Salmon’s Hamiltonian approach to balanced flow applied to a one-layer isentropic model of the atmosphere

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SUMMARY

Salmon’s Hamiltonian approach is applied to formulate a balanced approximation to a hydrostatic one-layer isentropic model of the atmosphere. The model, referred to as the parent model, describes an idealized atmosphere of which the dynamics is closely analogous to a one-layer shallow-water model on the sphere. The balance used as input in Salmon’s approach is a simplified form of linear balance, in which the balanced velocity $v_b$ is given by $v_b = k \times \nabla f^{-1}(M - \bar{M})$. Here $k$ is a vertical unit vector, $f$ is the Coriolis parameter, $M$ is the Montgomery potential and $\bar{M}$ is the value of the Montgomery potential at the state of rest. This form of balance is used in preference to standard geostrophic balance, $v_b = k \times f^{-1} \nabla M$, which forces the meridional wind velocity to be zero at the equator. Salmon’s Hamiltonian technique is applied to obtain an equation for the time rate of change of the balanced velocity that guarantees both the material conservation of potential vorticity as well as conservation of energy. New in this application of Salmon’s approach is a nonlinear relation between Montgomery potential and surface pressure (characteristic for an isentropic ideal gas in hydrostatic equilibrium) in combination with spherical geometry and a variable Coriolis parameter. We discuss how the unbalanced velocity $v_u$ can be calculated in a practical way and how the model can be stepped forward in time by advecting the balanced potential vorticity with the total velocity $v = v_b + v_u$. The balanced model is tested against a ten-day integration of the parent model.

KEYWORDS: Balanced model Hamilton’s principle Isentropic atmosphere

1. INTRODUCTION

In the present paper we apply Salmon’s Hamiltonian approach (Salmon 1983, 1985, 1988a, 1988b, 1996) to obtain a balanced approximation of a hydrostatic one-layer isentropic model of the atmosphere. The dynamics of an isentropic layer—a layer with uniform potential temperature—can be derived directly from the hydrostatic primitive equations on a rotating sphere (Verkley 2000a). For the one-layer (parent) model discussed in the present paper, the governing equations are: an equation for the time rate of change of horizontal velocity (horizontal momentum per unit mass), an equation relating the Montgomery potential to the surface pressure, and an equation stating the conservation of mass. Due to the assumed constancy of potential temperature and the assumption of hydrostatic equilibrium, the absolute temperature in the model decreases linearly with height following the dry adiabatic lapse rate. The pressure and density decrease with height according to simple power laws, whereas the Montgomery potential and the horizontal velocity are independent of height. The air in the model, therefore, moves column-wise between two material surfaces: a lower boundary with height $z_1$ (given by the earth’s orography) and an upper boundary with height $z_u$ (determined by the condition of zero pressure). Due to the linear decrease of temperature with height, the value of $z_u$ is finite. Apart from the nonlinear relation between Montgomery potential and surface pressure, the model equations are identical to the one-layer shallow-water equations on a rotating sphere with orography.

Our interest in this particular model, instead of the more familiar shallow-water model, is that discretization of the atmosphere in terms of isentropic layers strains reality less than discretization in terms of isopycnic layers. This is particularly true if one is interested in models with a relatively small number of layers. Although very idealized, a single-layer atmosphere with constant potential temperature, evolving in

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time according to the equations mentioned above, is an exact solution of the inviscid hydrostatic primitive equations. Our confinement to a single layer is not necessary but is motivated by practical considerations. We wish to concentrate on the new aspects that one encounters in applying Salmon’s method to this type of model: a nonlinear relation between Montgomery potential and surface pressure, spherical geometry, and a varying Coriolis parameter. Furthermore, instead of following Salmon by taking the standard geostrophic relationship \( v_0 = k \times f^{-1} \nabla M \) as input in the method, we use for this purpose a simplification of linear balance: \( v_0 = k \times \nabla f^{-1} (M - \bar{M}) \), where \( k \) is a vertical unit vector, \( f \) is the Coriolis parameter, \( M \) is the Montgomery potential and \( \bar{M} \) is the value of the Montgomery potential at the state of rest. The reason for doing this is that standard geostrophy, when applied globally, implies that the meridional wind velocity is zero at the equator. Our choice avoids this and, in addition, leads to a set of equations for the unbalanced velocity that is simpler than the corresponding set in the case of standard geostrophy.

In section 2 we give the equations that govern the parent model, discuss its conservation laws and give Hamilton’s principle for the model. In section 3 Salmon’s method is applied to construct a balanced approximation. The central results are the momentum equation (22) and the set of equations (30)–(31) that determine the unbalanced velocity. It is verified that the balanced model has equivalents of the original conservation laws. In section 4 we discuss in more detail how the balanced model can be integrated forward in time. We have chosen to use the material conservation of balanced potential vorticity as our basic prognostic equation, and are thus confronted with an inversion problem that consists of two parts: solving for the balanced velocity from the balanced potential vorticity (which is a nonlinear problem) and solving for the unbalanced velocity from the balanced variables (which is a linear problem). In this respect our model is analogous to the ‘slow equations’ of Lynch (1989). In section 5 we discuss a long integration of the parent model, concentrating on a particular period of ten days in which cyclogenesis occurs. It is checked whether the inversion procedure is able to reproduce from the potential vorticity field the flow field at the beginning of this period. We then integrate the balanced model forward in time for the same period of ten days. The balanced model describes the time evolution in this period very accurately. By performing an integration in which the unbalanced velocity is put to zero, we demonstrate that inclusion of the latter is crucial for maintaining high accuracy. A summary and discussion, in which the present work is placed in a broader perspective, can be found in section 6.

2. The Parent Model

The parent model is based on the hydrostatic primitive equations with no thermodynamic heating/cooling nor mechanical forcing/friction. These equations can be written in the form of three prognostic and three diagnostic equations. The prognostic equations are the conservation of thermodynamic energy, the conservation of mass and the time rate of change of horizontal momentum. The diagnostic equations are the definition of potential temperature, the ideal-gas law and the hydrostatic approximation. The basic assumption of the parent model is that the potential temperature \( \theta \) is uniform throughout the atmosphere. Here \( \theta = T (p_i / p)^{\kappa} \), where \( T \) is the absolute temperature, \( p \) is the pressure, \( p_i \) is a reference pressure of 1000 hPa and \( \kappa = R / c_p \), where \( R \) is the gas constant for dry air and \( c_p \) is the specific heat of dry air at constant pressure. By combining the definition of potential temperature with the ideal gas law (\( \rho = p / (RT) \)), where \( \rho \) is the density) and the hydrostatic approximation (\( \partial p / \partial z = -\rho g \), where \( z \) is the height above mean sea level and \( g \) is the acceleration due to gravity), it can be verified easily that
the absolute temperature $T$ decreases linearly with height according to the dry adiabatic lapse rate $-g/c_p$. The pressure and density in a layer with uniform potential temperature decrease according to simple power laws of the absolute temperature: $p/p_r = (T/\theta)^{1/\kappa}$ and $\rho/\rho_r = (T/\theta)^{1/(\kappa-1)}$. Here $\rho_r$ is the density of the air at the reference pressure $p_r$, i.e., $\rho_r = p_r/(R\theta)$. Because the absolute temperature decreases linearly with height, a natural upper boundary is the (finite) height $z_u$ at which the temperature, pressure and density are all equal to zero. The lower boundary is naturally formed by the height $z_1$ of the orography. A schematic cross-section of the model is given in Fig. 1.

