Evolution of disturbances and singular vectors in the shallow-water semi-geostrophic model

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(Received 27 September 1999; revised 8 March 2000)

SUMMARY

The first part of this work is devoted to exploring the instability mechanism of the shallow-water semi-geostrophic model by investigating the evolution of disturbances under constant-shear and cosine-type basic flows. For the constant-shear basic-flow case, the wave-packet analysis method is applied to study the evolution of potential and total energy and the meridional structure of disturbances under the assumption of a spatially slowly varying basic state. The results are found to be in good agreement with numerical calculations even when the assumption is slightly relaxed. For the cosine-type basic-flow case, both unstable normal modes and continuous spectra exist if the stability criterion is violated. Numerical results show that the relative importance of regular normal-mode and continuous-spectrum disturbances in the evolution process depends on the configuration of initial disturbances and their projections onto the normal modes and continuous spectra.

In the second part of this work, in order to search for initial disturbances based on which the growth of certain norms of disturbances can be optimized, singular vectors of different norms (metrics) are computed. It is found that of all the norms, the structures of singular vectors are mainly determined by the configuration of the zonal basic flow and less affected by model parameters such as the Rossby number, rotating Froude number, and amplitude of topography. However, when the shear of the basic flow is strong, the optimal singular value can be sensitive to the model parameters. When the stability criterion is violated, the optimal singular values of all the norms have clear wave-number preferences. The peaks of spectra of singular values of the energy-type norms, including potential, kinetic and total-energy norms, correspond to very weak normal-mode instability. However, the optimal singular value of the enstrophy norm has almost the same wave-number preference as unstable normal modes.

KEYWORDS: Continuous spectra Normal modes Singular values Singular vectors

1. INTRODUCTION

The shallow-water semi-geostrophic (SWSG) model is an important Hamiltonian balanced model in shallow-water dynamics. In the SWSG model the conservation properties of its parental model, the primitive shallow-water equations, remain (Roulston and Norbury 1994). Furthermore, in this model there are no restrictions on the value of the rotating Froude number or the amplitude of bottom topography as long as the layer depth of fluid is positive, and there is also no requirement that the Rossby number be vanishingly small. These special features make the SWSG model an important tool for investigating anisotropic motions in the atmosphere and oceans (Pedlosky 1987; Allen et al. 1990; Malardel et al. 1997).

In order to understand the stability properties of the SWSG model, the linearized SWSG equation, formulated in terms of the perturbation of free-surface height, was derived by Ren (1999). In that work, only the unstable growth rate of normal-mode disturbances was calculated numerically, and the evolution of other types of perturbation was left uninvestigated.

It is the purpose of this work to investigate the evolution of general disturbances in the SWSG model including regular normal-mode, continuous spectrum, and marginally stable normal-mode disturbances (Richtmyer 1978). Within the geophysical fluid-dynamics context, it has long been recognized that, in addition to the normal-mode instability mechanism which depends only on the configuration of basic-state variables, there is another important instability mechanism for continuous-spectrum disturbances which depends on the configuration of both basic-state variables and initial disturbances.

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(Farrell 1982, 1988). In the latter instability mechanism, continuous-spectrum disturbances can grow by extracting energy even from the basic flow satisfying the normal-mode stability criteria. Although continuous-spectrum disturbances die down over a long period of time (Case 1960; Lu et al. 1986), the mechanism of growth of continuous spectra is still important, as argued by Farrell (1982, 1988), in the initial stage of cyclogenesis and in the excitation of neutral Rossby waves in the atmosphere. In this work, we will examine this issue in the SWSG model so that the instability mechanisms of frontal waves observed in the atmosphere (Malardel et al. 1997) and coastal waves in the oceans can be better understood. We will study a stable basic-flow case in which only continuous-spectrum disturbances exist and an unstable basic-flow case in which both unstable regular normal-mode and continuous-spectrum disturbances exist. In the latter case, we are able to compare the relative importance of regular normal-mode and continuous-spectrum disturbances during the evolution process for different kinds of initial conditions.

Another important problem related to the instability mechanism of continuous-spectrum disturbances is to find initial disturbances which can optimize the growth of disturbances over certain time intervals. In doing so, one has to calculate singular vectors (SVs) with certain norms (metrics) and corresponding singular values (SVs) (Farrell 1988). In this work, this problem will be explored with a focus on the dependence of SVs and SVs on various factors such as the configuration of the basic flow and bottom topography, and the Rossby and rotating Froude numbers. This is important in light of recent work by Buizza and Montani (1999) on the application of the singular-vector theory in adaptive observations in order to improve data assimilation results in the atmosphere and oceans. It was suggested in their work that, by identifying the so-called dynamically sensitive areas based on the structures of the first few large SVs, the results of data assimilation could be improved by deploying extra observations in these regions. Since the key issue is to identify the location of SVs, it is important to understand the dependence of SVs and SVs on the background factors. The SWSG model is ideal for this purpose. Numerical evidence based on the primitive-equation atmospheric model shows that the optimal SV is primarily geostrophic (Errico 2000), therefore, the SWSG model is a suitable tool because the geostrophic component is dominant over the ageostrophic one, and free gravity waves are absent. More importantly, few restrictions on the background factors in the SWSG model make it possible to explore the relations between SVs/SVs and background factors such as the Rossby number and rotating Froude number, etc. It is impossible to do so in quasi-geostrophic (QG) dynamics in which the rotating Froude number is presupposed to be order one and the Rossby number to be vanishingly small. It is expected that the theoretical results can provide us some insight into the dependence of SVs and SVs on background factors in more realistic geophysical fluids.

