\infty K(y, x)n(y, t) \, dy \right\} \, dx.
\]  
(A.5)

Performing the variable change \( u = y - x \) leads to
\[
\mathcal{M}_3 = \int_0^\infty \left\{ \frac{1}{2} \int_0^\infty (u+x)K(x, u)n(x, t)n(u, t) \, dx \right\} \, du
- \int_0^\infty \left\{ \int_0^\infty x n(x, t)K(y, x)n(y, t) \, dy \right\} \, dx
= 0.
\]  
(A.6)
Equation (A.6) expresses the physically intuitive fact that coalescence is a mass conservative process.

APPENDIX B

Analytical calculation of collection tendencies

For the accretion process, only the last term of Eq. (14) is calculated. Based on Eq. (21), the tendency of the n-moment can be developed as:

$$I_{\text{ACC}} = \int_0^\infty D_i^n n_c(D_1, t) \int_0^\infty K_i(D_1^{3i} + D_2^{3i}) n_r(D_2, t) \, dD_2 \, dD_1,$$  \hspace{1cm} (B.1)

where the index $i$ is 1 or 2 according to the collection kernels in Eq. (20). In Eq. (B.1), the $D_2$-integral is the sum of two moments $M_r(0)$ and $M_r(3i)$ of the gamma function $n_r$ (Eq. 3). The result of the integration

$$I_{\text{ACC}} = \int_0^\infty D_i^n n_c(D_1, t) K_i N_r[D_1^{3i} M_r(0) + M_r(3i)] \, dD_1$$  \hspace{1cm} (B.2)

is once again an $n$-order and $n + 3i$-order moment of the generalized gamma function in $n_c$. The final integration leads to:

$$I_{\text{ACC}} = K_i N_r N_c[M_r(0) M_c(n + 3i) + M_r(3i) M_c(n)].$$  \hspace{1cm} (B.3)

APPENDIX C

Non-iterative adjustment at water saturation

Equation (11) is solved by seeking for a zero-crossing solution of

$$F(T) = T - T^* + \frac{L_v(T^*)}{C_{ph}} (r_{vs}(T) - r_{vs}^*),$$  \hspace{1cm} (C.1)

where $T^*$ and $r_{vs}^*$ result from integration of all the processes, the condensation/evaporation of the cloud droplets excepted. The procedure used for solving Eq. (C.1) follows Langlois (1973) with a quasi-second-order expansion of $F(T) = 0$ around $T^*$, namely

$$T \simeq T^* - \frac{F(T^*)}{F'(T^*)} \left\{ 1 + \frac{1}{2} \frac{F(T^*)}{F'(T^*)} \right\}.$$  \hspace{1cm} (C.2)

Given the definition of the saturated water-vapour mixing ratio, $r_{vs}(T)$

$$r_{vs}(T) = \frac{e_{vs}}{P - e_{vs}},$$  \hspace{1cm} (C.3)

where $e_{vs}$ is the saturation vapour pressure over water

$$e_{vs}(T) = \exp(\alpha_v - \beta_v / T - \gamma_v \ln(T))$$  \hspace{1cm} (C.4)

and

$$\alpha_v = \ln(e_{vs}(T_{00})) + \beta_v / T_{00} - \gamma_v \ln(T_{00}),$$  \hspace{1cm} (C.5a)

$$\beta_v = \frac{L_v(T_{00})}{R_v},$$  \hspace{1cm} (C.5b)

$$\gamma_v = \frac{C_{vv} - C_{pv}}{R_v},$$  \hspace{1cm} (C.5c)
leads to a useful expression for the derivatives \( r'_{\nu s} (T^*) \) and \( r''_{\nu s} (T^*) \):

\[
\begin{align*}
  r'_{\nu s} (T^*) &= A_w(T^*) r_{\nu s} (T^*) \left\{ 1 + \frac{r_{\nu s} (T^*)}{\varepsilon} \right\}, \quad (C.6a) \\
  r''_{\nu s} (T^*) &= r'_{\nu s} (T^*) \left\{ \frac{A'_w (T^*)}{A_w (T^*)} + A_w (T^*) \left( 1 + 2 \frac{r_{\nu s} (T^*)}{\varepsilon} \right) \right\}, \quad (C.6b)
\end{align*}
\]

with definitions

\[
A_w (T) = \frac{\beta_w}{T^2} - \frac{\gamma_w}{T} \quad \text{and} \quad A'_w (T) = -\frac{2 \beta_w}{T^3} + \frac{\gamma_w}{T^2}. \quad (C.7)
\]

Combining Eqs. (C.3)–(C.7) helps to rewrite Eq. (C.1) in the more convenient form:

\[
T = T^* - \Delta_1 (1 + \frac{1}{2} \Delta_1 \Delta_2), \quad (C.8)
\]

with

\[
\begin{align*}
  \Delta_1 &= \frac{F(T^*)}{F'(T^*)} = \frac{L_v (T^*)}{C_{ph} + L_v (T^*) r'_{\nu s} (T^*)} (r_{\nu s} (T^*) - r^*_{\nu s}), \quad (C.9a) \\
  \Delta_2 &= \frac{F''(T^*)}{F'(T^*)} = \frac{L_v (T^*)}{C_{ph} + L_v (T^*) r'_{\nu s} (T^*)} r''_{\nu s} (T^*). \quad (C.9b)
\end{align*}
\]