We assume that the earth is perfectly spherical with radius $a$ and that it rotates with angular velocity $\Omega$. Horizontal positions on the sphere will be denoted by $\lambda$ and $\phi$, where $\lambda$ is longitude and $\phi$ is latitude. The column-wise motion of the air is a consequence of the fact that the Coriolis parameter $f = 2\Omega \sin \phi$ and the Montgomery potential $M = gz + c_p T$ in the equation for the horizontal velocity $v$ are independent of height. The horizontal velocity can, therefore, also be assumed independent of height. As a consequence, the vertical advection of horizontal momentum is zero, so that the momentum equation reduces to

$$\frac{Dv}{Dt} + f \mathbf{k} \times \mathbf{v} + \nabla M = 0,$$

where $D/Dt$ is the horizontal advection operator for a two-dimensional vector field on a spherical surface with radius $a$. If the Montgomery potential is evaluated at $z = z_1$ we have

$$M = gz_1 + c_p \theta \eta_1^\kappa,$$

where we used the definition of potential temperature to express the absolute temperature at the lower boundary in terms of the pressure $p$. Variables at the upper and lower boundaries are denoted by the subscripts $u$ and $l$, respectively, and for ease of notation and dimensional convenience we introduced the normalized pressure

$$\eta \equiv p/p_r.$$  

An equation for the normalized surface pressure $\eta_1$ follows from mass conservation:

$$\frac{D\eta_1}{Dt} + \eta_1 (\nabla \cdot \mathbf{v}) = 0,$$

where $D/Dt$ now denotes the horizontal material derivative for a scalar. The significance of this equation can be appreciated by realizing that the air moves columnwise, with
material upper and lower boundaries, and that the total mass per unit horizontal area is given by \((p_i/g)\eta_l\) in a hydrostatic atmosphere with \(\eta_u = 0\) as an upper boundary. The model defined by (1), (2) and (4) forms a closed dynamical model in terms of \(v\), \(M\) and \(\eta_l\). It is closely analogous to a one-layer shallow-water model, to which it actually reduces by taking \(\kappa = 1\) and identifying \(\eta_l(c_p\theta)/g\) with the fluid depth \(H\) (uniform potential temperature is then equivalent with uniform density). In fact, in all equations that follow, we may obtain the shallow-water equivalents by substituting \(\kappa = 1\). More details on the dynamics of an isentropic layer in hydrostatic equilibrium—of which the present model is a particular case—can be found in Verkley (2000a).

(a) Conservation laws of the parent model

When we rewrite the mass conservation equation in flux form we have

\[
\frac{\partial \eta_l}{\partial t} + \nabla \cdot [\eta_l v] = 0. \tag{5}
\]

It then follows immediately that the total mass \(m\) is conserved, where \(m\) is given by

\[
m = \frac{p_r}{g} \int dS \eta_l. \tag{6}
\]

Here \(dS = a^2 \cos \phi \ d\lambda \ d\phi\) denotes an area element of the sphere. To investigate the conservation of potential vorticity and energy it is convenient to use the following formula for the material derivative of the horizontal velocity \(v\):

\[
\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + \zeta (k \times v) + \nabla \left( \frac{v \cdot v}{2} \right). \tag{7}
\]

The momentum equation can then be written as

\[
\frac{\partial v}{\partial t} + (f + \zeta)(k \times v) + \nabla \left( M + \frac{v \cdot v}{2} \right) = 0. \tag{8}
\]

By applying the operators \(k \cdot \nabla \times\) and \(\nabla \cdot\) to the equation above we obtain, respectively,

\[
\frac{\partial \zeta}{\partial t} + \nabla \cdot [(f + \zeta) v] = 0, \tag{9a}
\]

\[
\frac{\partial D}{\partial t} + \nabla \cdot \left[ (f + \zeta)(k \times v) + \nabla \left( M + \frac{v \cdot v}{2} \right) \right] = 0, \tag{9b}
\]

where \(\zeta = k \cdot \nabla \times v\) is the vorticity and \(D = \nabla \cdot v\) is the divergence of \(v\). From the vorticity equation (9a) one obtains an equation for the absolute vorticity \(f + \zeta\)

\[
\frac{D}{Dt} (f + \zeta) + (f + \zeta)(\nabla \cdot v) = 0, \tag{10}
\]

and, by eliminating the divergence from the mass conservation equation (4) and the absolute vorticity equation (10), we obtain

\[
\frac{DP}{Dt} = 0, \quad P = \frac{f + \zeta}{\eta_l}. \tag{11}
\]

This equation expresses the material conservation of the potential vorticity \(P\). It can also be verified, using the momentum equation in the form (8) and the mass conservation
equation in the form (5), that we have
\[
\frac{\partial}{\partial t} \left[ \eta \left( \frac{v \cdot v}{2} + g z_1 + \frac{c_p \theta}{\kappa + 1} \eta^\kappa \right) \right] + \nabla \cdot \left[ \left( M + \frac{v \cdot v}{2} \right) \eta v \right] = 0. \tag{12}
\]
More details can be found in appendix A of the paper by Verkley (2000b). When the equation above is integrated globally we obtain the conservation of total energy \(E\), where \(E\) is given by
\[
E = \frac{p_r}{g} \int dS \eta \left( \frac{v \cdot v}{2} + g z_1 + \frac{c_p \theta}{\kappa + 1} \eta^\kappa \right). \tag{13}
\]
These are the conservation laws of the parent model that we wish to retain in a balanced approximation.

