Accuracy of reduced-grid calculations

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SUMMARY

The accuracy of calculations on reduced grids is examined from a climate modelling perspective. It is demonstrated that an adiabatic Eulerian spectral-transform model run on an $\ell$-digit grid is accurate to $\ell$ digits for short integrations. The error introduced by the reduced grid grows as expected for a turbulent atmosphere, and the growth is not accelerated by the reduced grid. The errors from an adiabatic semi-Lagrangian spectral-transform model can be significantly larger than those from an Eulerian model because the interpolation required in the semi-Lagrangian method will not maintain more than a few digits of accuracy. Adiabatic model errors are put in perspective by illustrating errors introduced by arbitrary aspects of model specification such as the longitude of the first grid point. For the Eulerian adiabatic model there appears to be no justification in using a higher-digit grid than the 2-digit one. A higher-digit grid may be desirable for an adiabatic semi-Lagrangian model. Results from a cosine-bell advection test are presented. All reduced grids tested have the same error for Eulerian spectral-transform advection. The semi-Lagrangian advection shows some increase in error with greater reduction in the grid, but the variation is comparable with the difference in error from choosing different interpolants or different time steps. There is little reason to choose a higher-digit grid than the 4-digit grid. In fact the 3-, 2- and 1-digit grids might be considered satisfactory for semi-Lagrangian advection. Finally, multiple-year climate simulations with full and reduced grids are presented. The overall impression is that the mean climates produced by the models run on the reduced grids are all very similar. There is no indication that the reduced grids introduce pathological errors that contaminate the simulations. A few select fields which are particularly sensitive to model changes are shown to quantify the differences, and to indicate that the differences are comparable with those which arise from the natural variability of the model. The results indicate that even a 1-digit grid is suitable for climate modelling with both Eulerian and semi-Lagrangian spectral-transform models.

KEYWORDS: Eulerian models Gaussian grid $\ell$-digit grids Numerical errors Semi-Lagrangian models Spectral models Triangular truncation

1. INTRODUCTION

When global atmospheric models were first developed it was recognized that the straightforward choice of a uniform latitude–longitude grid meant points would cluster near the poles as the meridians converge. This leads to a very non-isotropic grid with wasteful resolution near the poles. In addition, because of the small longitudinal distance between grid points near the poles, the Courant–Friedrichs–Lewy stability condition imposes a very short time step on explicit schemes. To address these problems, Gates and Riegel (1962) introduced a grid in which the longitudinal grid increment (in degrees) increased at higher latitudes. In their grid, the longitudinal increment was always an integer multiple of the increment at the equator, and the longitudinal distance between grid points changed by up to a factor of 2 over the sphere. Kurihara (1965) developed a grid of nearly homogeneous density. Similar to the grid of Gates and Riegel, the longitudinal increment increased with increasing latitude, but it was not restricted to be an integer multiple of the equatorial increment. Neither approach was extremely successful for finite-difference approximations because the longer longitudinal increment introduced large truncation errors near the poles in gradient terms involving spherical coordinate vector components (Williamson and Browning 1973). Therefore, subsequent grid-point models were based on uniform latitude–longitude grids with a longitudinal filter near

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the poles to eliminate, or at least strongly damp, short waves which otherwise would impose a very short time step on the integration.

By the late 1970s as spectral-transform models were becoming generally accepted, it was again recognized that latitude–longitude Gaussian transform grids are highly non-isotropic due to convergence of meridians towards the poles, and inconsistent with triangular spectral truncation which is isotropic. Based on experiments with a 9-level spectral model using triangular truncation, Machenhauer (1979) concluded that a certain reduction of points along latitude circles could be made in middle to higher latitudes without significantly changing the integration results. Jarraud and Simmons (1983) stated that for large longitudinal wave number $m$ the associated Legendre functions $P_n^m$ “become vanishingly small as the poles are approached, and the contributions to spectral tendencies from polar regions become less than unavoidable round-off error for sufficiently large zonal wave numbers”. They simulated this effect by setting to zero the values of $P_n^m$ that fell below a given threshold, but performed the calculation on the full Gaussian grid.

Hortal and Simmons (1991) created a quasi-homogeneous grid for their spectral-transform model based on geometric arguments. At each latitude the longitudinal grid length was as long as possible without exceeding the grid length nearest the equator. They coined the name reduced grid for such grids. Their analysis of a series of 10-day T106 and 90-day T42 integrations revealed no evidence of an overall lowering of forecast accuracy in terms of skill scores, although they point out that their sample size was too small to be statistically significant. They also noted that mass and energy were conserved to the same degree with the reduced grid as with the full grid. The main difference between integrations on reduced and full grids was traced to subgrid-scale orographic variance which can be significantly larger at high latitudes with reduced grids. Envelope orography as defined from the grid-box surface geopotential mean and variance for their model is higher with the reduced grid and may be detrimental. In addition, they described a situation in which the larger subgrid-scale orographic variance in the reduced grid influenced a particular cut-off low through the parametrization of gravity-wave drag.

Practical experience led them to make the grid slightly finer at the poles to eliminate grid-scale noise in the polar region (Hortal 1991; Simmons 1991). Courtier and Naughton (1994) related that noise to the asymptotic properties of the associated Legendre functions, and showed that it could be eliminated by slightly increasing the number of points in the longitudinal direction near the pole, in agreement with the practical experiences of Hortal and Simmons.

Courtier and Naughton (1994) developed grids based on properties of the associated Legendre functions, following the arguments of Jarraud and Simmons (1983). They defined grids with a specified accuracy $\epsilon$, where $\epsilon$ is the error introduced by neglecting selected Legendre functions approaching the poles. Based on experimentation at Météo-France with an aqua-planet, they proposed the use of a grid with $\epsilon = 10^{-4}$, but speculated that a threshold of $10^{-2}$ might be a better compromise of cost versus performance. In the next section we define an ‘$\ell$-digit’ grid which is similar to their grid with $\epsilon = 10^{-\ell}$. Their work was with relatively high resolution T106 and T213 forecast models.

More recently, Naughton et al. (1996) determined the representation errors for a Gaussian bell centred at different locations and for different reduced grids. They studied a feature of size around 1000 km combined with low (T50 for climate) and high (T100 for forecast) resolutions. The representation error does increase approaching the poles, but they concluded that the variation is probably too small to be important compared with other sources of error.
In this paper we reconsider the accuracy of reduced grids from a climate-modelling perspective. We first demonstrate that an \( \ell \)-digit grid is in fact accurate to \( \ell \) digits for an adiabatic Eulerian spectral-transform model by comparing short reduced-grid integrations to a full-grid control integration. The error introduced by the reduced grid grows as expected for a turbulent atmosphere. The growth is not accelerated by the use of a reduced grid. These adiabatic Eulerian integrations also show that there are reduced grids for which the errors are less than round-off accumulation.