The rest of the paper is arranged as follows: The linearized SWSG equation and related wave-activity conservation are introduced in section 2. In section 3, the evolution of disturbances in the SWSG model is investigated. In this section, we first study the slowly (spatial) varying constant-shear basic-flow case by using the wave-packet method (e.g. Zeng 1983) and then the general case by the numerical method. The relative importance of regular normal-mode and continuous-spectrum disturbances in a (unstable) cosine-type basic flow are compared under some special initial conditions. In section 4, different norms (metrics) for SVs and SVs are derived. In section 5, SVs and corresponding SVs of different metrics are calculated for both constant-shear basic flow and cosine-type basic flow; their dependences on background factors are explored. Conclusions are given in the last section.
2. Model introduction

(a) Governing equations

The SWSG model is a reduced model from the shallow-water (SW) primitive equations. In the SWSG model, the continuity equation is exactly the same as that in the SW model while the momentum equation differs from that in SW dynamics. In the momentum equation in the SWSG model, geostrophic velocity replaces the full velocity after the substantial derivative in SW dynamics. Therefore, the SWSG model is a balanced model in which the divergence is only a diagnostic variable. In addition, the SWSG model keeps all the conservation properties of SW dynamics. These features of the SWSG model make it an ideal tool for the investigation of anisotropic motions in geophysical fluid.

The linearized equations for the SWSG model can be obtained by first formulating the perturbation of full velocity in the momentum equations in terms of geostrophic variables and then substituting it into the linearized continuous equation. In terms of the perturbation of free-surface height, \( \eta' \), the (non-dimensional) linearized SWSG equation can be formulated as (Ren 1999)

\[
\left( \frac{\partial}{\partial t} + U_G \frac{\partial}{\partial x} \right) q' + \Xi \frac{\partial \eta'}{\partial x} = 0,
\]

where \( U_G \) is the (geostrophic) basic flow,

\[
q' = \rho_1 \frac{\partial^2 \eta'}{\partial x^2} + \frac{\partial}{\partial y} \left( \rho_2 \frac{\partial \eta'}{\partial y} \right) - F \eta' \equiv \mathcal{L} \eta'
\]

is the perturbation of pseudo potential vorticity,

\[
\rho_1 \equiv \frac{H}{f^2}, \quad \rho_2 \equiv \frac{1}{fQ}, \quad \Xi \equiv -\frac{1}{Ro \frac{d}{dy} \left( \frac{1}{Q} \right)}.
\]

\( Q \equiv \Omega/H \) is the basic-state potential vorticity, \( \Omega = 1 - Ro \frac{dU_G}{dy} \) the basic-state vorticity, \( H = 1 + Ro F \tilde{\eta} - h_B \) the undisturbed depth of fluid, \( f \) is the Coriolis parameter, and \( h_B \) the bottom topography. \( Ro \) is the Rossby number which is not necessarily vanishingly small and \( F \) the rotating Froude number which can be any (positive) value as long as \( H \) is positive. Note the invertibility of \( q' \) requires that \( \rho_2 = 1/fQ > 0 \).

It is important to note that although the SWSG model is a balanced model, (slaved) ageostrophic components of velocity still exist and can be calculated diagnostically once \( \eta' \) is obtained from Eq. (2.1).

For mathematical convenience, we assume that \( \eta' \) is periodic along the \( x \)-direction, and choose the following non-normal geostrophic flow boundary conditions

\[
\frac{\partial \eta'}{\partial x} = 0 \quad \text{at} \ y = 0, 1.
\]

Under this boundary condition, \( \mathcal{L} \), defined in Eq. (2.2), is Hermitian.

(b) Conserved quantities

Although the linearized SWSG equation does not conserve energy and potential vorticity, it can be proved that there are two conservation equations for wave-activity invariants in the SWSG model (Ren (1999); for the finite-amplitude form of wave-activity
invariants, see Ren (2000a)). The first one is the pseudo-momentum conservation equation

\[
\frac{d}{dt} \int \frac{\bar{q}^2}{\Xi} \, dy = 0,
\]  

(2.4)

and the second is the pseudo-energy conservation equation

\[
\frac{d}{dt} \int \left( \bar{E} - \frac{U_G \bar{q}^2}{\Xi} \right) \, dy = 0,
\]

(2.5)

where variables with an overbar are averaged over one period in the \( x \)-direction, and

\[
E = \rho_1 (\partial_x \eta')^2 + \rho_2 (\partial_y \eta')^2 + F \eta'^2 > 0
\]

(2.6)

refers to the density of total energy. The evolution equation of total energy is

\[
\frac{d\bar{E}(t)}{dt} = 2 \int \rho_2 \frac{dU_G}{dy} \frac{\partial \eta'}{\partial x} \frac{\partial \eta'}{\partial y} \, dy,
\]

(2.7)

where \( \bar{E}(t) = \int \bar{E} \, dy \).

The stability criterion that \( \Xi \) is sign definite for the pseudo-momentum norm follows immediately from Eq. (2.4). For the pseudo-energy norm, the basic flow is stable under the condition that \( U_G / \Xi \) is negative definite or that \( U_G / \Xi \) is positive definite plus additional conditions (Ren 1999). Obviously, these criteria are norm dependent: They cannot guarantee the stability of disturbances of other norms. For instance, even if these criteria are satisfied, \( E \) can still grow, according to Eq. (2.7), if lines of constant \( \eta' \) slop north-west to south-east in regions where \( dU_G/\partial y > 0 \), and north-east to south-west in regions where \( dU_G/\partial y < 0 \). This is the physical reason for the growth of continuous-spectrum disturbances.