(b) **Hamilton’s principle for the parent model**

The basic idea of Salmon’s approach to balanced flow (Salmon 1983, 1985, 1988a, 1988b, 1996) is to cast the momentum equation in the form of Hamilton’s principle and to make approximations in the Lagrangian. To formulate Hamilton’s principle for the momentum equation of the parent model, Lagrangian column label fields \(\alpha\) and \(\beta\) are introduced. These labels are, by definition, materially conserved. By equating the total mass per unit horizontal area to the Jacobian of these label fields, i.e. by taking
\[
\left( \frac{p_r}{g} \right) \eta = \frac{1}{a^2 \cos \phi} \frac{\partial (\alpha, \beta)}{\partial (\lambda, \phi)}, \tag{14}
\]
where
\[
\frac{\partial (\alpha, \beta)}{\partial (\lambda, \phi)} = \frac{\partial \alpha}{\partial \lambda} \frac{\partial \beta}{\partial \phi} - \frac{\partial \alpha}{\partial \phi} \frac{\partial \beta}{\partial \lambda}, \tag{15}
\]
we find that the mass conservation equation (5) is satisfied. Indeed, it can be checked explicitly that (5) follows from (14), (15) and \(D\alpha/Dt = 0\) and \(D\beta/Dt = 0\). Note that the expressions above imply that the total mass above an area element \(dS\), which is given by \((p_r/g)\eta dS\), is given by \(dx\ d\beta\). Using the column label fields \(\alpha\) and \(\beta\) Hamilton’s principle for the parent model can be written as
\[
\delta \int \int \int d\tau \ d\alpha \ d\beta \ L = 0, \tag{16}
\]
where
\[
L = (u + \Omega \alpha \cos \phi) a \cos \phi \frac{\partial \lambda}{\partial \tau} + a \frac{\partial \phi}{\partial \tau} - H, \tag{17a}
\]
and
\[
H = \left( \frac{u^2 + v^2}{2} \right) + g z_1 + \frac{c_p \theta}{\kappa + 1} \eta^\kappa. \tag{17b}
\]
Here \(u\) and \(v\) are the zonal and meridional components of the horizontal velocity \(v\). The variable \(\tau\) is equal to the time \(t\), but we note that it has a different role depending on whether we use \((\lambda, \phi, \tau)\) as Eulerian independent variables or \((\alpha, \beta, \tau)\) as Lagrangian independent variables. A partial derivative in the first case is a local time derivative, whereas in the second case it is a material derivative. In the variational principle (16) the fields that are varied independently are \(u, v, \lambda, \phi\), considered as functions of \(\alpha, \beta\) and \(\tau\). Variations of \(u\) and \(v\) lead to the definitions of the zonal and meridional components of the horizontal velocity in terms of the material derivatives of \(\lambda\) and \(\phi\). Variations of \(\lambda\) and \(\phi\) give the zonal and meridional components of the momentum equation (1). More details can be found in appendix B of the paper by Verkley (2000b).
3. THE BALANCED MODEL

A Hamiltonian balanced approximation of the momentum equation is obtained by substituting for $u$ and $v$ the zonal and meridional components $u_b$ and $v_b$ of a balanced velocity $v_b$. Here 'balanced' is meant to denote any relation between wind field and mass field that eliminates gravity waves. As $u_b$ and $v_b$ are functions of $\lambda$ and $\phi$, the only variations to be considered are variations of $\lambda$ and $\phi$. For the balanced velocity we choose:

$$v_b = k \times \nabla \psi_b,$$  \hspace{1cm} (18)

where $\psi_b$ is the stream function of the balanced flow. The balanced stream function is related to the Montgomery potential by

$$M = \overline{M} + f \psi_b.$$  \hspace{1cm} (19)

Expressions (18) and (19) are a simplified form of linear balance and referred to by Daley (1983) as the simplest form of the geostrophic relationship. We note that (19) implies that the balanced stream function at the state of rest must be identically zero and that $M$ must behave smoothly at the equator.

Because the balanced velocity as defined above is non-divergent, there is no net balanced transport of air over any latitude circle, in particular over the equator. Standard geostrophy, however, is more restrictive; it implies that the meridional wind velocity is zero at the equator. To illustrate this we combine (18) and (19) to get $v_b = k \times \nabla f^{-1}(M - \overline{M})$. If $f$ were constant this velocity field would be equal to $v_b = k \times f^{-1} \nabla M$, the expression for standard geostrophic balance. However, when $f$ does vary—in particular when it goes to zero at the equator—it makes much difference whether the factor $f^{-1}$ is placed in front or after the gradient operator. Indeed, by calculating the divergence of the latter expression we readily verify that we have: $v_b = -a \tan \phi (\nabla \cdot v_b)$. Because the divergence of any velocity field should be finite, this expression implies that $v_b = 0$ at $\phi = 0$. So, standard geostrophic balance forces the equator to be a rigid impenetrable barrier between the hemispheres for the balanced part of the flow. Now, the unbalanced part of the flow might allow for cross-equatorial flow as is the case in the semi-geostrophic model used by Mawson and Cullen (1992) in their study of the Indian Monsoon. However, we do not wish to rely on the unbalanced part of the flow to account for interhemispheric exchange of air and, therefore, prefer (18) and (19) in favour of standard geostrophic balance.

(a) *Hamilton’s principle for the balanced model*

An equation for the time-dependence of the balanced velocity follows from Hamilton’s principle:

$$\delta \int \int \int d\tau \ d\alpha \ d\beta \ L_b = 0,$$  \hspace{1cm} (20)

where

$$L_b = (u_b + \Omega a \cos \phi) a \cos \phi \frac{\partial \lambda}{\partial \tau} + v_b a \frac{\partial \phi}{\partial \tau} - H_b,$$  \hspace{1cm} (21a)

$$H_b = \frac{(u_b^2 + v_b^2)}{2} + g z_1 + \frac{c_p \theta}{\kappa + 1} \eta^\kappa.$$  \hspace{1cm} (21b)
Calculating the variations with respect to \( \lambda \) and \( \phi \) in (20) is a tedious task and for details we refer to appendix B of the paper by Verkley (2000b). The result, however, is quite transparent and the variations with respect to \( \lambda \) and \( \phi \) lead to the zonal and meridional components of the following momentum equation:

\[
\frac{D_b v_b}{Dt} + f k \times v_b + \nabla M + (f + \zeta_b) k \times v_a + \nabla [(\xi_1 f)^{-1} k \cdot \nabla \times (\eta v_a)] = 0. \tag{22}
\]

Here \( D_b/Dt \) is the material derivative in which \( u \) and \( v \) are replaced by \( u_b \) and \( v_b \). A subscript \( b \) means, as usual, that the field has to be evaluated at the lower boundary. The velocity \( v_a \) is the unbalanced velocity defined by \( v_a = v - v_b \), where \( v \) is the true velocity in terms of material derivatives of \( \lambda \) and \( \phi \). We furthermore introduce, for ease of notation,

\[
\xi \equiv \frac{\eta^{1-k}}{\kappa_c \rho \vartheta}. \tag{23}
\]

When compared with the momentum equation (1) of the parent model we see that two additional terms have arisen: an extra Coriolis term and an extra gradient term, both of which are zero if the unbalanced velocity is zero. When the unbalanced velocity is non-zero, these terms are needed for potential vorticity and energy conservation.

(b) The unbalanced velocity

The unbalanced velocity can be obtained from the balanced fields by a diagnostic relation. This relation is obtained by first using (7) for the material derivative of the balanced velocity (with \( v \) replaced by \( v_b \)) to rewrite the momentum equation as

\[
\frac{\partial v_b}{\partial t} + (f + \zeta_b)(k \times v) + \nabla \left[ M + \frac{v_b \cdot v_b}{2} + (\xi_1 f)^{-1} k \cdot \nabla \times (\eta v_a) \right] = 0. \tag{24}
\]

We then use the definition of the balanced velocity (18) and (19), in combination with expression (2) for \( M \) and the mass conservation equation (5), to obtain

\[
\frac{\partial v_b}{\partial t} = -k \times \nabla [(\xi_1 f)^{-1} \nabla \cdot (\eta v)]. \tag{25}
\]