We repeat these experiments with an adiabatic semi-Lagrangian spectral-transform model. Here the error can be significantly larger because the commonly used cubic interpolation only maintains a few digits of accuracy if the sampled data points are changed, even though the underlying field is identical in spectral space. For the adiabatic models, we put the errors in perspective by illustrating errors introduced by arbitrary aspects of model design which provide a lower bound on significant errors. One such aspect is the longitude of the first grid point.

We next present results from a cosine-bell advection test to expose errors accumulated over time. In this case all the reduced grids have the same error for Eulerian spectral-transform advection. The semi-Lagrangian advection shows some increase in error with greater reduction in the grid, but the variation is comparable with the difference in error from choosing different interpolants or different time steps and is therefore not significant.

Finally, multiple-year climate simulations with full and reduced grids are presented. The simulations are compared to identify any systematic accumulation of errors.

2. Definition of \( \ell \)-Digit Reduced Grids

The full Gaussian grid is defined as the set of points \((\lambda_i, \varphi_j)\) where \(\lambda\) is longitude and \(\varphi\) is latitude, and \(i = 1, \ldots, I\) for \(\lambda_i\) from 0 to \((2\pi - \Delta\lambda)\) and \(j = \pm 1, \ldots, \pm J\) for \(\varphi_j\) from the first rows off the equator to the poles. For a triangular truncation, data on the grid \(q(\lambda_i, \varphi_j)\) are determined from spectral coefficients \(q_n^m\) by

\[
q(\lambda_i, \varphi_j) = \sum_{n=0}^{N} \sum_{m=-n}^{n} q_n^m P_n^m(\mu_j) e^{im\lambda_i},
\]

(2.1)

where \(i = \sqrt{I}\) and \(P_n^m\) are associated Legendre functions, \(N\) is the truncation limit, and \(\mu_j = \sin \varphi_j\). The coefficients are determined from discrete data on the Gaussian grid by

\[
q_n^m = \sum_{j=\pm 1}^{\pm J} q^m(\mu_j) P_n^m(\mu_j) w_j,
\]

(2.2)

where \(w_j\) are the Gaussian weights and \(q^m(\mu_j)\) is the Fourier coefficient of wave number \(m\) at latitude \(\varphi_j\),

\[
q^m(\mu_j) = \frac{1}{I} \sum_{i=1}^{I} q(\lambda_i, \mu_j) e^{-im\lambda_i}.
\]

(2.3)

The relationship between the number of grid points and spectral coefficients for triangular truncation and a quadratically unaliased grid is

\[
J \geq (3N + 1)/2
\]

(2.4)
and

\[ I \geq (3M + 1) \]  

(2.5)

with \( M = N \).

Following Courtier and Naughton (1994) we define a reduced grid based on values of the associated Legendre functions and the error made by ignoring their contribution to the spectral transform when they are sufficiently small. As stated earlier, for large \( m \) the associated Legendre functions \( P_n^m \) become small approaching the pole \( (\mu_j \to 1) \). Thus their contribution to either (2.1) or (2.2) becomes small for large \( m \) and \( |j| \). This assumes that the other components of (2.1) and (2.2) do not become large at the same time. In fact they do not become large. The Fourier coefficients \( q^m(\mu_j) \) of atmospheric fields such as temperature and wind usually decrease with increasing \( m \), or at least do not increase. Atmospheric fields themselves represented on the grid \( q(\lambda, \mu_j) \) usually vary reasonably uniformly. An exception is water vapour which decreases in polar regions. In addition, the Gaussian weights \( w_j \) in (2.2) decrease approaching the poles.

To define a reduced grid we first specify the desired accuracy. Then for each Fourier wave number \( m \) we determine the latitude poleward of which the relative magnitude of the associated Legendre functions falls below that accuracy for all \( n \). For each \( m \) define \( \mu_s(m) \) to be the latitude closest to the equator such that

\[ \max_{\mu_j \geq \mu_s} |P_n^m(\mu_j)| \leq 10^{-\epsilon} \max_{\mu_j < \mu_s} |P_n^m(\mu_j)| \]  

(2.6)

for all \( n \). Thus if \( P_n^m \) is dropped from the sum (2.2) poleward of \( \mu_s(m) \), the error introduced will be less than \( 10^{-\epsilon} \). Since \( P_n^m \) is dropped, the Fourier coefficient \( q^m(\mu_j) \) is not needed for that \( m \) poleward of \( \mu_s(m) \) for the calculation of \( q_n^m \). Defining \( M_j \), the largest \( m \) retained at latitude \( \mu_j \), to be the minimum \( m \) such that \( \mu_s(m_j) \leq \mu_s(m_j) \) for \( m_j > M_j \), the Fourier transform (2.3) for the reduced grid becomes

\[ q^m(\mu_j) = \frac{1}{I_j} \sum_{i=1}^{I_j} q(\lambda_i, \mu_j) e^{-im\lambda_i} \]  

(2.7)

for \( |m| = 0, \ldots, M_j \). To retain unaliased quadratic terms we choose the number of points in longitude to be the smallest integer satisfying

\[ I_j \geq (3M_j + 1), \]  

(2.8)

which is suitable for the fast Fourier transform (FFT). In our case the FFT requires factors of only 2, 3 and 5 with at least one factor of 2. From (2.2) the coefficients need only be summed to the last latitude \( (J_m) \) containing Fourier wave number \( m \);

\[ q_n^m = \sum_{j=-J_m}^{J_m} q^m(\mu_j) P_n^m(\mu_j) w_j. \]  

(2.9)

We note that for small \( m \), \( J_m = J \) and all the original Gaussian latitudes must be retained in the reduced grid.

To calculate values in physical space on the reduced grid, first the Fourier coefficients are calculated at each latitude

\[ q^m(\mu_j) = \sum_{n=1}^{N} q_n^m P_n^m(\mu_j) \]  

(2.10)
for \(|m| = 0, \ldots, M_j\). Then the inverse Fourier transform is performed and

\[
q(\lambda_i, \varphi_j) = \sum_{m=-M_j}^{M_j} q^m(\mu_j) e^{im\lambda_i}
\]

for \(i = 1, \ldots, I_j\).

We refer to the grid and truncation as defined above as an unaliased \(\ell\)-digit grid. It is very similar to the \(\varepsilon = 10^{-\ell}\) grid of Courtier and Naughton (1994). The difference is that they use \(\varepsilon\) on the right side of our (2.6) without the \(P_{n}^{m}\) factors.

Figure 1 presents the properties of the reduced grids for \(\ell = 1\) to \(\ell = 12\). Figure 1(a) shows the ratio of longitudinal grid length at each latitude to the grid length at the first row of points poleward of the equator. All grids are finer than the geometric reduced grid, and the polar grid interval of the 1-digit grid is about half the equatorial interval.