3. EVOLUTION OF DISTURBANCES

(a) Solution type in the SWSG model

The linearized equation of the SWSG model can be rewritten in terms of \( q' \) as

\[
\frac{\partial q'}{\partial t} + U_G \frac{\partial q'}{\partial x} = -\Xi \mathcal{L}^{-1} \frac{\partial q'}{\partial x} \equiv -Gq'.
\]

(3.1)

As long as \( \Xi \) is not a constant, \( G \) is not a self-adjoint operator. Consequently, the linearized equation of the SWSG model contains three kinds of solutions (Richtmyer 1978): Continuous spectra whose shapes are not preserved during the evolution and whose amplitudes go to zero as time goes to infinity; regular normal modes, including unstable, damped and neutrally stable regular normal modes, which have the following form

\[
q'(x, y, t) = \phi(y) \exp\{ik(x - ct)\},
\]

where \( \phi \) is the amplitude of \( q \), and marginally stable modes whose phase speed is equal to the value of basic flow at the points where \( \Xi = 0 \). In what follows, we include marginally stable modes in the catalogue of continuous spectra.

As with the QG model, because \( G \) is not a self-adjoint operator, there is no orthogonality in the linear SWSG model; this is a mathematical reason why continuous-spectrum disturbances can grow by extracting energy from a stable basic flow.
(b) Evolution of disturbances

By introducing the integral propagator $L = L(t, t_0)$, where $t_0$ is the initial time, the solution of $\eta'$ at time $t$ can be formally written as

$$\eta'(t) = L(t, t_0)\eta'(t_0).$$  \hfill (3.2)

According the discussion above, $L$ can be further decomposed as $L = L_{\text{nm}} + L_{\text{cs}}$, where $L_{\text{nm}}, L_{\text{cs}}$ correspond to integral propagators of regular normal modes and continuous spectra, respectively.

Let us consider the special case in which basic flow is (normal mode) stable, namely, that $dQ/dy$ is sign definite within the domain. In this case continuous spectra can still grow. Unlike the exponential growth of regular normal-mode disturbances, the growth of continuous-spectrum disturbances is algebraic. Obviously, according to Eq. (2.4), this kind of growth is initial-value dependent and bounded. The upper bounds on the growth of enstrophy and energy can be estimated, respectively, from Eq. (2.4) as

$$\int \tilde{q}^2(t_0) \, dy \leq \frac{\Sigma_{\text{max}}}{\Sigma_{\text{min}}} \int \tilde{q}^2(t_0) \, dy,$$  \hfill (3.3)

$$\mathcal{E}(t) \leq \frac{\Sigma_{\text{max}}}{\Sigma_{\text{min}}} \int \frac{\tilde{q}^2(t_0)}{\kappa_0^2} \, dy$$  \hfill (3.4)

where $\kappa_0 = \min (\int E \, dy / \int \eta^2 \, dy)$.

(c) Evolution of continuous spectra in a slowly varying background

If the change of basic-state variables is rather slow over the spatial scale, namely, that $H$ remains approximately unchanged, and

$$U_G \approx U_{G0} + \gamma y + \cdots,$$  \hfill (3.5)

where $U_{G0}$ and $\gamma$ are constants, then $\Sigma, \rho_1$ and $\rho_2$ can be regarded as constants, approximately. Under these approximations, the wave-packet theory can be applied (for details of the application of this theory in QG dynamics, one can refer to the work of Zeng (1983) and Tung (1983)). According to this theory, the dispersion relation of a wave packet with wave number peaking around $k$ and $n$ in the linearized SWSG model is

$$\omega = k \left( U_G - \frac{\Sigma}{\mu^2} \right),$$  \hfill (3.6)

where $\mu^2 = \rho_1 k^2 + \rho_2 n^2 + F$. Thus, the evolution equations for ridges and troughs are

$$\frac{dx}{dt} = -\frac{\partial \omega}{\partial x} = 0,$$  \hfill (3.7)

$$\frac{dn}{dt} = -\frac{\partial \omega}{\partial y} = -k \frac{dU_G}{dy} = -\gamma k,$$  \hfill (3.8)

where

$$\frac{dg}{dt} = \frac{\partial}{\partial t} + c_{gx} \frac{\partial}{\partial x} + c_{gy} \frac{\partial}{\partial y},$$

and

$$c_{gx} = \frac{\partial \omega}{\partial k}, \quad c_{gy} = \frac{\partial \omega}{\partial n}$$
are group velocities in the $x$ and $y$ directions respectively.

For an observer moving with the wave packet, according to Eq. (3.8)

$$n(t) = n_0 - \gamma kt,$$  \hspace{1cm} (3.9)

where $n_0$ is the initial value of $n$. This relation indicates that, during the evolution, the meridional structure of disturbances becomes *globalized* during the time interval $(0, n_0/\gamma k)$ due to the decrease in meridional wave number, and becomes *localized* during the time interval $(n_0/\gamma k, \infty)$ due to the increase in meridional wave number. With these relations and the conservation equation (2.4), the evolution of potential energy and energy $E$ of the wave packet can be described approximately as follows.

According to the pseudo-momentum conservation equation

$$\int \int \mu(t)^4 \eta_{wp}'(x, y, t)^2 \, dx \, dy = \int \int \mu(0)^4 \eta_{wp}'(x, y, 0)^2 \, dx \, dy,$$  \hspace{1cm} (3.10)

where $\mu^2(t) = \rho_1 k^2 + \rho_2 (n_0 - \gamma kt)^2 + F$, $\eta_{wp}'$ is the perturbation of the free surface in the wave packet. Notice the approximation that $\Xi \approx \text{const}$ has been used. From Eq. (3.10), one obtains

$$P_{wp}(t) = \int \int F \eta_{wp}'(x, y, t)^2 \, dx \, dy = \left( \frac{\rho_2 n_0^2 + \rho_1 k^2 + F}{\rho_1 k^2 + \rho_2 (n_0 - \gamma kt)^2 + F} \right)^2 P_{wp}(0),$$  \hspace{1cm} (3.11)