By eliminating \( \partial v_b/\partial t \) from both equations we find:

\[
-k \times \nabla [(\xi_1 f)^{-1} \nabla \cdot (\eta v)] + (f + \zeta_b)(k \times v) + \nabla \left[ M + \frac{v_b \cdot v_b}{2} + (\xi_1 f)^{-1} k \cdot \nabla \times (\eta v_a) \right] = 0, \tag{26}
\]

which is the desired expression. To find a practical way to obtain the unbalanced velocity from this expression, we note that the first term on the left-hand side of (26) is divergenceless and that the third term is rotationless. By applying the Helmholtz decomposition to the middle term and using the vector identity \( k \cdot \nabla \times A = -\nabla \cdot (k \times A) \), we observe that (26) is equivalent to the following pair of scalar equations:

\[
(\xi_1 f)^{-1} \nabla \cdot (\eta v) - \nabla^2 \nabla \cdot [(f + \zeta_b) v] = 0, \tag{27a}
\]

\[
\nabla^2 \nabla \cdot [(f + \zeta_b)(k \times v)] + f \psi_b + \frac{v_b \cdot v_b}{2} + (\xi_1 f)^{-1} k \cdot \nabla \times (\eta v_a) = 0. \tag{27b}
\]

To obtain these expressions we assumed that the result should also be valid at the state of rest, where \( v = 0 \) and \( M = \bar{M} \). We also used that \( M = \bar{M} + f \psi_b \) and defined the
inverse of the Laplacian to have a zero average. If we write \( \mathbf{v} = \mathbf{v}_b + \mathbf{v}_a \), multiply by \( \xi_1 f \), use the balanced potential vorticity defined in (33) and carry out some rearranging, we can put these equations in the form:

\[
\nabla \cdot (\eta_1 \mathbf{v}_a) - \xi_1 f \nabla^{-2} \mathbf{v} \cdot [P_b \eta_1 \mathbf{v}_a] = S_x,
\]

(28a)

\[
\mathbf{k} \cdot \nabla \times (\eta_1 \mathbf{v}_a) + \xi_1 f \nabla^{-2} \nabla \cdot [P_b \mathbf{k} \times \eta_1 \mathbf{v}_a] = S_\psi,
\]

(28b)

where the terms \( S_x \) and \( S_\psi \) are defined by

\[
S_x = -\nabla \cdot (\eta_1 \mathbf{v}_b) + \xi_1 f \nabla^{-2} \mathbf{v} \cdot [(f + \zeta_b) \mathbf{v}_b],
\]

(29a)

\[
S_\psi = -\xi_1 f \left( f \mathbf{v}_b + \frac{\mathbf{v}_b \cdot \mathbf{v}_b}{2} \right) - \xi_1 f \nabla^{-2} \mathbf{v} \cdot [(f + \zeta_b)(\mathbf{k} \times \mathbf{v}_b)].
\]

(29b)

If we finally use the Helmholtz decomposition of \( \eta_1 \mathbf{v}_a \),

\[
\eta_1 \mathbf{v}_a = \mathbf{k} \times \nabla \psi'_a + \nabla \chi'_a,
\]

(30)

we can write

\[
\nabla^2 \chi'_a - \xi_1 f \nabla^{-2} \nabla \cdot [P_b (\nabla \chi'_a + \mathbf{k} \times \nabla \psi'_a)] = S_x,
\]

(31a)

\[
\nabla^2 \psi'_a - \xi_1 f \nabla^{-2} \nabla \cdot [P_b (\nabla \psi'_a - \mathbf{k} \times \nabla \chi'_a)] = S_\psi.
\]

(31b)

This is a linear system of equations that one can solve for \( \chi'_a \) and \( \psi'_a \). By dividing (30) by \( \eta_1 \) and again applying a Helmholtz decomposition, we can obtain the fields \( \psi_a \) and \( \chi_a \) of the Helmholtz decomposition of the unbalanced velocity itself.

(c) Conservation laws of the balanced model

Salmon’s Hamiltonian approach guarantees that both energy and potential-vorticity conservation have their counterparts in the balanced model. Mass remains to be conserved because the mass conservation equation (4) is part of the balanced model. The material conservation of balanced potential vorticity can be verified easily. From the momentum equation in the form (24) we can obtain an equation for the balanced absolute vorticity by operating with \( \mathbf{k} \cdot \nabla \times \) on this equation:

\[
\frac{D}{Dt} (f + \zeta_b) + (f + \zeta_b)(\nabla \cdot \mathbf{v}) = 0.
\]

(32)

This equation can be combined, in the usual way, with the mass conservation equation (4) to give the material conservation of the balanced potential vorticity,

\[
\frac{DP_b}{Dt} = 0, \quad P_b = \frac{f + \zeta_b}{\eta_1}.
\]

(33)

Concerning energy conservation, it is shown in appendix A of the paper by Verkley (2000b) that we have the following counterpart of the local energy conservation law (12)

\[
\frac{\partial}{\partial t} \left[ \eta_1 \left( \frac{\mathbf{v}_b \cdot \mathbf{v}_b}{2} + g \zeta_1 + \frac{c_p \theta}{\kappa + 1} \eta_1 \right) \right] + \nabla \cdot \left[ \left( M + \frac{\mathbf{v}_b \cdot \mathbf{v}_b}{2} \right) \eta_1 \mathbf{v} \right] + \nabla \cdot [(\xi_1 f)^{-1} \mathbf{k} \cdot \nabla \times (\eta_1 \mathbf{v}_a)] = 0.
\]

(34)

Integrated over the whole sphere, this equation implies that

\[
E_b = \frac{p_r}{g} \int dS \eta_1 \left( \frac{\mathbf{v}_b \cdot \mathbf{v}_b}{2} + g \zeta_1 + \frac{c_p \theta}{\kappa + 1} \eta_1 \right)
\]

is conserved. The quantity \( E_b \) will be called the balanced energy.
4. Time integration of the balanced model

In the balanced model we have three prognostic scalar equations: Eq. (4) for the normalized surface pressure, Eq. (32) for the balanced absolute vorticity, and Eq. (33) for the balanced potential vorticity. In the balanced model (subject to a particular condition) each one of these equations implies the validity of the others, as shown by Verkley (2000b). This is interesting from the viewpoint of time stepping, because it allows one to use either the mass conservation equation, the balanced absolute-vorticity equation or the balanced potential-vorticity equation to step the model forward in time. Time stepping in terms of the normalized surface pressure confronts us with the difficulty of finding \( \psi_b \) from \( \eta_1 \) because one has to divide by \( f \). Although not impossible, it means that extra smoothness conditions on \( M \) have to be maintained. Time stepping in terms of the balanced absolute vorticity avoids this difficulty because the stream function can then be readily obtained by inverting the Laplacian and finding the normalized surface pressure only involves a multiplication with \( f \). We will, however, choose the third possibility because advection as the basic time stepping process allows the use of Lagrangian or semi-Lagrangian integration procedures which offer the best perspectives for a fast performance of the model.