Figure 1(b) shows the ratio of the total number of grid points in the reduced grids to the number of points in the full grid, and thus the potential computational savings from using the reduced grids. The 1-digit grid has 28% fewer points while the 4-digit grid has 20% fewer.

### 3. Adiabatic General-Circulation Model Integrations

In this section we determine the error introduced by the reduced grid in 2-day integrations of Eulerian and semi-Lagrangian adiabatic baroclinic models. The error is defined as the difference between integrations on a reduced grid and on the full grid, both starting from the same initial conditions. The models are based on the National Center for Atmospheric Research (NCAR) Community Climate Model (CCM3). The Eulerian version is described by Kiehl et al. (1996) and the semi-Lagrangian version by Williamson and Olsson (1994). All physical parametrizations are eliminated in the adiabatic versions except horizontal diffusion, which we consider part of the dynamics.
The partial correction of the horizontal diffusion to pressure surfaces is excluded, however, as is the frictional heating from the momentum diffusion. The adiabatic model includes the normal surface orography. The integrations are with 18 vertical levels and T42 spectral truncation. The full Gaussian grid has 128 longitude by 64 latitude points, the standard configuration used for CCM3 climate simulations. The time step for all integrations is 20 minutes. An adiabatic initial state was created by integrating the adiabatic model for three days from a state taken from a long simulation with the complete diabatic general-circulation model (GCM) on the full grid. The reduced-grid initial conditions were produced from the full-grid initial conditions by transforming to Fourier series in longitude at each latitude, truncating the series to that appropriate for the particular reduced grid at that latitude (2.8), and transforming the truncated series back to physical space on the reduced grid.

Figure 2 shows the root mean square (r.m.s.) and maximum absolute temperature differences between integrations with the adiabatic Eulerian model on the full grid and on unaliasied 1- to 8-, 10- and 12-digit reduced grids. The error is calculated on the full grid after expanding the reduced-grid simulations to the full grid using Fourier transforms at each latitude, the inverse of the procedure described above is used to create the reduced-grid initial conditions. The maximum absolute error is just the maximum absolute grid-point difference over all grid points at all levels. The r.m.s. error is defined as

\[
RMS(T) = \left( \frac{\sum_i \sum_j \sum_k (T(i, j, k) - T_r(i, j, k))^2 \Delta p(i, j, k) \omega_j}{\sum_i \sum_j \sum_k \Delta p(i, j, k) \omega_j} \right)^{1/2},
\]

where \((i, j, k)\) denotes the grid point \((\lambda_i, \varphi_j, \eta_k)\), \(\eta\) is the vertical coordinate, and \(k\) denotes the discrete vertical grid level. The sums are taken over all full model Gaussian grid points in longitude, latitude, and height; \(T\) and \(T_r\) are temperatures from the fullgrid and reduced-grid simulations, respectively; \(\Delta p\) is the average of the local vertical pressure difference between model levels from the two cases at the grid point:

\[
\Delta p = ((\Delta p)_r + (\Delta p)_f)/2.
\]
The two values \((\Delta p)_f\) and \((\Delta p)_r\) are not necessarily the same, given the hybrid \((\sigma - p)\) (where \(\sigma\) is pressure divided by surface pressure) vertical coordinate adopted in CCM3. Considering the full grid as providing the reference solution, we refer to the r.m.s. and maximum differences as the 'errors' introduced by the reduced grid.

Figure 2 shows that the r.m.s. error increases by one order of magnitude for each digit decrease in the grid for \(\ell < 10\). The 10- and 12-digit grids, however, have the same error. The 10- and larger-digit grids provide examples where the contributions to spectral tendencies of large \(m\) from polar regions are indeed smaller than computational round-off errors. Figure 2 of Rosinski and Williamson (1997) shows the 2-day growth of rounding errors in the adiabatic CCM2, which has essentially the same numerical approximations as CCM3 for the adiabatic dynamics. The initial perturbation was the smallest recognizable by the computer given the 64-bit Institute of Electrical and Electronics Engineers representation that was used. The error grows from \(10^{-14}\) to \(10^{-12}\) in 2 to 3 hours, then grows more slowly as expected with a turbulent flow. Growth during the first 3 hours in Fig. 2 of Rosinski and Williamson is just the accumulation of round-off error. In Fig. 2 here, the 12-digit grid error grows rapidly from \(10^{-14}\) to \(10^{-12}\) also during the first few hours, again the accumulation of round-off error, after which it grows much more slowly. The 10-digit grid error is essentially the same as the 12-digit grid error after the first few hours. Thus the errors from these grids are dominated by the accumulation of rounding errors, and additional errors introduced by the reduced grid do not increase the overall accumulation.

The r.m.s. errors in Fig. 2 are less than expected from the definition of the grids because the errors are introduced only where the grid is reduced, rather than throughout the domain. In addition the error is not introduced at \(10^{-\ell}\) everywhere since some neglected functions can be substantially less than \(10^{-\ell}\). The maximum error also shown in Fig. 2 is more consistent with that expected from the definition of the grids since a uniform 300 K is removed from the temperature field before the spectral transform.

Figure 3 shows the r.m.s. and maximum absolute errors for the adiabatic semi-Lagrangian model on the reduced grids. In this case, the control integration is from the semi-Lagrangian model on the full grid, with the adiabatic initial conditions created as described above for the Eulerian model, but using the adiabatic semi-Lagrangian model for the 3-day integration. Unlike the Eulerian case, the r.m.s. errors from the different grids span just two orders of magnitude. The error with the 1-digit grid is a little larger than that for the Eulerian model on the same grid. The larger errors with the adiabatic semi-Lagrangian model are not surprising since they are introduced by the interpolations in the semi-Lagrangian algorithm. Any change in grid-point location will introduce a difference in the interpolations even though the spectral representation of the field being interpolated is the same.

The pattern shown in Fig. 2 for the Eulerian model with the error decreasing an order of magnitude with each increasing digit of accuracy in the grid definition is reassuring but of no practical significance. For example, Fig. 4 shows the errors introduced by shifting the grid in longitude relative to the initial conditions and surface orography. The integrations use the adiabatic Eulerian and semi-Lagrangian models and the adiabatic initial conditions described above. Each error curve involves a single model, a single type of grid, and the same spectral representation of the initial conditions. Only the relative grid-point locations with respect to the phase of the initial conditions differ, i.e. the longitude of the first grid point. One member of each full-grid model pair has its initial conditions and mountains arbitrarily shifted by \(0.25\Delta\alpha\) relative to the other via Fourier analysis. The solution is then shifted back \(-0.25\Delta\alpha\) after the integration to calculate the difference. Thus, when compared, the two solutions have the same phase.
and the error is introduced only because the two forecasts are calculated on different sets of grid points which are shifted relative to each other. The reduced-grid integrations use a 4-digit grid and the shift is $3.0\Delta\lambda_{eq}$ where $\Delta\lambda_{eq}$ denotes the longitudinal grid interval at the latitude closest to the equator. This shift was chosen so that the points are the same in the two grids at latitudes where the grid is not reduced, with the phase shift relative to the grid only introduced in the reduced domain, but throughout the reduced domain.