and

$$\mathcal{E}_{wp}(t) = \int \int \mu(t)^2 \eta_{wp}'(x, y, t)^2 \, dx \, dy = \frac{\rho_2 n_0^2 + \rho_1 k^2 + F}{\rho_1 k^2 + \rho_2 (n_0 - \gamma kt)^2 + F} \mathcal{E}_{wp}(0),$$  \hspace{1cm} (3.12)

where $P_{wp}(0)$ and $\mathcal{E}_{wp}(0)$ are, respectively, the initial potential energy and initial total energy of the wave packet. Evidently, both of them increase during the time interval $(0, n_0/\gamma k)$, and decrease during $(n_0/\gamma k, \infty)$. It will be shown later that, although Eqs. (3.11) and (3.12) are only the results of wave packets based on the special assumptions, they are, however, in good agreement with the following numerical results based on relaxed assumptions about the variation of the basic state.

(d) Numerical calculations

In this part the bottom topography is assumed absent. Two kinds of zonal basic flow, constant-shear basic flow, $U_G = y$, $y \in [0, 1]$, and cosine-type basic flow, $U_G = U_s[1 + \cos \pi (y - 1)]$, $y \in [0, 2]$ (where $U_s \text{ is the amplitude of } U_G$), will be used in the numerical calculations. In the constant-shear basic-flow case, there is no normal-mode instability because

$$\frac{dQ}{dy} = \frac{R \beta + fFQy}{H},$$  \hspace{1cm} (3.13)

where $\beta$ is the derivative of $f$ in the $y$-direction, and is sign definite. For the cosine-type basic-flow case, however, the basic state might be unstable because

$$\frac{dQ}{dy} = \frac{R \beta + \pi^2 U_s \cos \pi (y - 1) + fFQU_s[1 + \cos \pi (y - 1)]}{H}$$  \hspace{1cm} (3.14)
is sign indefinite if

\[-1 < \frac{b + (f F Q/\pi^2)_{\text{min}}}{1 + (f F Q/\pi^2)_{\text{max}}} < 1,\]

where \(b = \beta/U_s \pi^2\), and

\[
H(y) = 1 - RoF U_s \left( y(1 + 0.5 Ro\beta y) + \frac{1}{\pi} (1 + Ro\beta y) \sin\{\pi(y - 1)\} \right.
\]

\[
+ \frac{Ro\beta}{\pi^2} \cos\{\pi(y - 1)\} + 1 \bigg). \tag{3.15}
\]

We study the constant-shear basic flow first. In this case, only continuous-spectrum disturbances can grow. In order to test the validity of relations (3.11) and (3.12), which are obtained from the wave-packet theory, we first choose, in the following numerical experiments, \(Ro = 0.2\), \(F = 1\), \(\beta = 1\) so that the variation of

\[
H = 1 - RoF \left( 0.5 y^2 + \frac{\beta Ro y^3}{3} \right) \tag{3.16}
\]

is very small. In the first experiment, \(n_0 = 4\pi, k = 4\). According to Eqs. (3.11) and (3.12), both potential energy and total energy would increase during the time interval \((0, \pi)\) and decrease after \(t = \pi\). The solid lines in Figs. 1(a) and (b) corresponding to these values show that the results agree with this conclusion to a very high degree. Another relation, Eq. (3.9) from the wave-packet theory, is also verified by the solid line
in Fig. 1(c). In Fig. 1(c) the meridional wave number reaches its minimum (\(\pi\)) at \(t = \pi\), and then decays. In the second experiment, \(k = 1, n_0 = 4\pi\), the results (not shown) show that indeed both potential energy and total energy reach their maximum at \(t = 12 \approx 4\pi\) and then decay.

To test the validity of the relations (3.11) and (3.12) when the underlying assumptions are relaxed further, we choose, in the following two experiments, \(Ro = 0.2, F = 5\), and \(\beta = 1\). With these parameters, the variation of \(H\) is large. With the same values of \(n_0\) and \(k\) as in the previous experiment, the results plotted as dashed lines in Fig. 1 show that Eqs. (3.11) and (3.12) are still good approximations for the evolution of the potential and total energy of continuous-spectrum disturbances. A close comparison between the solid lines and the dashed lines in Fig. 1 indicates that for a larger \(F\), the time at which the energy reaches a maximum slightly increases; thus \(n_0/k\) is still a good measurement of the time-scale of the growth of continuous-spectrum disturbances.

Now we consider the cosine-type basic-flow case. In this case we choose \(\beta = 1, F = 1, Ro = 0.2\) and \(U_s = 0.8\). With these parameters, \(dQ/dy\) changes its sign within the domain. The numerical results show that regular normal modes with wave numbers between 0 and 4.5 are unstable. The growth rate of regular normal modes has a wavenumber preference; At wave number 2.5, the unstable growth rate reaches its maximum.

The basic state in this case changes rapidly in the domain, therefore, the wave-packet theory is no longer valid. Due to the coexistence of regular normal modes and continuous spectra, potential/total energy contains three parts, namely, the energy corresponding to regular normal modes, continuous spectra, and the interaction between them.
To show the relative importance of regular normal modes and continuous spectra during the evolution, two numerical experiments are carried out. In the first experiment, we choose the initial value of $\eta'$ with horizontal wave number $k = 2.5$ corresponding to the fastest growing regular normal modes. Therefore, regular unstable normal modes might dominate the evolution process. Fig. 2 indicates that this is indeed the case; total and potential energy corresponding to continuous spectra (Figs. 2(a) and (b)) is even smaller than that corresponding to the interaction between regular normal modes and continuous spectra. The total and potential energy of whole disturbances increase due to the growth of normal modes (Figs. 2(b) and (d)). Fig. 3, showing the structures of regular normal-mode, continuous-spectrum, and total disturbances, indicates that regular normal-mode disturbances have a global structure while continuous-spectrum disturbances have a localized structure most of the time during their evolution.