So, we step the model forward in time by advecting the balanced potential vorticity \( P_b \) with the full velocity \( \mathbf{v} = \mathbf{v}_b + \mathbf{v}_a \). We now describe in more detail how—if such a time step is made—the new velocity \( \mathbf{v} \) can be obtained from the new potential vorticity \( P_b \). Rewriting the definition of the potential vorticity we have

\[
\nabla^2 \psi_b = P_b \eta_1 - f.
\]

For \( \eta_1 \) we have, combining (2) and (19),

\[
\eta_1 = \left( \frac{\overline{M} + f \psi_b - g z_1}{c_p \theta} \right)^{1/\kappa}.
\]

so that

\[
\nabla^2 \psi_b = P_b \left( \frac{\overline{M} + f \psi_b - g z_1}{c_p \theta} \right)^{1/\kappa} - f.
\]

This is a nonlinear equation that relates \( \psi_b \) to \( P_b \). The value \( \overline{M} \) is the value of the Montgomery potential at the state of rest. From the definition of the Montgomery potential we see that the state of rest is characterized by the following normalized surface pressure distribution:

\[
\overline{\eta}_1 = \left( \frac{\overline{M} - g z_1}{c_p \theta} \right)^{1/\kappa}.
\]

Now, if we use the fact that the total mass \( m \) of the atmosphere is a given constant, the expression above fixes \( \overline{M} \) in terms of \( m \) because we have

\[
m = \frac{p_r}{g} \int dS \, \eta_1 = \frac{p_r}{g} \int dS \, \overline{\eta}_1 = \frac{p_r}{g} \int dS \left( \frac{\overline{M} - g z_1}{c_p \theta} \right)^{1/\kappa}.
\]

In the numerical simulation to be discussed, we start from the state of rest, where \( \overline{\eta}_1 \) is given by (39) with \( \overline{M} = c_p \theta \). It is ensured in the integration that the mass does not change in time, so that \( \overline{M} \) is and remains equal to \( c_p \theta \).
We have not been able to establish the conditions under which (38) is formally solvable. However, experience has shown that an iteration procedure always leads to a solution. Furthermore, the resulting solution is unique to the extent that different first guesses do not lead to substantially different solutions*. The iteration procedure, as actually implemented, starts with \( \psi_b = 0 \) as a first guess on the right-hand side of (38). Then \( \psi_b \) on the left-hand side is obtained by inverting the Laplace operator; subsequently \( \psi_b \) is updated by adding a fraction \( r \) of the difference between this and the previous field to \( \psi_b \), after which the result is substituted back in the right-hand side of (38). We then again invert the Laplacian, etc., and continue the procedure until the required accuracy is obtained. From the resulting stream function \( \psi_b \) the fields \( \xi_b, v_b, \eta_l \) and \( M \) can be obtained straightforwardly.

Having obtained the balanced flow variables we may solve Eqs. (31a) and (31b) to determine the unbalanced velocity \( v_a \). We note that, without the second terms on the left-hand side, the system (31) consists of two uncoupled Poisson equations that are clearly solvable. Whether the system is formally solvable if these terms are present is another open question, but also here experience has shown that an iterative procedure always leads to a solution that is apparently unique. As first guesses for \( \psi'_a \) and \( \chi'_a \) we take the fields that result from solving (31) without the second terms on the left-hand side and with \( S_\chi = -\nabla \cdot (\eta_l v_b) \) and \( S_\psi = 0 \). So the first guesses of \( \chi'_a \) and \( \psi'_a \) satisfy

\[
\nabla^2 \chi'_a + \nabla \cdot (\eta_l v_b) = 0, \quad (41a)
\]
\[
\nabla^2 \psi'_a = 0. \quad (41b)
\]

In the iteration procedure, the second terms on the left-hand side of (31) are treated as source terms in which fields \( \psi'_a \) and \( \chi'_a \) from the previous iteration are substituted. The new \( \psi'_a \) and \( \chi'_a \) are obtained by inverting the corresponding Laplacians, as in (38), after which the fields are updated by adding a fraction \( r \) of the difference between the new fields and the previous fields. The procedure is repeated as many times as is considered necessary to get the required accuracy.

5. Numerical integrations

To have a reference against which the balanced model can be tested, we performed a long numerical integration of a spectral implementation of the parent model. Within this period a shorter period was selected during which cyclogenesis took place. In this paper we check to what extent the balanced model is able to predict this phenomenon from an earlier initial state.

We first give some details on the spectral method that is used. We recall that the parent model is originally given by the momentum equation (1), expression (2) for the Montgomery potential and the mass conservation equation (4). The velocity field \( v \) is Helmholtz-decomposed in terms of a divergenceless and a rotationless part, \( v = k \times \nabla \psi + \nabla \chi \), where the stream function \( \psi \) and the velocity potential \( \chi \) are related to the vorticity \( \xi \) and the divergence \( D \) by \( \nabla^2 \psi = \xi \) and \( \nabla^2 \chi = D \). Instead of the momentum equation, we use Eqs. (9a) and (9b) for the relative vorticity and the divergence. We, furthermore, use the mass conservation equation in the form (5) and expression (2) for the Montgomery potential. The parent model is thus formulated in

* All this is true even if the product \( f P_h \) is slightly negative, which happens occasionally around the equator. Apparently our choice of basic balance is able to cope with a mild form of inertial instability. Until the exact solvability conditions of (38) are known, this remains an open issue.
terms of the scalar fields $\zeta$, $D$, $\eta_1$ and $M$. In the numerical integration we will make use of $\Omega^{-1}$ and $a$ as the units of time and length, respectively.

The scalar fields will be represented by spherical harmonics $Y_{mn}(\lambda, \phi)$, where $m$ and $n$ are integers with $n \geq |m|$. The spherical harmonics are normalized such that they are orthonormal with respect to the usual inner product $\langle \psi, \chi \rangle = 1/(4\pi) \int dS \psi^* \chi$ for functions $\psi$ and $\chi$ on a sphere, where the asterisk denotes the complex conjugate and the integral is over the whole sphere. We will use a triangular T42 truncation, i.e., we use all spherical harmonics up to $n = 42$. Spherical harmonics are eigenfunctions of the Laplace operator with eigenvalues $-n(n+1)$, so that $\nabla^2 \psi$ and its inverse $\nabla^{-2} \psi$ can be represented easily and exactly in terms of a diagonal matrix. The other operators are approximated by applying the operator on a Gaussian grid (equidistant in $\lambda$, Gaussian quadrature points for $\sin \phi$) and projecting the result on $Y_{mn}$ by a summation over this grid. These operators are $k \cdot \nabla \psi \times \nabla \chi$, $\nabla \psi \cdot \nabla \chi$, $\psi \chi$, $\psi^\mu \chi$ and $\psi^\mu$, where $\mu$ is a positive real number. All operators encountered (also in the context of the balanced model) can be—and are actually—reduced to one of the operators mentioned above. We use a Gaussian grid of $128 \times 64$ points, as a result of which the projection is exact for the operators $k \cdot \nabla \psi \times \nabla \chi$, $\nabla \psi \cdot \nabla \chi$ and $\psi \chi$. For more details on the spectral method for a sphere we refer to Machenhauer (1979).