The full- and reduced-grid semi-Lagrangian pairs indicate the level of uncertainty introduced by the interpolations inherent in the semi-Lagrangian method. Differences between semi-Lagrangian integrations such as those in Fig. 3 which are below the semi-Lagrangian levels in Fig. 4 are less than the inherent error in the numerical approach and are not significant.
The Eulerian pairs have differences well above rounding. These differences are introduced by aliasing. If the model included only linear and quadratic terms the difference would be at the rounding level of the 10- and 12-digit grids in Fig. 2. However, the pressure gradient in the momentum equation, the vertical advection in the momentum and thermodynamic equations, the energy-conversion term in the thermodynamic equation, and the vertical integrals in the surface pressure-tendency equation all have higher degrees of nonlinearity and therefore contribute to the difference.

The cause of the error is easily illustrated by a one-dimensional example involving the interaction of two waves. Define a discrete grid

\[ x_j = (j - 1)\Delta x, \quad j = 1, \ldots, J \]  

(3.3)

where \( \Delta x = 2\pi / J \). Consider the product of two waves defined on the grid with wave numbers \( k \) and \( l \) and suppose \( k + l > J / 2 \), the smallest wave representable on the grid. Define \( k' = J - (k + l) \). Then

\[ e^{ikx_j} e^{ilx_j} = e^{i(k+l)x_j} = e^{i(k+l)(j-1)\Delta x} \equiv (e^{i2\pi(j-1)\Delta x}) e^{-i k' x_j} = e^{-i k' x_j}, \]  

(3.4)

the normal aliasing relationship. If the structures are shifted a fraction of a grid interval \( \alpha \Delta x \) after the product is calculated, the result on the grid is

\[ e^{-ik'(x_j-\alpha \Delta x)} = e^{-ik'x_j} e^{+ik'\alpha \Delta x}. \]  

(3.5)

If the structures are shifted \( \alpha \Delta x \) before the product is calculated, the result on the grid is

\[ e^{ik(x_j-\alpha \Delta x)} e^{il(x_j-\alpha \Delta x)} = e^{i(k+l)x_j} e^{-i(k+l)\alpha \Delta x} = e^{-ik'x_j} e^{+ik'\alpha \Delta x} e^{-iJ\alpha \Delta x}. \]  

(3.6)

The difference between the right-hand sides of (3.5) and (3.6) is due to differences in the aliasing on the two discrete grids. In the CCM3 experiments in Fig. 4, the grid is chosen to prevent quadratic aliasing. Thus for quadratic terms no difference is introduced by shifting the phase before or after the calculation. However, the higher-order terms will alias and are responsible for the differences seen in Fig. 4. This provides some indication of the level of the aliasing error in the integration. Differences between Eulerian integrations, such as those in Fig. 2, which are below the levels in Fig. 4 are less than the error inherent in either model from aliasing, and are likely to be unimportant unless introduced in some systematic or pathological manner.

The r.m.s. error for the Eulerian model on the full grid in Fig. 4 is several orders of magnitude larger than the r.m.s. error for the reduced grid. It is tempting to ascribe this to the error being introduced over a larger fraction of the globe. However, the maximum error also differs by several orders of magnitude. In reality the difference is due to the shift, and thus the aliasing being different in the two cases. The error in the full-grid case happens to be largest within the region where the reduced grid is not reduced, i.e. equatorward of the first reduced latitude, and also occurs in the reduced grid if the shift is 0.25\( \Delta \lambda_{eq} \) rather than 3.0\( \Delta \lambda_{eq} \). This illustrates that the aliasing error can be rather capricious and that a large sample might be needed to study the error level in a statistically significant manner.

Given this aliasing error, it might seem surprising that the Eulerian high-digit grids did as well as they did in Fig. 2. However, the aliasing in those comparisons is only different at latitudes where the grid is reduced, i.e. where \( J_m \) in (2.9) differs from \( J \) in (2.2). The error which dominates the full-grid Eulerian case in Fig. 4 occurs at latitudes where \( J_m = J \) for most of the reduced grids. Thus the aliasing errors are the same in both the full and reduced cases in that region and do not contribute to a difference between the
integrations. We also note that the Eulerian reduced-grid error in Fig. 4 is for a 4-digit grid and is less than the error in the 4-digit reduced grid in Fig. 2. That difference in aliasing is contributing to the 4-digit grid error in Fig. 2 but it is less than the primary error introduced by the reduced grid.

The 2-digit grid error of the Eulerian model is comparable with the aliasing error exposed above with the full grid (Fig. 4 vs. Fig. 2). So for an adiabatic Eulerian model there appears to be no justification in using a higher-digit grid than the 2-digit one. In the semi-Lagrangian model, Fig. 4 shows that the interpolation error overpowers the aliasing error, as indicated by the Eulerian model on the full grid, by one order of magnitude. However, the interpolation error in the reduced grid is larger enough than the error estimated by shifting the grid, that the 5- or 6-digit grid errors are comparable with the aliasing error. Therefore a higher-digit grid may be desirable for an adiabatic semi-Lagrangian model.

4. Pure Advection Tests

To evaluate the adequacy of the various reduced grids further we consider a simple standard advection test case. This is Case 1 of the shallow-water tests of Williamson et al. (1992), which involves pure advection of a cosine bell once around the sphere without change of shape. The advection wind is solid body rotation about an axis which is at an angle \( \alpha \) from the polar axis of the spherical coordinate system. Four orientations of the advection winds are specified in the test definition: around the equator (\( \alpha = 0 \)), up a meridian and directly over the pole (\( \alpha = \pi/2 \)), and minor shifts from these two orientations to avoid symmetries (\( \alpha = 0.05 \) and \( \alpha = \pi/2 - 0.05 \)). We also include 14 other orientations equally spaced between the equator and pole (\( \alpha = k\pi/30, k = 1, \ldots, 14 \)).

The initial cosine-bell test pattern is

\[
h(\lambda, \varphi) = \begin{cases} 
    \left( h_0/2 \right) \left( 1 + \cos(\pi r/R) \right) & \text{if } r \leq R \\
    0 & \text{if } r > R
\end{cases}
\]  

(4.1)

where \( h_0 = 1000 \) m and \( r \) is the great circle distance between \((\lambda, \varphi)\) and the centre of the cosine bell, specified initially as \((3\pi/2, 0)\). The radius is taken to be \( R = a/3 \) where \( a \) is the radius of the earth. The bell’s radius is about 6.8 T42 Gaussian latitudinal grid intervals. The advection velocity is chosen such that the pattern takes 12 days to go once around the globe. The time step is 20 minutes, resulting in 432 sequential \( 2\Delta t \) time steps.