In the second experiment, the horizontal wave number of initial value of $\eta'$ is taken as 4 so that the normal-mode instability is very weak. With this wave number, continuous spectra dominate the evolution process. Numerical results (not shown) clearly show that the total and potential energy corresponding to regular normal-mode disturbances is much smaller than that corresponding to continuous-spectrum disturbances, and the interaction between them varies a lot. Thus, we conclude from the two experiments that the relative importance of regular unstable normal modes and continuous spectra in the evolution process is determined by the choice of initial value.
4. SINGULAR VECTORS AND SINGULAR VALUES

As mentioned above, an important feature of the continuous-spectrum instability mechanism is its dependence on the configuration of initial disturbances. A natural question is whether it is possible to find initial disturbances based on which the growth of some variables with a (sign-definite) quadratic form, such as energy and enstrophy, can be optimized over a finite time interval. Detailed analyses (Lorenz 1965; Farrell 1988) show that the initial value is equivalent to the eigenvector with the largest eigenvalue of the singular operator (see the definition below).

(a) Inner product

First let \((\cdot, \cdot)\) denote an inner product, namely

\[
(x, y) = (E_1 x, E_2 y),
\]

where \((\cdot, \cdot)\) is the canonical Euclidean scalar product, and \(E_1, E_2\) are some weight factors. In this work, the definitions for \(E_1, E_2\) are different in different norms (metrics).

(b) Norm definitions

As with other models in geophysical fluid dynamics, the choice of norm in the SWSG model is not unique (Palmer et al. 1998). Since \(\eta'\) is the only variable in Eq. (2.1), it is easy to construct the following norms.

(1) Potential-energy norm:

The definition of the potential-energy norm is

\[
(\eta', \eta')_p = (\eta', \eta') = (L\eta'(t_0), L\eta'(t_0)) = (L^* L \eta'(t_0), \eta'(t_0)),
\]

where \(\eta'(t_0)\) is \(\eta'\) at initial time, \(L^*\) is the complex-conjugate of \(L\), and \(L^* L\) is called the singular-value operator of the potential-energy norm, which is Hermitian.

(2) Enstrophy norm:

The definition of the enstrophy norm is

\[
(q', q')_e = (q', q') = (\mathcal{L} \eta'(t_0), \mathcal{L} \eta'(t_0))
\]

\[
= (\mathcal{L} \mathcal{L}^{-1} q'(t_0), \mathcal{L} \mathcal{L}^{-1} q'(t_0)) = (L_e^* L_e q'(t_0), q'(t_0)),
\]

where \(L_e = \mathcal{L} \mathcal{L}^{-1}\). Again \(L_e^* L_e\) is Hermitian. According to the conservation of pseudo-momentum, \(L_e^* L_e\) is approximately an unitary operator when the variation of \(\Xi\) is very small.

(3) Energy norm:

Since \(\mathcal{L}\) is a real self-adjoint operator, it can be (Cholesky) factorized as

\[
\mathcal{L} = B^T B.
\]

Therefore, the energy norm can be expressed in the following form

\[
(\eta', \eta')_e = -(\eta', \mathcal{L} \eta') = -(B \eta', B \eta') = -(L_B^* L_B B \eta'(0), B \eta'(0)),
\]

where \(L_B = B L B^{-1}\), and \(L_B^* L_B\) is a Hermitian operator.

(4) Kinetic-energy norm:

The definition of the kinetic-energy norm is almost the same as the definition above except that \(\mathcal{L}\) is replaced by \(\mathcal{L} - F\).
(5) Other norms:

The total energy of geostrophic disturbances can be expressed as

\[ E_g = H(u_g'^2 + v_g'^2) + F\eta'^2. \]

With this formula the geostrophic-energy norm can be defined as

\[ \langle \eta', \eta' \rangle_{ge} = -\langle \eta', \tilde{L}\eta' \rangle = -\langle \tilde{B}\eta', \tilde{B}\eta' \rangle = -\langle \hat{L}_B^*\hat{L}_B B\eta'(0), B\eta'(0) \rangle, \quad (4.5) \]

where

\[ \tilde{L} = -k^2 H + \frac{\partial}{\partial y} \left( H \frac{\partial}{\partial y} \right) - F, \quad \hat{L} = \hat{B}^T \hat{B}. \quad (4.6) \]

(c) Singular vectors and singular values

Once the singular-value operator \( L^*L \) is obtained, SVs and SVEs can be computed from the following equation

\[ L^*Lv = \sigma v, \quad (4.7) \]

where \( \sigma \) is the square of the SVs and \( v \) the initial SVEs for optimization time (Noble and Daniel 1977). Since \( L^*L \) is a Hermitian operator, \( \sigma \) must be a distinct real number, and \( v \) must be orthogonal and the set of \( v \) is complete. Obviously, \( \sigma \) and \( v \) are norm dependent. Among all the singular vectors, we are only interested in a few dominant SVEs with large SVs at initial time and at optimization time (SVE at optimization is \( Lv \)).

The largest SV is a measure of the ratio of the corresponding norm at optimization time to that at initial time, namely,

\[ \langle \eta', \eta' \rangle(t) = (L^*L\eta', \eta')(0) \leq \sigma_1(\eta', \eta')(0). \quad (4.8) \]

Therefore, the growth of disturbances can be optimized if the largest singular vector is used as the initial disturbance.

Since \( L \) is determined by the basic state, according to Eq. (4.7), \( \sigma_1 \) is also related to the basic state. To explore this relation, we first derive two inequalities for the energy and enstrophy norms.