(a) A numerical integration of the parent model

Using a fourth-order Runge–Kutta time-stepping scheme with a time step of 5 min, the model is integrated in time for a total period of 500 days. To suppress the emergence of spurious small-scale structures at the truncation limit a hyperviscosity term is added to Eqs. (9a) and (9b). More specifically, the evolution equations for the relative vorticity and the divergence become

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot [(f + \zeta)v] + \frac{1}{\tau_h} \nabla^{12} \zeta = 0,$$  \hspace{1cm} (42a)

$$\frac{\partial D}{\partial t} + \nabla \cdot \left[(f + \zeta)(k \times v) + \nabla \left(M + \frac{v \cdot \nabla v}{2}\right)\right] + \frac{1}{\tau_h} \nabla^{12} D = 0,$$  \hspace{1cm} (42b)

where $\tau_h$ is chosen such that the spectral coefficients of $\zeta$ and $D$ with $n = 42$ are damped with an e-folding time of three hours. Furthermore, in order to spin up the model, it is forced to a zonal surface pressure distribution $\eta_1^f$. The mass conservation equation (5), therefore, becomes

$$\frac{\partial \eta_1}{\partial t} + \nabla \cdot [\eta_1v] + \frac{1}{\tau_f} (\eta_1 - \eta_1^f) = 0,$$  \hspace{1cm} (43)

where the e-folding time $\tau_f$ is 15 days and the surface pressure distribution $\eta_1^f$ is given by

$$\eta_1^f(\phi) = \alpha^f + \beta^f \left[ \cos 2\phi (\sin^2 2\phi + 2) - \frac{13}{16} \right].$$  \hspace{1cm} (44)

Here $\alpha^f$ is the mean value of the pressure distribution and $\beta^f$ the strength of its meridional variation. For the parameters $\alpha^f$ and $\beta^f$ we have taken 0.97482 and 0.05, respectively. A graph of $\eta_1^f$ as a function of $\phi$ is given in Fig. 2(a).

The initial state of the integration is the state of rest with $\zeta = 0$, $D = 0$ and $\eta_1$ given by (39), with $\overline{M} = c_p \theta$. Note that with this choice the pressure field $\bar{\eta}_1$ at the state of rest has the value 1 at $z_1 = 0$. The total mass of the atmosphere follows from (40). For
the field $z_1$ we use the T42 representation of the earth’s orography. A plot of $g z_1$ in units of $a^2 \Omega^2$ is shown in Fig. 2(b). With $c_p = 1005 \text{ J K}^{-1} \text{kg}^{-1}$, $R = 287.04 \text{ J K}^{-1} \text{kg}^{-1}$ (values taken from the paper by Dutton (1986, appendix 3)) and $\theta = 300 \text{ K}$, we have $\kappa = R/c_p = 0.28561$ and $c_p \theta = 1.39694 a^2 \Omega^2$. The mean value of $\bar{\eta}_1$ is 0.97482; this is the value taken for $\alpha^f$ in expression (44). As a result, the forcing to the prescribed surface pressure, expressed by (43), does not change the total mass of the model atmosphere and, therefore, does not change the value of $\bar{M}$ which thus remains $c_p \theta$.

In the period from 83 to 93 days after the start of the integration a cyclogenesis process occurs as a result of interaction with the orography. We concentrate on this period. To obtain a reference integration that is as close as possible to an inviscid integration we rerun the model for 10 days from day 83 with the forcing term in the mass conservation equation turned off. (The hyperviscosity term was retained in order to suppress small-scale structures.) The potential vorticity $P$ and the velocity field $v$ are shown in Fig. 3 at six times in this period of ten days: at 0, 2, 4, 6, 8 and 10 days after the rerun from day 83. The potential vorticity is calculated by evaluating the quotient of $f + \zeta$ and $\eta_1$ on the Gaussian grid and projecting the result back to T42 by a summation over the grid—as with the other nonlinear operators. The differences between the original simulation are not very large, except that in the original simulation the vortices that form in the process of wave breaking are dissipated somewhat faster. The cyclogenesis process manifests itself in the pinching off of potential vorticity around day 8. In Fig. 4 we show the other fields at day 0: the relative vorticity $\zeta$; the stream function $\psi$; the divergence $\mathcal{D}$; the velocity potential $\chi$; the normalized surface pressure $\eta_1$; and the Montgomery potential $M$. The divergence is about four times smaller than the relative vorticity and is concentrated around the extrema in the orography. Furthermore, it can be seen that the relative vorticity has concentrated into well-defined bands (with corresponding jets), in particular at the northern hemisphere.

(b) A numerical integration of the balanced model

The balanced model is governed by the advection of balanced potential vorticity, as expressed by (33). We recall that this equation is based on the combination of the equations for the normalized surface pressure and the balanced absolute vorticity. When
Figure 3. The potential vorticity (colours) and velocity (arrows) given by the numerical integration of the parent model after (a) 0, (b) 2, (c) 4, (d) 6, (e) 8 and (f) 10 days from the start of the rerun, initialized with the fields after 83 days of the original integration. The rerun differs from the original integration in that the forcing to the prescribed zonal pressure distribution is turned off. Here, and in other plots of a similar type, the potential vorticity is plotted on a regular longitude-latitude grid of $256 \times 128$ points; the velocity vectors are displayed on the same type of grid but with $32 \times 16$ points. The unit of potential vorticity in the plots is $\Omega$ (the earth's angular velocity) and the contour interval is 0.5; the values from $-3.0$ to $-2.5$ are coloured deep blue, the values from $-2.5$ to $-2.0$ somewhat lighter blue, etc., until the values from $-0.5$ to 0 that are coloured white; then the values from 0 to 0.5 are coloured light yellow, the values from 0.5 to 1.0 somewhat darker yellow, etc., until the values from 2.5 to 3.0 that are coloured deep red. This particular period was chosen because of the cyclogenesis process that occurred within this period, leading to a cyclone at ($90^\circ$W, $45^\circ$N) around day 8.

we add the forcing term to the equation for the normalized surface pressure and the hyperviscosity term to the equation for the balanced absolute vorticity (the $\nabla^{12}$ operator acting on the balanced relative vorticity), we may combine these equations into an equation for the balanced potential vorticity. This equation then becomes (writing out
Figure 4. (a) The relative vorticity \( \zeta \times 10 \), (b) the stream function \( \psi \times 1000 \), (c) the divergence \( \nabla \times 10 \), (d) the velocity potential \( \chi \times 1000 \), (e) the normalized surface pressure \( \eta \), and (f) the Montgomery potential \( M \) at day 0 of the model run shown in Fig. 3. The convention here, and in other plots of a similar type, is that solid isolines denote positive and zero values and that dashed isolines denote negative values. In plots of the divergence the zero contour is deleted. Note that the relative vorticity—in particular in the northern hemisphere—has concentrated into a well-defined band, and that the divergence is about four times smaller than the relative vorticity. The divergence is largest around high values of the orography. The units of the variables plotted are: \( \Omega \) for the relative vorticity and divergence, \( a^2 \Omega \) for the stream function and velocity potential, and \( a^2 \Omega^2 \) for the Montgomery potential (\( a \) is the earth's radius and \( \Omega \) its angular velocity).

The material derivative

\[
\frac{\partial P_b}{\partial t} + \mathbf{v} \cdot \nabla P_b = \delta, \tag{45}
\]

where the source term \( \delta \) is given by

\[
\eta \delta = -\frac{1}{\tau_h} \nabla^\perp \zeta_b + \frac{1}{\tau_f} P_b (\eta - \eta^f). \tag{46}
\]
In the integration to be discussed, we used (45) and (46) with the second term in the forcing \(\delta\) put to zero.