The lowest curves in Fig. 5 show the normalized \( l_2 \) and \( l_\infty \) errors (as defined by Williamson et al. (1992)) for Eulerian spectral-transform advection as a function of the orientation, \( \alpha \), of the advection wind, after 12 days. Reduced-grid errors are calculated by expanding the reduced-grid data to the full grid via Fourier transforms in longitude. The spectral-transform integrations have the same error for all the reduced grids and the error is independent of the orientation of the wind, as might be expected for triangular truncation. The errors are identical to those in Fig. 4.4 of Jakob-Chien et al. (1995) for the T42 case.

Figure 5 also shows the normalized \( l_2 \) and \( l_\infty \) errors for semi-Lagrangian transport with three different interpolants as a series of lines, one for each digit grid. (Unlike the previous adiabatic model results, these errors are for pure semi-Lagrangian transport without spectral transforms.) The error decreases monotonically with increasing digit grids, although for the higher-digit grids the errors are very close to the full-grid errors.
The interpolants are monotonic Hermite cubic polynomial, monotonic Hermite quasi-cubic polynomial, and Lagrange quasi-cubic polynomial (Williamson and Rasch 1989). The interpolants are all of tensor product form with four longitudinal interpolations performed first, followed by one latitudinal interpolation. The quasi-cubic interpolants use the appropriate cubic polynomial interpolation for the two interior longitudinal interpolations, and linear for the two outer interpolations. The latitudinal interpolation is cubic. The error was calculated on the full grid by expanding the reduced-grid simulations at each latitude to the full grid using an interpolant consistent with the particular model. Thus a monotonic Hermite cubic polynomial in longitude was used with the two monotonic schemes, and a Lagrange cubic polynomial in longitude was used in the non-monotonic case.

The error for the monotonic Hermite cubic polynomial on the full grid and with $\alpha = \pi/2$ is very similar to that seen in Williamson and Rasch (1989) for the same transport scheme and orientation (their Fig. 11(a)), but not identical since their radius was around seven T42 Gaussian latitudinal grid intervals, and they applied a longer time step requiring only 256 iterations to circumnavigate the globe. Surprisingly, the monotonic quasi-cubic scheme produces less error than the monotonic cubic. The variation with different grids is very similar between the two. The quasi-cubic polynomial interpolation has smaller errors than the monotonic cases since the quasi-cubic interpolant does not damp structures such as the cosine bell as much as do the monotonic interpolants. The error for the full grid with $\alpha = \pi/2$ is very similar to that seen in Williamson and Rasch (1989) for the same transport scheme (their Fig. 10(a)). The error decreases as the orientation of the flow moves away from the pole or equator, unlike the monotonic interpolants in which the error increases. The dependence of the error on $\alpha$ is easily explained based on the characteristics of the interpolants and the local Courant number. The amplitude error of the Lagrange cubic polynomial interpolant is maximum when the departure point falls in the centre of a grid interval, and decreases as it approaches a grid point. The time step of these experiments with $\alpha = 0$ or $\alpha = \pi/2$ leads to a departure
point $0.3\Delta\lambda$ or $0.3\Delta\varphi$ from a grid point. When $\alpha = \pi/4$ it is $0.2\Delta\lambda$ or $0.2\Delta\varphi$. Thus the interpolation error is less for $\alpha = \pi/4$. The monotonic interpolant on the other hand clips a round structure that is moving along grid lines less than one that is moving at an angle $\pi/4$ through the grid. Thus the errors for a monotonic interpolant are larger for $\alpha \sim \pi/4$ than for $\alpha = 0$ or $\alpha = \pi/2$. We have no explanation for why the monotonic quasi-cubic error is less than the monotonic cubic error for this case.

Figure 5 indicates about a 10% variation in the errors of the monotonic semi-Lagrangian schemes between the 1-digit grid and the 4-digit grid for flows with $\alpha$ between $50^\circ$ and $80^\circ$, with only minor degradation for higher-digit grids. Also notice that the differences between errors with the monotonic cubic and quasi-cubic interpolants are comparable with the differences between different reduced grids with the same interpolant, and that the variation in accuracy with time step or Courant number is also comparable with the difference between the various reduced grids. Thus there is little reason to choose a higher-digit grid than the 4-digit grid, and the 3- and 2-digit grids might be considered satisfactory.

We have repeated the above advection tests with a cosine bell of half the radius, about 3.4 T42 Gaussian latitudinal grid intervals. In the semi-Lagrangian cases the errors are larger than with the wider bell, but the spread of the error with different reduced grids is less. The normalized $l_2$ error for the monotonic cubic case varies from around 0.7 for $\alpha = 0$ to around 0.83 for $\alpha = 40$ to $70^\circ$, and down to 0.75 for $\alpha = 90^\circ$. The Eulerian spectral-transform normalized $l_2$ error is 0.26 and remains independent of flow orientation and particular digit grid.

5. **Atmospheric General-circulation model simulations**

We now consider simulations with the complete model, i.e. including the diabatic forcing. The initial conditions are taken from a previous simulation with the full grid and therefore represent a state from the model's climatology. Atmospheric initial conditions for the reduced grid are obtained from the full-grid initial conditions by Fourier analysis in longitude as described above with the adiabatic experiments, except for water vapour which is interpolated in longitude by monotonic Hermite cubic interpolation. This choice is consistent with the numerical approximations in the model.

The diabatic model with subgrid-scale physical parametrizations introduces the additional complication of specifying the underlying surface on the reduced grid. Land-surface type datasets, including soil texture, soil colour, land type, and inland water are obtained for the reduced grid from observation-based datasets following the same procedure as used to produce them for the full grid. For each dataset this involves area-weighted averaging or determining the area-weighted dominant surface type as appropriate for the particular dataset (Bonan 1996). The observation-based datasets are on either $1^\circ \times 1^\circ$ or $0.5^\circ \times 0.5^\circ$ grids. Climatological sea surface temperature (SST) and sea ice concentration are interpolated from a $2^\circ \times 2^\circ$ dataset. Definition of land–ocean boundary and surface orography are obtained from a $10' \times 10'$ dataset. The subgrid-scale variance required by the gravity-wave drag parametrization is also obtained for the full grid from the $10' \times 10'$ dataset. The full-grid variance is interpolated to the reduced grid with a monotonic Hermite cubic interpolant.