For the energy norm, it can be proved, according to Eq. (2.7), that

\[
\frac{d\mathcal{E}(t)}{dt} \leq \int \sqrt{\frac{\rho_2}{k^2 \rho_1 + F}} \frac{dU_G}{dy} \left[ k^2 \rho_1 \eta'^2 + \rho_2 (\partial_y \eta')^2 + F \eta'^2 \right] dy \\
\leq \left( \sqrt{\frac{\rho_2}{k^2 \rho_1 + F}} \frac{dU_G}{dy} \right)_{\text{max}} \mathcal{E}(t). \quad (4.9)
\]

Therefore,

\[ \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \leq \exp \left( \sqrt{\frac{\rho_2}{k^2 \rho_1 + F}} \frac{dU_G}{dy} \right)_{\text{max}} t. \quad (4.10) \]

The right-hand side of Eq. (4.10) can be regarded as an approximate estimate of the first singular value of the energy norm, therefore, analyzing this term may give us some insight into the relation between the first singular value of the energy norm and the basic state (see section 5(c)).
Figure 4. For the constant-shear basic-flow case, the singular values at different optimization time with (a) the potential-energy norm, (b) the enstrophy norm, (c) the kinetic-energy norm, and (d) the total-energy norm. Black circles, triangles, and squares correspond to the first, second, and third singular values, respectively.

For the enstrophy norm, we have the following inequality based on Eq. (2.1)

$$\frac{d}{dr} \int \bar{\theta}^2 \, dy = \frac{d\mathcal{V}(t)}{dr} = \int \rho_2 \frac{d\Xi}{dy} \frac{\partial_x \eta' \partial_y \eta'}{\rho_1 + F} \, dy \leq \left( \frac{\rho_2}{k^2 \rho_1 + F} \frac{d\Xi}{dy} \right)_{\text{max}} \mathcal{E}(t).$$

(4.11)

Using the relation (3.4) one has

$$\frac{\mathcal{V}(t)}{\mathcal{V}(0)} \leq \exp \left( \frac{1}{k_0} \sqrt{\frac{\rho_2}{k^2 \rho_1 + F}} \frac{d\Xi}{dy} \right)_{\text{max}} t.$$

(4.12)

Again, the right-hand side of Eq. (4.11) can be regarded as an approximate estimate of the first singular value of the enstrophy norm. If the spatial variation of $\Xi$ is slow then the SVs of the enstrophy norm are approximately one.

A comparison between Eqs. (4.10) and (4.11) indicates that the first singular values of the energy norm and enstrophy norm would be quite different: The former does not depend on the gradient of basic-state potential vorticity while the later does. The numerical results in the next section will show this is indeed the case.

5. Numerical calculations of singular vectors and singular values

(a) Singular vectors and singular values of different norms

First, we calculate the SVs and SVs for the constant-shear basic-flow case with $Ro = 0.2$, $\beta = 1$, $F = 1$ and $k = 3$. Fig. 4 shows the singular values of different norms.
Figure 5. Amplitude of the first singular vectors in the constant-shear basic-flow case with (a) the potential-energy norm, (b) the enstrophy norm, (c) the kinetic-energy norm, and (d) the total-energy norm. The solid lines and dashed lines correspond, respectively, to initial time and optimization time $t = 1$. The dashed lines in (a), (c), and (d) have been scaled by 0.15.

at different optimization times. Among Figs. 4(a)–(d), which correspond to the SVs of the potential-energy, enstrophy, kinetic-energy, and total-energy norms, respectively, the SVs of the total-energy norm increase most rapidly, and the SVs of the enstrophy norm remain almost constant with time. The latter fact can be easily explained by Eq. (4.11). In this case the right-hand side of Eq. (4.11) is almost unity due to the small variation of $E$, therefore, the first singular value of the enstrophy norm changes very little over time.

A common feature of Figs. 4(a), (b) and (c) is that the first SV always dominates the other SVs of the same norm after optimization time 2.

Figure 5 shows the amplitudes of the first SVs of different norms. For the energy-type norms, including the potential-, kinetic-, and total-energy norms, the amplitude of the first SV has a global structure with only one maximum whose location shifts to the north with optimization time. Although many small variations in the first SV of the enstrophy norm can be found from Fig. 5(b), the amplitude as a whole also has a global structure. For the second and third SVs (not shown), the amplitude has two and three maxima respectively for the energy-type norms. Again, for the enstrophy norm, many small variations of amplitude of the second SV are found superimposed on two maxima, and there is no clear maxima for the third SV.

The work of Palmer et al. (1998) on the structure of the first singular vector using real data shows when the jet in the atmosphere is strong, the position of the maximum of the vertical shear of basic flow is an index of the location of the first few SVs. To see if this is also true for horizontal shear in the SWSG model, the cosine-type basic flow
will be used in the following experiments. In this case, $|dU_G/dy|$ reaches its maximum at two points: $y = 0.5$ and $y = 1.5$.

We first calculate the SVs of different norms at different optimization times. The results are shown in Fig. 6. Like the constant-shear basic-flow case, the first SVs of all the norms increases much faster than the rest of the SVs. Among all the first SVs of the energy-type norms, the first SV of the potential-energy norm increases most rapidly. However, unlike the constant-shear basic-flow case, the first SV of the enstrophy norm does not remain as a constant, but increases with time (Fig. 6(c)) due to the sign indefiniteness of $\Xi$.