Before discussing the result of the integration we will consider how well the balancing procedure is able to recover the fields as displayed in Fig. 4 from the potential vorticity field at day 0 (shown in Fig. 3(a) and identified with \(P_b\)). We begin by discussing the balanced fields. The balanced stream function \(\psi_b\) is calculated from (38) using the iteration procedure that is described in section 4. The iteration procedure is stopped if the norm (the square root of the inner product) of the update becomes smaller than 0.5% of the norm of the present field. With a relaxation factor \(r = 0.3\), an optimal value, the procedure finished after 14 iterations—the result is shown in Fig. 5. We recall that \(M\) is calculated from \(\psi_b\) using (19), with \(\overline{M} = c_0 \theta\), whereas \(\eta\) is calculated from \(M\) using (2). When we compare Figs. 5(a)–(d) with Figs. 4(a), (b), (c) and (f), respectively, we see that the fields obtained from \(P_b\) are close to their original counterparts. To quantify the resemblance we calculated the maximum absolute value of the difference between the fields in Fig. 5 and their counterparts in Fig. 4, using the 256 \(\times\) 128 grid values of the plots. For the difference between the fields in Figs. 4(a), (b), (e) and (f) and Figs. 5(a)–(d), respectively, we obtain 0.0308 (2%), 0.00587 (15%), 0.0318 (3%) and 0.0109 (1%) (the results in brackets are percentages of the maximum absolute values of the fields in Fig. 4).

By definition the balanced flow has zero divergence. The divergence field is obtained by solving (31) for the unbalanced part of the flow, using the method outlined in the previous section. We use a relaxation factor \(r = 0.5\) and stop the iteration when the norms of both updates are smaller than 0.5% of the norms of the corresponding fields. The procedure needed eight iterations to reach this criterion. From the fields \(\psi'_a\) and \(\chi'_a\)
we obtain $\psi_a$ and $\chi_a$ using the method outlined after Eq. (31). The divergence field is given by $D_a = \nabla^2 \chi_a$. The results are shown in Fig. 6. By comparing Figs. 6(c) and (d) with Figs. 4(c) and (d) we see that the divergence field is also recovered well. To quantify the degree of resemblance we again calculated the maximum absolute differences. For Figs. 4(c) and 4(d) and Figs. 6(c) and 6(d), respectively, the results are 0.0208 (6%) and 0.000185 (10%).

To summarize, Figs. 5(c) and (d) show the reconstruction of the fields $\eta_l$ and $M$, respectively, and should be compared with Figs. 4(e) and (f). Figures 6(c) and (d) show the reconstruction of the fields $D$ and $\chi$, respectively, and are to be compared
with Figs. 4(c) and (d). Figures 6(a) and (b) show the unbalanced contributions to the fields $\zeta$ and $\psi$ of which the balanced contributions are shown in Figs. 5(a) and (b). For completeness we give the sum of the balanced and unbalanced contributions to $\zeta$ and $\psi$ in Figs. 7(a) and (b). If we calculate the maximum absolute difference between the fields shown in Fig. 7 and their counterparts in Fig. 4 we now obtain 0.0588 (3%) and 0.00188 (5%), respectively. Measured in this particular way the relative vorticity deteriorates somewhat, although a more detailed analysis shows that it both slightly improves and deteriorates, depending on the position. The stream function, though, generally improves in the sense that the difference with the original field is reduced considerably. In Fig. 8 we show how well the velocity field $v$ of the numerical integration is recovered by $v_b$ and $v_a$. The maximum value is 6.13 m s$^{-1}$ which is 10% of the maximum velocity in Fig. 3(a). We see that the balanced velocity field gives a reasonable approximation to $v$, apart from the equatorial region and the regions with high values of the orography. It is here that the unbalanced velocity field helps to improve the resemblance, as can be seen from Fig. 8(b), where we show $v - v_b - v_a$. Here the maximum value is 1.40 m s$^{-1}$ or 2% of the maximum value of Fig. 3(a).

From the potential vorticity $P$ at day 0, identified with the balanced potential vorticity $P_b$, we have integrated our balanced model forward in time for ten days. The results are very close to the results displayed in Fig. 3. Instead of showing the results in the format of Fig. 3, we show the result at day 10 and the difference with the parent integration at day 10 in Figs. 9(a) and (b), respectively. The maximum absolute value of the difference between the potential vorticity of the balanced model and the potential vorticity of the parent model is 0.54 (or 20% of the maximum absolute value of the latter field). For the velocity field the maximum absolute value of the difference is 14.1 m s$^{-1}$ (or 21% of the maximum value of the latter field). In order to see to which extent the unbalanced velocity is important in this respect, we have repeated the integration with the unbalanced velocity $v_a$ put to zero. The results are given in Figs. 9(c) and (d). The maximum absolute values of the difference in potential vorticity and velocity are now 1.53 and 38.8 m s$^{-1}$, respectively, i.e. 51% and 59% of the maximum values of the parent fields. So, inclusion of the unbalanced velocity is essential for the performance of the model.

To give an idea of how the differences between the parent integration and the two balanced integrations evolve in time we show in Fig. 10 the following mean squared
Figure 9. (a) The result at day 10 of the integration with the balanced model. The result is shown in terms of potential vorticity (colours) and velocity (arrows). In (b) the difference is shown between this end state and the end state of the parent integration, again in terms of potential vorticity (colours) and velocity (arrows). In (c) and (d) the results are shown for a balanced integration in which the unbalanced velocity is put to zero. Plotting conventions as in Fig. 3.

differences as a function of time:

\[ D_{P_c, P} = \frac{1}{4\pi} \int dS (P_c - P)^2, \]  \hspace{1cm} (47a)  
\[ D_{v_c, v} = \frac{1}{4\pi} \int dS (v_c - v)^2, \]  \hspace{1cm} (47b)  

where the unsubscripted fields refer to the parent model and the fields with a subscript c refer to the two balanced models. Using partial integration and the inner product we can write

\[ D_{P_c, P} = \langle P_c - P, P_c - P \rangle, \]  \hspace{1cm} (48a)  
\[ D_{v_c, v} = -\langle \psi_c - \psi, \zeta_c - \zeta \rangle - \langle \chi_c - \chi, D_c - D \rangle, \]  \hspace{1cm} (48b)  

which expressions were used in the actual computations. Notice that \( P_c \) is always equal to \( P_b \), but \( v_c = v_b + v_a \) in the full balanced integration whereas \( v_c = v_b \) in the integration with zero unbalanced velocity. In Fig. 10(a) we show the value of \( D_{P_c, P} \) as function of time for the two different models. We see that the differences are very substantial, in particular at the end of the integration period. The same behaviour can be seen in a graph of \( D_{v_c, v} \) as a function of time, shown in Fig. 10(b). Note that the graphs differ already at \( t = 0 \) because the unbalanced velocity is not included in the second case.
6. SUMMARY AND DISCUSSION