Perhaps a word is needed here to justify our choice of interpolating the variance to the reduced grid rather than recalculating the variance directly on the reduced grid, especially as Hortal and Simmons (1991) reported that recalculating the variance affected their forecasts. Like all parametrizations, gravity-wave drag is an attempt to represent the effect of unresolved scales on the resolved scales. Unlike other parametrizations it
requires an explicit parameter determined from unresolved scales, the so-called subgrid-scale orographic variance. The unresolved scales in a trian-gularly truncated global spectral model are very different from the subgrid-scales of a full Gaussian grid. As the adiabatic model experiments show, the resolved spectral scales are well represented by the reduced grid. The orographic variance for the standard CCM3 is calculated over the full Gaussian grid box from the difference of 10' high-resolution data and the grid-box average of these data. If the high-resolution data represent point values, then approaching the poles this variance includes smaller and smaller unresolved longitudinal scales and excludes more and more longer unresolved scales. This approach to defining the variance is at odds with the philosophy of the parametrization. A more consistent procedure would base the unresolved variance on a uniform range of scales everywhere. Such a variance would not vary greatly between full and reduced grids. Since the standard CCM3 does not use such a procedure and we want to use the standard CCM3 as our reference model, we chose the expedient of interpolation to produce similar full- and reduced-grid variances. Recalculating the variance on the different grids as described above from unfiltered high-resolution data would introduce artificial differences in simulations from the parametrization which are not relevant to the issue at hand.

Land-surface initial conditions are harder to map between different grids. When the grid is changed there is the possibility of different categories dominating, in which case simple interpolations do not work. Therefore we use the arbitrary land initial condition option in the model which sets the soil temperature and soil moisture to values not totally unrelated to the simulated climate. Thus, although the land is not completely in balance with the atmosphere initially, its evolution behaves as land would be expected to and it does not introduce wild oscillations into the integrations which might dominate initial error growth. For practical purposes the land comes into equilibrium within one year.

(a) Initial error growth

Figure 6 shows the growth of the mass-weighted r.m.s. temperature errors (3.1) from the Eulerian model plotted with two different scales. The initial conditions were all based on the Eulerian adiabatic initial conditions described above. The lowest curve shows the growth of rounding sized differences introduced into the initial conditions with the full-grid model. This error grows to almost $10^{-2}$ K in two days. The fast growth is caused by the physical parametrizations, not the fluid flow itself, and is dominated by the land-surface model and the Zhang and McFarlane (1995) convective parametrization. The upper set of curves present the errors for 1- to 4-digit grids. The error increases monotonically for decreasing digit grids. The errors at day 0 are calculated from the fields after the convective parametrization is applied, and therefore are not the same values as in Fig. 2 or Fig. 3. The middle set of curves present the growth of a perturbation in the full-grid model equivalent to the error introduced initially by the reduced grids, i.e. the full-grid initial conditions are mapped to the reduced grid and then are mapped back to the full grid, using Fourier interpolation in longitude for the dynamical variables and monotonic cubic interpolation for water vapour. Again, the convective parametrization is applied before the day 0 error calculation. The line codes in Fig. 6(b) for the reduced perturbations match those for the reduced grids. The rounding perturbation curve indicates the best that could be expected from the model given any initial difference, and provides a lower bound on significant errors. The reduced perturbation initial errors are all the size of the rounding perturbation error at two days, which is in the range where the parametrizations are still causing fast growth, and therefore could not be considered significant. Those reduced-grid perturbations grow one order of magnitude relatively rapidly the first day, then grow
at the same rate as the rounding perturbation at the same error level. The reduced-grid error accumulates for two orders of magnitude to a value one order larger than that at which the parametrizations no longer dominate the growth, then grow at the same rate as the rounding perturbation. The value at which the transition in behaviour occurs (0.05 to 0.1 K) is still below the uncertainty in determining the state of the atmosphere at any one time. For example, the r.m.s. differences between the European Centre for Medium-Range Weather Forecasts operational analyses and the National Centers for Environmental Prediction re-analyses for the mid-1990s are 1.5 to 1.8 K when calculated for the same season as the model experiments.

To put the reduced-grid error growth of the full atmospheric general-circulation model (AGCM) into perspective, we once again look for errors introduced by uncertainties in, or arbitrary aspects of, the model definition. One such aspect is the detailed gridded land-surface specification obtained from the observation-based datasets, including land–ocean boundaries which are identified with grid-box edges. To exploit this uncertainty we set the first model grid point to be at $-2^\circ$ longitude rather than the normal $0^\circ$. If the atmospheric model was based on a $2^\circ$ grid this would have no affect on the simulation since all the datasets used to build the model grid data have points every $2^\circ$ and everything would just be shifted one $2^\circ$ interval. However, the grid for the T42 model is not a $2^\circ$ grid, and therefore the surface definitions will differ due to the different registration of the atmosphere grid to the grids which specify the underlying land characteristics. These differences are within the modelling uncertainty. Figure 7 contrasts the growth of the difference between the unshifted and the shifted simulation with the reduced-grid ‘errors’. The difference with the shifted simulation is slightly larger than even the 1-digit grid error at day 1, but the difference grows more slowly than the reduced-grid errors or a perturbation within the same model. The error is larger initially because the shifted-grid data when shifted back by interpolation have more points which are interpolated between land and ocean points than do the reduced grids. In addition to such interpolated points, there is the possibility of a point representing ocean in one model and land in the other. In the 1-digit grid, the reduction begins around
Figure 7. Mass-weighted root-mean-square (r.m.s.) temperature differences between original and shifted full-grid integrations (solid) and the four reduced-grid integrations in Fig. 6.

±30° so that over roughly half the domain the definition of land is identical to that in the full grid. The different definitions lead to fairly large initial differences ranging up to 20 K in the surface temperature at isolated points, for example off the east coasts of North America and Asia where the sea surface temperatures can be 20 K colder than nearby land temperatures, especially with the arbitrary land initialization used in these experiments. The differences propagate into the lower atmosphere rapidly. The growth of the atmospheric difference between the shifted and original grid runs is not due to the same processes as the growth of a difference within a single model. Therefore it is not unreasonable that the growth rates differ.

Figure 7 shows that during the entire integration period the 4-digit reduced-grid model error is less than the modelling uncertainty as determined from the shifted integration. The 1- to 3-digit reduced-grid model errors are less than the modelling uncertainty for 5 to 10 days. Thus even the 1-digit grid may be suitable for climate modelling if it does not introduce pathological errors into the simulated climate.

(b) Simulated climates

The Eulerian and semi-Lagrangian simulations on the full grid, and 4-, 3-, 2-, and 1-digit reduced grids and shifted grid were all run for 5.5 years. The initial date was 1 September, and we arbitrarily label it year 0. The simulations were run to 1 March of year 6. We have analysed climatological 5-year December–January–February (DJF) averages for years 2 to 6, and annual averages of years 1 to 5. In addition to the 5.5-year run with the full grid carried out for this study, we also consider the original 10-year control simulation made with the Eulerian CCM3 when the model code was frozen.