**APPENDIX D**

**List of symbols**

- \( A \): \( 4 \sigma_w / a R_v T_p \),
- \( B(a, b) \): Beta function,
- \( c \) and \( d \): Parameters of the fall-speed/diameter relationship for the water drops,
- \( C \): Activation spectrum coefficient,
- \( c_p \): Specific heat at constant pressure,
- \( C_{vv} \): Heat capacity at constant volume of water vapour,
- \( C_{pd}, C_{pv} \) and \( C_w \): Heat capacity at constant pressure of dry air, water vapour and liquid water,
- \( C_{ph} \): Heat capacity at constant pressure of cloudy air: \( C_{pd} + r_v C_{pv} + (r_c + r_r) C_w \),
- \( D, D_1 \) and \( D_2 \): Drop diameters,
- \( D_c, D_r \): Mean-volume drop diameter for cloud-droplet and raindrop distributions,
- \( D_{crit} \): Critical diameter of newly nucleated droplets,
- \( D_m \): Diameter of separation between the cloud-droplet mode and the raindrop mode (here 32 \( \mu \)m),
- \( D_{lag} \): Minimum diameter in the Lagrangian scheme of sedimentation.
$D_v$  Diffusivity of water vapour in the air
$e_v$  Water vapour pressure
$e_{vs}$  Saturation vapour pressure over water
$E_c$  Collection efficiency
$f$  Ventilation factor
$F$  Ventilation coefficient
$F(a, b, c; x)$  Hypergeometric function
$g$  Acceleration due to gravity
$G(D, T, P)\quad \frac{1}{\rho_v} \left\{ \frac{R_v T}{e_{vs}(T) D_v} + \frac{L_v(T)}{k_a T} \left( \frac{L_v(T)}{R_v T} - 1 \right) \right\}^{-1}$
i  Cloud droplet (c) or raindrop (r)
k  Activation spectrum coefficient
$k_a$  Heat conductivity of air
$K(x, y)$ and $K(D_1, D_2)$  Collection kernels
$K_1$ and $K_2$  Long's collection kernel coefficients
$L$  Autoconversion water mass
$L_v$  Latent heat of vaporization
$M_i(p)$  $p$-moment of the $i$-drop size distribution
$max(a, b)$  Function returning the maximum value between $a$ and $b$
n, $n_c$ and $n_r$  Total, cloud-droplet and raindrop size distributions
$N_0$  Intercept parameter of an exponential distribution law
$N_c, N_r$  Cloud-droplet and raindrop number concentration
$N_h, N_a$  Condensation nuclei and activated CCN number concentration
$N_{CCN}$  Total activated CCN number concentration
$P$ and $P_{00}$  Pressure and reference pressure (1000 hPa)
r$_v$, r$_c$ and r$_r$  Water-vapour, cloud-water and rainwater mixing ratios
$r_{vs}$  Saturated-vapour mixing ratio
$R_d$ and $R_v$  Gas constant for dry air and water vapour
$Re$  Reynolds number
$s_{v, w}$  Supersaturation ($= e_v/e_{vs} - 1$)
$s_{v, w_{max}}$  Maximum supersaturation
t  Time
$T$ and $T_{00}$  Temperature and reference temperature (273.16 K)
$V(D)$  Fall-speed of drop with diameter $D$
w  Updraught velocity
$x$ and $y$  Drop mass
z  Height or vertical coordinate
$\alpha_c, \alpha_r$  Dispersion parameter of the generalized gamma distribution law for the cloud droplets and the raindrops
$\beta$  Activation spectrum coefficient
$\delta t$  Time step
$\Gamma(a)$  Complete gamma function
$\epsilon$  $R_v/R_d$
$\theta$  Potential temperature
$\lambda_c, \lambda_r$  Slope parameter of the generalized gamma distribution law for the cloud droplets and the raindrops
Activation spectrum coefficient
Dispersion parameter of the generalized gamma distribution law for the cloud droplets and the raindrops
Kinematic viscosity of air
\( (P / P_0) R_d / C_p \)
Air and liquid-water densities
Air density at \( P = P_0 \) and \( T = T_0 \)
Standard deviation of cloud-droplet distribution
Surface tension of water over the air
Time-scale for autoconversion

\[
\begin{align*}
\psi_1(T, P) &= \frac{g}{TR_d} \left( \frac{L_v}{c_p T} - 1 \right) \\
\psi_2(T, P) &= \left( \frac{P}{\epsilon \nu s(T)} + \frac{\epsilon L_v^2}{R_d T^2 c_p} \right)
\end{align*}
\]


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