The amplitude of the first SVs of different norms is shown in Fig. 7. For all norms, the amplitude of the first SV at initial time has only one maximum located at the (southern) maximum of $|dU_G/dy|$. As time increases this location of the energy-type norm shifts to the location of the peak of the basic flow ($y = 1$). However, for the first SV of the enstrophy norm, this location does not change much with time (Fig. 7(b)). For the second SV of all the norms (not shown), the location of the maximum of amplitudes shifts to another location of the (northern) maximum of $|dU_G/dy|$. For the third SV (not shown), two maxima of amplitude of SVs of the energy-type norms emerge around the (southern) location of the maximum of $|dU_G/dy|$ while two maxima of amplitude of the third SV of the enstrophy norm appear at the northern part of the jet. Thus, our results seem to be different from those of Palmer et al. Our results suggest that the maximum of the first SVs of the energy-type norm tends to match the centre of the jet, but not the centre of the shear of the jet. Nevertheless, for the second and third SVs, and SVs of the enstrophy norm, our results agree with the results of Palmer et al. In order to
understand the relation between the vertical shear of the basic flow and the centre of the optimal SVe, a two-layer shallow-water semi-geostrophic model (Ren 2000a) should be used.

\((b)\) Wave-number preference of singular vectors and singular values

In section 3 we showed that unstable normal modes in the cosine-type basic-flow case have a clear wave-number preference. In this part we will investigate the preference of the SVs and SVs of different norms for wave number.

Figure 8 shows the spectrum of SVs for the constant-shear basic-flow case. It shows clearly that for the energy-type norms, large SVs correspond to small scales. For the enstrophy norm, however, the SVs do not have a clear wave-number preference due to the small variation in \(E\). Results also show that the amplitudes of the first SVe of the energy-type norms do not change much with wave number. For the enstrophy norm, however, the amplitude of the first SVe at small scales is quite different from that at large scales.

The spectrum of the first three SVs of different norms in the cosine-type basic-flow case is shown by Fig. 9. It is clear from this figure that the peak of the spectrum of the first SVA corresponds to wave number 4.5 for the energy-type norms, and to wave number 2.5 for the enstrophy norms. A close comparison between Fig. 9 and the preference of unstable normal modes (section 3(d)) indicates that unstable normal modes have almost the same wave-number preference as the first SVA of the enstrophy
norm, but have totally different wave-number preference from the first SVAs of the energy-type norms.

In spite of the large variation of the first SVAs of the energy-type norms with wave number, the amplitude and location of the corresponding SVEs do not change much with wave numbers (not shown). However, the amplitudes of the first SVE of the enstrophy norm with different wave numbers are quite different.

(c) The impact of the Rossby and rotating Froude numbers on singular vectors and singular values

Two important parameters in the SWSG model are the Rossby and rotating Froude numbers whose variation can affect $\rho_1$, $\rho_2$ and $\Xi$ in the governing equation and thus affect SVEs and SVAs. This has been shown very clearly by Eqs. (4.10) and (4.12) for the energy and enstrophy norms, respectively.

In order to keep the validity of the SWSG model, the Rossby number in the following experiments is less than 0.4 and the rotating Froude number is within the range in which $H$ is positive.

We first discuss the influence of the Rossby number on the SVAs and SVEs in the constant basic-flow case. According to Eq. (4.10), the first SVA of the total-energy norm is approximately proportional to the ratio of $\rho_2$ to $\rho_1$ because $k^2 \rho_1 \gg 1$. Since $\rho_2/\rho_1$ increases as $Ro$ increases, the first singular value of the total-energy norm is expected to increase with the Rossby number. Also, according to Eq. (4.11), the first SVA of the enstrophy norm will remain almost unchanged. The numerical results of (b) and (d)
in Fig. 10 confirm these two conclusions. In Fig. 10(d), the optimal SVa of the total-energy norm indeed increases with the Rossby number. However, its sensitivity to the Rossby number is weaker than that of the optimal SVa of the potential-energy norm to the Rossby number. The numerical results also show that the change in the Rossby number has little impact on the structures of the SVes (not shown).

In the cosine-type basic-flow case, however, the first SVa of all the norms changes rapidly with Rossby numbers larger than 0.3, as shown by Fig. 11. (The first SVa corresponding to $Ro = 0.4$ is many times larger than that corresponding to $Ro = 0.1$.) In this case it is hard to apply Eqs. (4.10) and (4.11) due to the complex forms of $U_G$ and $E$.

Variation of the Rossby number has some impact on the structure of the first SVs at initial times of different norms: The large Rossby number makes the amplitude of the first SVes of enstrophy, total- and kinetic-energy norms less concentrated around $y = 1.5$.

To investigate the influence of the rotating Froude number on the first SVs of the total-energy norm in both constant-shear and cosine-type basic-flow cases, we notice that in Eqs. (3.15) and (3.16), the increase in $F$ is equivalent to the decrease in $H$. Thus, according to Eq. (4.10), the large $F$ corresponds to the small first SVs of the energy norm. Figures 12(a) and 13(a) show that this is exactly the case. The results in the two figures also show that this is also true for the first SVs of kinetic- and potential-energy norms (see (c) and (d) in Figs. 12 and 13). Figure 12(b) shows that in the constant-shear basic-flow case, the first SVa of the enstrophy norm increases with the increase in $F$ and this agrees with the analysis of Eq. (4.11). On the right-hand side of Eq. (4.11), the term in the square root decreases with the increase in $F$; however, $dE/dy$
Figure 10. The variation of singular values with the Rossby number in the constant-shear basic-flow case. (a)–(d) correspond to the potential-energy norm, the enstrophy norm, the kinetic-energy norm, and the total-energy norm, respectively.

Figure 11. As in Fig. 10 except for the cosine-type basic-flow case.
Figure 12. The variation of singular values with the rotating Froude number in the constant-shear basic-flow case. (a)–(d) correspond to the potential-energy norm, the enstrophy norm, the kinetic-energy norm, and the total-energy norm, respectively.