Salmon’s method (Salmon 1983, 1985, 1988a, 1988b, 1996) of constructing balanced approximations of geophysical fluid systems guarantees the existence of conservation laws that correspond to the conservation laws of the original (parent) systems. In the present paper Salmon’s method is applied to a one-layer isentropic model of the atmosphere. Although very idealized, this is a physically consistent simplification of the atmosphere, i.e. a simplified but exact solution of the inviscid hydrostatic primitive equations. As discussed in section 2, the model is analogous to a one-layer shallow-water model and is governed by an equation for the time rate of change of horizontal momentum (1), an equation relating the Montgomery potential to the surface pressure (2), and an equation for the conservation of mass (4). Besides mass, the model conserves potential vorticity and energy. Following Salmon, we derive in section 3 a balanced approximation of this model by formulating the momentum equation in terms of Hamilton’s principle and then substituting a balanced velocity field into the Lagrangian. The balanced velocity field that is chosen, given by (18) and (19), is a simplification of linear balance. In contrast to standard geostrophy in which the balanced velocity field is given by \( \mathbf{v}_b = \mathbf{k} \times f^{-1} \nabla M \), our choice, \( \mathbf{v}_b = \mathbf{k} \times \nabla f^{-1} (M - \bar{M}) \), allows for cross-equatorial flow. Application of Hamilton’s principle leads to a balanced approximation (22) of the horizontal momentum equation. Combined with the original mass conservation equation it gives a diagnostic equation (26) for the unbalanced velocity field \( \mathbf{v}_a \). This equation can be transformed into two scalar equations (31). It is verified that the balanced model respects analogues of the conservation of potential vorticity and energy. Mass conservation is, by construction, incorporated in the balanced model.

The central results of this paper are (22), the balanced approximation of the momentum equation (1), and the set of equations (30)–(31) that determine the unbalanced velocity. The balanced model that we have obtained can be compared with other balanced models by considering the last term on the left-hand side of (22). If this term is zero, one obtains the geostrophic vorticity approximation discussed—among many other approximations—by Allen et al. (1990a, 1990b) and Barth et al. (1990). It has many properties in common with our approximation, although it has not been proved that an energy invariant exists. If standard geostrophic balance had been used, the last
term on the left-hand side of (22) would have been
\[ \nabla [\xi^{-1}_l \mathbf{k} \cdot \nabla \times (\eta f^{-1}_l \mathbf{v}_a)]. \] (49)

The difference concerns the position of the factor \( f^{-1} \). The same holds for (25) where the factor \( f^{-1} \) would have appeared in front of the gradient operator. The resulting equation (26) would, of course, reflect these changes. We note that transforming this vector equation into scalar equations is now more involved because the first term on the left-hand side is no longer divergenceless. The most widely used alternative with which any balance approximation needs to be compared is the geostrophic momentum approximation (Hoskins 1975). In this case the last term on the left-hand side of (22) would have been, as can be easily verified from Eqs. (10) of the latter reference,
\[ \mathbf{v}_a \cdot \nabla \mathbf{v}_b - \zeta_0 \mathbf{k} \times \mathbf{v}_a. \] (50)

In Hoskins's (1975) formulation \( \mathbf{v}_b \) is given by standard geostrophic balance. Also here, it is quite difficult to obtain the unbalanced velocity. To solve this problem Hoskins (1975) introduced a coordinate transformation such that fluid particles move with the geostrophic velocity. In an \( f \)-plane context this has been a very fruitful approach, allowing solutions of many important problems. Generalizing these semi-geostrophic equations to spherical geometry, however, meets with the problem that the coordinate transformation is not trivial for a variable Coriolis parameter. It is the author's opinion that, despite notable attempts to solve these and other issues related to a variable Coriolis parameter (Shutts 1989; Magnusdottir and Schubert 1990, 1991; Mawson and Cullen 1992; Mawson 1996), a completely satisfactory solution has not yet been found. On an \( f \)-plane, standard geostrophic balance is identical to the approximation of linear balance that we have used. In this case the dynamics of the balanced model is the isentropic generalization of Salmon's \( L_1 \)-dynamics. The performance of Salmon's \( L_1 \)-dynamics in comparison with other approximations is investigated very thoroughly for shallow-water flow on an \( f \)-plane by Allen et al. (1990a, 1990b) and Barth et al. (1990). Many references to related work can be found there.

In section 4 we discuss how the balanced model that we developed can be integrated forward in time. We take the material conservation of balanced potential vorticity as the basic prognostic equation. The surface pressure and balanced velocity can be obtained by solving a nonlinear equation relating the balanced potential vorticity to the balanced stream function. It is pointed out how this equation and the equation for the unbalanced velocity can be solved by iteration. To produce a benchmark against which the balanced model can be tested, we discuss in section 5 a long integration with a forced and damped spectral implementation of the parent model. Within this period we select a shorter period of 10 days in which cyclogenesis occurs as a result of interaction with the model's orography. This period is rerun with the unforced and undamped parent model and taken as a reference. The result, in terms of potential-vorticity and velocity fields, is shown in Fig. 3. We investigate, for the initial state, how well the underlying basic fields can be reconstructed from the potential vorticity. In this context Fig. 8 is of particular interest; this figure shows how well the velocity field is reconstructed from the potential vorticity. The figure shows that the unbalanced velocity field is important around the equator and close to high values of the orography. We then discuss a ten-day integration of the balanced model. The results are summarized in Fig. 9. The balanced integration stays close to the parent integration as can be seen from Figs. 9(a) and (b). The differences at day 10 are of the order of 20% of the maximum values of the fields in the parent integration. Figures 9(c) and (d) show that the unbalanced velocity is crucial in maintaining this degree of accuracy. This is shown in a different manner in Fig. 10.
With a time step of two hours the balanced model in its present state of numerical sophistication is about as fast as the parent model when a time step of 15 min is used. This time-step value might possibly be taken somewhat larger, but not larger than 30 min as the model becomes numerically unstable in that case. The question arises whether the balanced model, as developed here, is an attractive alternative for the parent model. The answer depends on the accuracy that is required. For numerical weather prediction, high accuracy is important as local weather conditions are closely tied to the details of the flow. On the other hand, for climate simulations these details matter less as long as the relevant statistics are reproduced with sufficient accuracy (Opsteegh et al. 1998). If balanced models of the type discussed in the present paper are accurate enough for a given purpose, they have an advantage over primitive-equation models in that the state of the atmosphere is at any time completely given in terms of the potential vorticity field. This makes it easier to grasp the model's dynamics (Hoskins et al. 1985). Furthermore, as the basic prognostic equation is advection of balanced potential vorticity the contour-advective semi-Lagrangian (CASL) algorithm developed by Dritschel and Ambaum (1997) could be used to step such models forward in time. Because in this algorithm the potential vorticity is advected using contour advection while the inversion of potential vorticity is carried out in an Eulerian framework, the algorithm combines advective accuracy with computational efficiency. When the balanced procedure, as developed here, is extended to a multilayer isentropic model, a balanced Lagrangian weather prediction model comes in sight. If the efficiency of the inversion could be improved—and this certainly seems possible—such a model could be of much use in assisting the forecaster in monitoring the numerical weather prediction process and in understanding the structure and evolution of weather systems.

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