We have examined many aspects of the climates of the various simulations. Our overall impression is that the mean climates of the simulations are all very similar. The 5-year samples are too short to enable one to establish statistically that the climates are the same. However, the differences are small enough that it is also difficult to establish that they are significantly different. Below we show a few select fields to illustrate the differences, and to point out that the differences are comparable with those due to the natural variability of the model. We choose fields which in our experience are particularly sensitive to model changes.
To produce contour maps of fields and differences, the fields from the reduced grids are interpolated to the full grid via linear interpolation in longitude, which is consistent with the basis of many contouring routines. Similarly, the fields from the grid with the first point at $-2^\circ$ longitude are linearly interpolated to the standard grid in which the first point is at $0^\circ$ longitude. The zonal mean fields and zonal r.m.s. differences we show are calculated from these linearly interpolated fields, even though the zonal averages could have been computed directly on the original reduced and shifted grids.

Figures 8(a) and (b) show the 5-year DJF zonal average low-cloud fraction from the full- and reduced-grid simulations from the Eulerian and semi-Lagrangian models respectively. Also shown in Fig. 8(a) are the mean ± one standard deviation of the DJF zonal average from the 10-year control integration with the Eulerian CCM3 and in Fig. 8(b) the 5-year DJF average of the full-grid semi-Lagrangian integration ± one standard deviation of the Eulerian run. (We do not have a matching 10-year semi-Lagrangian simulation.) There are also five solid lines in each figure from the full grid,
and 4-, 3-, 2-, and 1-digit reduced-grid 5-year DJF averages. All of the 5-year DJF averages fall within one standard deviation of the 10-year run. Of course this is the zonal average and the agreement could result from a cancellation of local positive and negative differences. Figures 8(c) and (d) show the zonal r.m.s. differences between the four reduced grids and the full grid, i.e. the 'error' in the reduced grids, as the four solid lines for (c) the Eulerian and (d) the semi-Lagrangian models. The two dashed lines in the Eulerian case are the zonal r.m.s. difference between different 5-year samples from the 10-year Eulerian simulation. One curve is the difference between years 1–5 and 6–10 and the other is the difference between the five odd-numbered years and the five even-numbered years. Both indicate the natural variability in 5-year samples. Clearly, the differences between the reduced grids and the full grid are comparable with the natural variability of the Eulerian model. The differences between the reduced and full grids in the semi-Lagrangian case are similar to those in the Eulerian case and we can therefore assume that they are also comparable with the natural variability. The 'error' in the shifted-grid integration (not shown) falls within the envelope of the curves in Fig. 8(c) poleward of $+20^\circ$ and $-30^\circ$. From $+10^\circ$ to $-20^\circ$ the values are around 0.05, exceeding the 'errors' from the reduced grids.
Figure 9 shows precipitation in the same format as Fig. 8. Figure 8 and Fig. 9 show that there is nothing to indicate that the low cloud and precipitation from the reduced grids are different from that produced by the full-grid model. In addition, the differences between the reduced-grid and the full-grid models are comparable with, or less than, errors introduced by arbitrary aspects of the model definition, such as the longitude of the first grid point. The zonal r.m.s. 'error' with the shifted-grid integration (not shown) also falls within the envelope of the curves in Fig. 9(c), except at the two latitude grid rows north of the equator where the r.m.s. values almost reach 1.44 mm day$^{-1}$.

Although Fig. 8 and Fig. 9 indicate that the atmospheric simulations are very similar, there are in fact local differences as would be expected just because the underlying surface is defined differently on the reduced grids. Figures 10(a) and (b) show the zonal average surface temperature and zonal r.m.s. difference of surface temperature, respectively, from the Eulerian simulations. The zonal average is within the natural variability of the model but the zonal r.m.s. difference exceeds our simple measure of the natural variability between $+30^\circ$ and $+60^\circ$. Examination of the difference fields themselves (not shown) shows that this is due to isolated points where the full-grid model is land and the reduced-grid model is dominated (after interpolation back to the full grid) by ocean, or vice versa. These pointwise differences are as large as 12 to 15 K due to the strong coastal gradients in surface temperature, particularly along the eastern coasts of continents. Similar differences occur with the shifted-grid integration (not shown), in which case the zonal r.m.s. difference exceeds the natural variability as far south as $-30^\circ$. The magnitude of the difference decreases toward the tropics, being around 1.0 K at $+30^\circ$ and closer to 0.5 K from $+10^\circ$ to $-30^\circ$. Poleward of $-30^\circ$ the zonal r.m.s. differences are comparable with natural variability because specified SSTs dominate to $-60^\circ$, after which sea ice and snow-covered land dominate, with smaller east–west gradients to introduce differences. The reduced-grid models do not show differences equatorward of $\pm30^\circ$ since none of them is reduced in this region. Thus, although local differences do occur with the reduced grid, they are no larger than errors introduced by arbitrary aspects of the model definition.
TABLE 1. FIVE-YEAR AVERAGE ENERGY IMBALANCE (W m\(^{-2}\)) FROM VARIOUS MODEL TERMS FOR FULL AND REDUCED GRIDS WITH THE EULERIAN MODEL

<table>
<thead>
<tr>
<th>Grid</th>
<th>Full</th>
<th>4-digit</th>
<th>3-digit</th>
<th>2-digit</th>
<th>1-digit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total residual</td>
<td>0.306</td>
<td>0.304</td>
<td>0.305</td>
<td>0.303</td>
<td>0.304</td>
</tr>
<tr>
<td>Diffusion</td>
<td>-0.021</td>
<td>-0.027</td>
<td>-0.025</td>
<td>-0.027</td>
<td>-0.031</td>
</tr>
<tr>
<td>Mass fixer</td>
<td>-0.0092</td>
<td>-0.0088</td>
<td>-0.0091</td>
<td>-0.0086</td>
<td>-0.0059</td>
</tr>
<tr>
<td>Time filter</td>
<td>-0.017</td>
<td>-0.017</td>
<td>-0.017</td>
<td>-0.017</td>
<td>-0.017</td>
</tr>
</tbody>
</table>

TABLE 2. FIVE-YEAR AVERAGE ENERGY IMBALANCE (W m\(^{-2}\)) FROM VARIOUS MODEL TERMS FOR FULL AND REDUCED GRIDS WITH THE SEMI-LAGRANGIAN MODEL

<table>
<thead>
<tr>
<th>Grid</th>
<th>Full</th>
<th>4-digit</th>
<th>3-digit</th>
<th>2-digit</th>
<th>1-digit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total residual</td>
<td>-5.13</td>
<td>-5.12</td>
<td>-5.12</td>
<td>-5.13</td>
<td>-5.09</td>
</tr>
<tr>
<td>Diffusion</td>
<td>0.0048</td>
<td>0.0013</td>
<td>0.0029</td>
<td>-0.0026</td>
<td>0.00007</td>
</tr>
<tr>
<td>Mass fixer</td>
<td>-0.346</td>
<td>-0.336</td>
<td>-0.331</td>
<td>-0.327</td>
<td>-0.321</td>
</tr>
<tr>
<td>Time filter</td>
<td>-0.012</td>
<td>-0.012</td>
<td>-0.012</td>
<td>-0.011</td>
<td>-0.011</td>
</tr>
</tbody>
</table>