Figure 13. As in Fig. 12 except for the cosine-type basic-flow case.
is approximately proportional to $F$ and thus increases more rapidly with the increase in $F$. Therefore, large $F$ corresponds to the large first SVs of the enstrophy norm. In the cosine-type basic-flow case, however, the variation of $d\Sigma/dy$ is more complicated due to the complicated spatial structure of $U_G$. Detailed analysis shows that $d\Sigma/dy$ can decrease with $F$ with some choices of $Ro$ and $\beta$. With the parameters used in Fig. 13 $d\Sigma/dy$ obviously decreases with $F$ at some points within the domain and consequently the first SVs of the enstrophy norm decreases with $F$. The numerical results also show that the rotating Froude number has almost no impact on the amplitude of SVs (not shown).

(d) The impact of topography on singular vectors and singular values

Buiuza (1995) studied the impact of orographic forcing on barotropic unstable SVs. He found that the presence of orographic forcing has both direct and indirect impacts on the structure of SVs. Since in our work the basic flow is a given steady variable, only the direct impact of the topography on SVs can be studied. In the following experiments, the topography is assumed to have the following form

$$h_B = h_m \{1 - \frac{1}{2}(y - 0.5)^2\},$$

where $h_m$ is the amplitude of the topography. As mentioned above, the advantage of the SWSG model over the QG model is that the height of topography can be arbitrary as long as $h$ is positive. Thus, by changing $h_m$, we can investigate the impact of $h_B$ on SVs. Since the increase in $h_B$ is equivalent to the increase in $F$ in both constant-shear
and cosine-type basic-flow cases, it is expected that the change in the first SVAs of all norms with $h_B$ would follow the same pattern as Fig. 12 and Fig. 13.

In the following experiments, we change the amplitude of the topography from 0.05 to 0.4 for both constant-shear basic-flow and cosine-type basic-flow cases. It turns out that in the first case, the amplitude of the topography has a small impact on the first three SVAs of all the norms (Fig. 14). In the second case, however, the first SVAs of the enstrophy, total- and kinetic-energy norms change by about 50% and about 20% for the potential-energy norm when $h_m$ varies over 0.05 to 0.4 (Fig. 15). In a comparison between Fig. 11 and Fig. 13, one can see easily that, except in the total-energy norm case, the change of the first SVAs with $h_B$ has the same pattern as the variation pattern of the first SVAs with $F$. However, Fig. 12 and Fig. 14 show clearly that for all norms the change of the first SVAs with $h_B$ in the cosine-type basic-flow case has the same pattern as that with $F$. In both cases the change of $h_m$ has little impact on the structures of SVEs of different norms (not shown).

6. Conclusions

In order to understand the instability mechanism of disturbances in coastal flow in the oceans as well as in the initial development stage of frontal waves in the atmosphere, we have investigated in this paper the evolution of general disturbances in the SWSG model. As with the QG theory, there are two kinds of instability mechanism in the SWSG model: Algebraical growth of continuous-spectrum disturbances and exponential growth of unstable normal-mode disturbances. The first kind of growth depends on the configuration of the initial value of $\eta'$ (the ratio of meridional wave number to horizontal
wave number) as well as the shear of the basic flow while the second kind of growth depends on the basic-state only. Both can be important in anisotropic motions.

For slowly varying basic flow satisfying the stability criterion, the amplification of potential and total energy as well as the evolution of the meridional structure of disturbances can be described reasonable well by the wave-packet theory. For unstable basic flow, however, the relative importance of continuous-spectrum and regular normal-mode disturbances can only be identified using numerical methods.

The special properties of the SWSG model make it a good tool to explore the relation between the optimal growth of disturbances and model parameters such as the Rossby and rotating Froude numbers, and the amplitude of bottom topography on the structure of SVs and corresponding SVs.

Numerical results show that the structures of SVs of different norms are largely determined by the configuration of the basic flow, and the variations of parameters have little impact on them. In the cosine-type basic-flow case the optimal SVs of the energy-type norms always concentrate around the peak of the basic flow. The optimal SVs of the enstrophy norm tend to appear at the location where the shear of the basic flow reaches its maximum. Since the Rossby number is the measure of the ratio of the (slaved) ageostrophic component of velocity to the geostrophic component one, our results seem to support the conclusion of Errico (2000) which is based on a non-balanced model that the ageostrophic effect does not have a great impact on the structure of SVs.

Our results also show that SVs are sensitive to the choice of metrics (norms). Among all the optimal SVs of different norms, the optimal SVa of the enstrophy norm changes slowly while the optimal SVs of the energy-type norm change rapidly. For all norms, the optimal SVa is most sensitive to the variations of the model parameters. Both analytical results and numerical experiments on the sensitivity of SVs to the model parameters indicate that SVs can grow rapidly in an environment with a large Rossby number, strong jet strength, and small rotating Froude number.

Finally, it is found in our work that the optimal SVs of the energy-type norms have different wave-number preferences from unstable regular normal modes. In the cosine-type basic-flow case, the peak of the spectra of the optimal SVs of the energy-type norms occurs at the wave number corresponding to weak unstable normal modes or to neutrally stable normal modes. However, the optimal SVa of the enstrophy norm has almost exactly the same wave-number preference as the unstable normal modes; both of them reach the peak at the same wave number, although they have totally different spatial structures.

In the SWSG model, baroclinicity is only partially represented by the variation of the free surface. To fully understand the relation between SVs/SVs and baroclinicity, especially the relation between the vertical shear of the basic flow and the structure of the optimal SVa, a two-layer SWSG model (Ren 2000b) should be used. This will be done in our future work.

**ACKNOWLEDGEMENTS**

I am grateful to Dr Saroja Polavarapu for valuable discussions and continual encouragement. This work is supported by the Natural Sciences and Engineering Research Council (NSERC) Visiting Fellowship and the Meteorological Service of Canada.

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