(c) Conservation characteristics

The energy conservation aspects of AGCMs are very important when they are a component of a coupled climate model. The conservation characteristics of the numerical approximations in the Eulerian and semi-Lagrangian models are unaffected by the reduced grids. Table 1 lists the energy imbalances averaged every iteration over five years of the simulations. The total residual includes the errors introduced by the dynamical approximations and physical parametrizations. The imbalance arises primarily from the dynamical approximations. Some small contributions to the imbalance are due to the subgrid-scale parametrizations in which the effect of water vapour on the energy balance is not completely consistent, to the lack of mass conservation, and to the horizontal diffusion. The term in the table labelled ‘diffusion’ is the contribution to the total energy residual from horizontal diffusion. Although the horizontal momentum diffusion is converted to heating to conserve energy, temperature is diffused rather than potential temperature, leading to a slight energy inconsistency. The term labelled ‘mass fixer’ shows the difference in energy before and after application of the mass fixer. Finally the row labelled ‘time filter’ shows the difference in the global energy before and after application of the time filter. All of these are minimal compared with the dynamics, but illustrate that it would be difficult to achieve energy conservation better than a few hundredths of a W m\(^{-2}\) without applying a fixer. Table 2 shows the energy statistics from the semi-Lagrangian integrations. The total residual is significantly larger than the residual in the Eulerian model, but the reduced grids have no impact on the energy conservation characteristics.

The error growth, the energy conservation characteristics, and the simulated climates all indicate that all the reduced grids examined are suitable for climate simulations, including the 1-digit grid. In all cases the errors introduced by the reduced grids are smaller than those attributable to arbitrary aspects of the model definition. In addition, the reduced-grid errors do not manifest themselves in any pathological manner.
6. CONCLUSIONS

The term reduced grid generally refers to a grid based on latitude and longitude circles in which the longitudinal grid increment increases at latitudes approaching the poles so that the longitudinal distance between grid points is reasonably constant (Hortal and Simmons 1991). The actual definition of a reduced grid is arbitrary. The first reduced grids were based on geometric principles and attempted to define the longitudinal grid lengths as uniformly as possible. It was observed in application (Hortal 1991; Simmons 1991) and shown theoretically (Courtier and Naughton 1994) that reduced grids for spectral-transform models should be finer near the poles than such geometric grids.

We defined an $\ell$-digit reduced grid for spectral-transform models following Courtier and Naughton (1994). In these grids the relative error made by omitting certain associated Legendre functions from the Gaussian quadrature poleward of some latitude is less than $10^{-\ell}$. The associated Legendre functions omitted from the quadrature are those which have unrepresentable Fourier wave numbers at a latitude because of the reduced number of longitudinal points at that latitude.

We first determined the error introduced by the reduced grid in 2-day integrations with Eulerian and semi-Lagrangian baroclinic adiabatic models. The error is defined as the difference between integrations on a reduced grid and integrations on the full grid. The global r.m.s. error in the Eulerian model increases by one order of magnitude for each digit decrease in the grid for $\ell < 10$ as expected by the definition of the grids. For the 10-digit and more accurate grids the errors introduced by the reduced grid are less than computational round-off error with 64-bit arithmetic. Global r.m.s. errors from the semi-Lagrangian model span just two orders of magnitude from the 1-digit to the 12-digit grid, and the error with the 1-digit grid is a little larger than in the equivalent Eulerian model. The larger errors with the adiabatic semi-Lagrangian model are introduced by the interpolations in the semi-Lagrangian algorithm.

These errors were put into perspective by comparing them with modelling uncertainties such as aliasing error. We obtained an indication of aliasing errors by comparing the full-grid model runs with integrations in which the location of the first longitudinal grid point was shifted relative to the control integration. For the adiabatic Eulerian model, there appears to be no justification in using a higher-digit grid than the 2-digit one. However, a higher-digit grid may be desirable for an adiabatic semi-Lagrangian model.

We also evaluated the accuracy of calculations on reduced grids with a simple standard advection test case in which a cosine bell is advected once around the globe without change of shape. The advecting wind was solid body rotation about an axis which is at an angle $\alpha$ from the polar axis of the spherical coordinate system. We considered angles $\alpha$ from 0, giving advection around the equator, to $\pi/2$, giving advection along a meridian and directly over the poles. The errors for Eulerian spectral-transform advection are the same for all angles of the flow and for all reduced grids from 1- to 12-digit. Thus the 1-digit grid is adequate for the Eulerian advection. The errors for semi-Lagrangian advection are larger than those for the Eulerian scheme and decrease monotonically with increasing digit grids by about 10% between the 1-digit and 4-digit grids. There is very little improvement when using higher-digit grids. Thus there is little reason to choose a higher-digit grid than the 4-digit grid, and the 3-, 2- and 1-digit grids might be considered satisfactory for semi-Lagrangian advection.

We considered short and long integrations with a diabatic AGCM with a complete suite of subgrid-scale parametrizations, the NCAR CCM3, with both Eulerian and semi-Lagrangian approximations. We first considered the error growth over 30 days and compared it with the growth of small errors such as those introduced by rounding errors. We also put the reduced-grid error growth into perspective by comparing it
with the growth of errors introduced by uncertainties in, or arbitrary aspects of, the model definition. In this case we changed the longitude of the first grid point. These comparisons showed that the errors introduced by all grids are less than the modelling uncertainty, and even the 1-digit grid may be suitable for climate modelling if it does not introduce pathological errors into the simulated climate.

Finally, we carried out 5.5-year simulations with the models on the full grid, 4-, 3-, 2-, and 1-digit reduced grids, and the shifted grid. We examined many aspects of the climates of the various simulations, but selected only a few variables which are generally sensitive to model changes to show in the paper. Our overall impression is that the mean climates produced by all the reduced grids are all very similar. We saw no indication that the reduced grids introduce pathological errors that contaminate the simulations. The 5-year seasonal means used to examine the climates are too short to be able to establish statistically that the climates are the same. However, the differences are small enough that it is also difficult to establish that the climates are significantly different. The few statistics shown illustrate these small differences, and show that the differences are comparable with those due to the natural variability of the model.

Our results indicate that even a 1-digit grid is suitable for climate modelling. This is perhaps surprising, but not inconsistent with Hortal and Simmons (1991) who noticed no evidence of an overall degradation of forecast accuracy with the geometric grid. As illustrated in Fig. 1, the 1-digit reduced grid has an equal or finer longitudinal grid interval at all latitudes than the geometric reduced grid of Hortal and Simmons (1991). In fact, the ratio of the longitudinal interval in the 1-digit grid to that in the geometric grid approaches 0.5 at the poles.

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